Applied Mathematics, 2022, 13, 753-759
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

An Estimate on Linear Functionals' Kernels in Banach Spaces, and Regularity of Convex Functionals

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How to cite this paper: Okochi, H. (2022) An Estimate on Linear Functionals' Kernels in Banach Spaces, and Regularity of Convex Functionals. *Applied Mathematics*, **13**, 753-759.
https://doi.org/10.4236/am.2022.139047

Received: July 29, 2022
Accepted: September 16, 2022
Published: September 19, 2022

Abstract

Motivated to obtain the second critical point of a nonlinear differential equation, which is expressed by derivatives of convex functional defined on a Banach space, an estimate with $\|f\|, \|f-g\|$ is given to see the relation between $f^{-1}(0)$ and $g^{-1}(0)$. And both the Fréchet differentiability and the continuity of Fréchet derivative of every convex functional defined on an open subset of a Banach space are shown.

Keywords

Banach Space, Convex Functional, Subdifferential, Frèchet Derivative, Gâteaux Derivative, Deformation Lemma, Mountain Pass Theorem

1. Introduction

Many important differential equations are concerned with derivatives of convex functional defined on real Banach spaces.

This paper's research is motivated to fined u in a Banach space $(X, \|.\|)$ such that

$$\{\partial\varphi(u)\}^{-1}(0) = \{\mathrm{d}\psi(u)\}^{-1}(0), \tag{1.1}$$

where $\mathrm{d}\psi$ denotes the Fréchet differential of a functional ψ and $\partial\varphi$ is the subdifferential of a lower semi-continuous convex functional φ. In general, for a proper convex functional φ and $u \in X$, the subgradients of φ at u are the elements $f \in X^*$ satisfying

$$\varphi(u) \le \varphi(w) + f(u-w), \quad \forall w \in X,$$

and the subdifferential $\partial\varphi(u)$ is the set of all subgradients of φ at u (see [1]).

For a most interesting example, put $X = L^q(\Omega)$, $\Omega \subset \mathbf{R}^n$, and

$$\varphi(w) = \frac{1}{p}\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial w}{\partial x_i}(x)\right|^p \mathrm{d}x,\ w \in D(\varphi),\ \ \psi(w) = \frac{1}{q}\|w\|^q. \tag{1.2}$$

If we can find u as a solution of (1.1) with (1.2), then there is $\alpha > 0$ such that αu is a critical point of $\varphi - \psi$. In this example, to find the second critical point is very interesting since the mountain pass theorem is not so useful.

The author wants to verify the following assertion.

Assertion 1.1. *Fix* $\tilde{\lambda} > \inf\varphi$, $\tilde{\mu} > \inf\psi$. *Assume that there are* $\delta, k > 0$ *satisfying*

$$\left|\varphi(v) - \tilde{\lambda}\right| < \delta, \left|\psi(v) - \tilde{\mu}\right| < \delta \Rightarrow \left\|\frac{\partial\varphi(v)}{\|\partial\varphi(v)\|} \pm \frac{\mathrm{d}\psi(v)}{\|\mathrm{d}\psi(v)\|}\right\| > \delta, \tag{1.3}$$

$$\left|\psi(v) - \tilde{\mu}\right| < \delta \Rightarrow \|\mathrm{d}\psi(v)\| \ge k. \tag{1.4}$$

Then, for $\forall\delta' \in (0,\delta)$, $C(\varphi,\tilde{\lambda}) \cap \{v : \psi(v) \ge \tilde{\mu} - \delta'\}$ *and* $C(\varphi,\tilde{\lambda}) \cap \{v : \psi(v) \ge \tilde{\mu} + \delta'\}$ *are homeomorphic. Here,*

$$C(\varphi,\lambda) := \{w \in X : \varphi(w) \le \lambda\},\ \ \lambda \in \mathbf{R}$$

and $\alpha \pm \beta$ *means both* $\alpha + \beta$ *and* $\alpha - \beta$.

Assertion 1.1 is a kind of Morse lemma in the sense that the contraposition implies the existence of solutions of (1.1) in the case where $C(\varphi,\lambda)$ is compact and $\psi \in C^1(X)$ (cf. [2] [3]). In trying to prove Assertion 1.1, the author obtained a number of propositions, and some of them seem to be useful in other mathematical researches. This paper's theorems are obtained in such process.

2. Results

Assertion 1.1 is proved if we can define a Lipschitz continuous vector field $Z(.)$ such that the flow $(\mathrm{d}/\mathrm{d}t)v(x,t) = Z(v(x,t))$, $v(x,0) = x$ defines a homeomorphism between $C(\varphi,\tilde{\lambda}) \cap \{v : \psi(v) \ge \tilde{\mu} - \delta'\}$ and $C(\varphi,\tilde{\lambda}) \cap \{v : \psi(v) \ge \tilde{\mu} + \delta'\}$. For example, $Z(.)$ is expected to satisfy the following property.

(a) If $v \in C(\varphi,\tilde{\lambda}) \cap \{v : \psi(v) \ge \tilde{\mu} + \delta'\}$, then $Z(v) = 0$.

(b) If $v \in C(\varphi,\tilde{\lambda}) \cap \{v : \tilde{\mu} - \delta' \le \psi(v) \le \tilde{\mu} + (1/2)\delta'\}$,

(b-1) if $\varphi(v) \in \left[\tilde{\lambda} - \frac{\delta'}{2}, \tilde{\lambda}\right]$, then $(\mathrm{d}/\mathrm{d}t)\varphi(v(t)) = 0$ and $(\mathrm{d}/\mathrm{d}t)\psi(v(t)) = 1$, or

$$Z(v) \in \left(\frac{\partial\varphi(v)}{\|\partial\varphi(v)\|}\right)^{-1}(0) \cap \left(\frac{\mathrm{d}\psi(v)}{\|\mathrm{d}\psi(v)\|}\right)^{-1}\left(\frac{1}{\|\mathrm{d}\psi(v)\|}\right)$$

(b-2) if $\varphi(v) \in \left[\min\varphi, \tilde{\lambda} - \delta'\right]$, then $(\mathrm{d}/\mathrm{d}t)\psi(v(t)) = 1$, or

DOI: 10.4236/am.2022.139047

$$Z(v) \in \left(\frac{\mathrm{d}\psi(v)}{\|\mathrm{d}\psi(v)\|}\right)^{-1}\left(\frac{1}{\|\mathrm{d}\psi(v)\|}\right)$$

(b-3) if $\varphi(v) \in \left[\tilde{\lambda}-\delta', \tilde{\lambda}-\frac{\delta'}{2}\right]$, then $Z(.)$ is continuous with (b-1) (b-2).

(c) If $v \in C(\varphi, \tilde{\lambda}) \cap \left\{v : \tilde{\mu}+\frac{\delta'}{2} \le \psi(v) \le \tilde{\mu}+\delta'\right\}$, $Z(v)$ is continuous with (a) (b).

In general, we cannot construct a Lipschitz continuous vector field $Z(.)$ with (a)-(c). The author has constructed a sequence $\{Z_n(.)\}$ such that each $Z_n(.)$ is Lipschitz continuous with the constant L_n, and that $\lim_{n\to\infty} Z_n(v) = \exists Z_\infty(v)$ local uniformly. To do this, assumptions (1.3) (1.4) play an important role to see

$$\operatorname{dist}\left(0, \left(\frac{\partial\varphi(v)}{\|\partial\varphi(v)\|}\right)^{-1}(0) \cap \left(\frac{\mathrm{d}\psi(v)}{\|\mathrm{d}\psi(v)\|}\right)^{-1}\left(\frac{1}{\|\mathrm{d}\psi(v)\|}\right)\right) \le \exists M. \tag{2.1}$$

Here, for $w \in X$ and $A \subset X$,

$$\operatorname{dist}(w, A) := \inf_{\xi \in A} \|w-\xi\|.$$

Hence, in the case where $\{Z_n(.)\}$ satisfies

$$L_n \|Z_n(v) - Z_\infty(v)\| \to 0, \quad \text{local uniformly,} \tag{2.2}$$

then for $\forall x$ and $\forall t > 0$ the convergence $v_n(x,t) \to \exists v_\infty(x,t)$ holds and $v_\infty(.,T)$ is the aimed homeomorphism with some $T > 0$.

The author thinks, at this moment, that (2.2) can be hold if the following assumptions are satisfied.

(i) φ is even, or equivalently $-C(\varphi,\lambda) = C(\varphi,\lambda)$.

(ii) $\exists \rho : [0,\infty) \to \mathbf{R}$ such that $\varphi(rw) = \rho(r)$, for $\forall r > 0$, $\forall w \in \varphi^{-1}(1)$.

(iii) (i) (ii) together mean that putting $\|rw\|_Y := r$ for $\forall w \in \varphi^{-1}(1)$ defines a norm in the linear space $Y := D(\varphi)$. Suppose that this norm is uniformly convex and uniformly smooth in *Y*.

(iv) $C(\varphi,\lambda)$ is compact in *X*.

In constructing the sequence $\{Z_n(.)\}$ required above, the following theorem plays important roles, and seems to be useful in many other mathematical researches.

Theorem 2.1. *Let X be a real Banach space, and* X^* *be the dual space of X. For* $f, g \in X^* \setminus \{0\}$ *and* $v \in g^{-1}(0)$,

$$\frac{\operatorname{dist}\left(v, f^{-1}(0)\right)}{\|v\|} \le \frac{\|f-g\|}{\|f\|}. \tag{2.3}$$

In the next two theorems, the differentiability and continuity of derivatives of convex functionals are shown.

For a moment, we recall definitions of derivatives (cf. Masuda [2]). Let $F : U \to W$, where V, W are normed vector spaces and U is an open subset of

V.

Definition 2.1. (*Fréchet derivative*) *F is called Fréchet differentiable at* $x \in U$ *if there is a bounded linear operator* $A: V \to W$ *such that*

$$\lim_{\|\xi\|\to 0} \frac{\|F(x+\xi) - F(x) - A\xi\|}{\|\xi\|} = 0.$$

Equivalently, the first-order expansion holds, in Landau notion

$$F(x+\xi) = F(x) + A\xi + o(\xi).$$

Definition 2.2. (*Gâteaux derivative*) *The Gâteaux differential* $\mathrm{d}F(x;\xi)$ *of F at* $x \in U$ *in the direction* $\xi \in V$ *is defined as*

$$\mathrm{d}F(x;\xi) := \lim_{\tau\to 0} \frac{F(x+\tau\xi) - F(x)}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} F(x+\tau\xi)\Big|_{\tau=0}$$

If the limit exists for all $\xi \in V$, *then one calls F is Gâteaux differentiable at* x.

The Gâteaux differential may fail to be linear, unlike the Fréchet derivative. Even if linear, it may fail to depend continuously on ξ.

In the following, let $\varphi: X \to \mathbf{R} \cup \{+\infty\}$ be a lower semi-continuous convex functional. The set

$$D(\varphi) := \{x \in X : \varphi(x) < +\infty\}$$

is called the effective domain of φ.

Remark 2.1. *For* $x \in D(\varphi)$, *put*

$$Y(x) := \{\xi \in X : \exists \delta = \delta(\xi) > 0 \text{ such that } x + \delta\xi, x - \delta\xi \in D(\varphi)\}.$$

Suppose that the Gâteaux differential $\mathrm{d}\varphi(x;\xi)$ *for every direction* $\xi \in Y(x)$ *exists. Then, since* φ *is convex,* $Y(x)$ *is a linear subspace of X and* $\mathrm{d}\varphi(x;\xi)\big|_{\xi\in Y(x)}$ *is linear with respect to* ξ.

Theorem 2.2. *Let* $x \in U \subset D(\varphi)$. *If U is open in X and* φ *is Gâteaux differentiable at x, then* φ *is Fréchet differentiable at* x.

Remark 2.2. *In Theorem* 2.2, *the openness of U is needed. For example, put*

$$\varphi(x) = \|x\|^2 \text{ if } x \in C; \ = +\infty \text{ otherwise},$$

where C is a closed convex subset of X. Then φ is *a lower semi-continuous convex functional on X. As is noted in Remark 2.1, the Gâteaux differential* $\mathrm{d}\varphi(0,\xi)\big|_{\xi\in Y(0)}$ *exists for all* $\xi \in Y(0)$ *and is linear on* $Y(0)$, *where*

$$Y(0) := \{\xi \in X : \exists \delta = \delta(\xi) > 0 \text{ such that } \delta\xi, -\delta\xi \in C\}.$$

If 0 *is not an inner point of C, or equivalently* $Y(0) \neq X$, *then* φ *is not Fréchet differentiable at* 0.

Theorem 2.3. *Suppose* $U \subset D(\varphi)$, *U is open in X, and* φ *is Fréchet differentiable on U. Then the Fréchet derivative of* φ *is continuous on U.*

3. Proof of Theorem 2.1

Throughout this paper, the following symbols are used.

$$B(z,r) := \{\xi \in X : \|\xi - z\| < r\},\ S(z,r) := \{\xi \in X : \|\xi - z\| = r\}.$$

For any $f, g \in X^*$, $(f-g)^{-1}(1)$ is expressed by

$$(f-g)^{-1}(1) = \bigcup_{t\in\mathbf{R}} \left[t\{f^{-1}(1) \cap g^{-1}(0)\} + (1-t)\{f^{-1}(0) \cap g^{-1}(-1)\} \right]. \quad (3.1)$$

Let $v \in g^{-1}(0) \cap S(0,1)$. Take $\alpha \in \mathbf{R}$ such that $\alpha v \in (f-g)^{-1}(1)$. Then, since $\alpha v \in g^{-1}(0)$,

$$\alpha v \in (f-g)^{-1}(1) \cap g^{-1}(0).$$

Noting the relation $g^{-1}(0) \cap g^{-1}(-1) = \varnothing$ in (3.1) implies

$$\alpha v \in (f-g)^{-1}(1) \cap g^{-1}(0) = f^{-1}(1) \cap g^{-1}(0). \quad (3.2)$$

Since $f^{-1}(0)$ is a linear subspace, $\operatorname{dist}(\alpha v, f^{-1}(0)) = |\alpha| \operatorname{dist}(v, f^{-1}(0))$. Therefore,

$$\frac{1}{\|f\|} = \operatorname{dist}(f^{-1}(1), f^{-1}(0)) = \operatorname{dist}(\alpha v, f^{-1}(0)) = |\alpha| \operatorname{dist}(v, f^{-1}(0)).$$

On the other hand, the relation $\alpha v \in (f-g)^{-1}(1)$ implies

$$|\alpha| = \|\alpha v\| \geq \operatorname{dist}(0, (f-g)^{-1}(1)) = \frac{1}{\|f-g\|}.$$

Thus, Theorem 2.1 is proved.

4. Proof of Theorem 2.2

Let U_0 be an open subset of U satisfying $x \in U_0 \subset \overline{U_0} \subset U$.

We verify that the linear functional $\mathrm{d}\varphi(x;.)$ is bounded. For $\lambda \in \mathbf{R}$, put

$$C(\varphi;\lambda) := \{x \in \overline{U_0} : \varphi(x) \leq \lambda\}.$$

Since φ is lower semi-continuous, $C(\varphi;\lambda)$ is closed in $\overline{U_0}$. By Baire category theorem, the inclusion relationship $\bigcup_{n\in\mathbf{N}} C(\varphi;n) = \overline{U_0}$ implies that, for some $n_0 \in \mathbf{N}$, $C(\varphi;n_0)$ has an inner point. Therefore, $\{\mathrm{d}\varphi(x;.)\}^{-1}(n_0 - \varphi(x))$ is not dense in *X*. Hence also $\{\mathrm{d}\varphi(x;.)\}^{-1}(0)$. This means that $\{\mathrm{d}\varphi(x;.)\}^{-1}(0)$ is closed in *X*, or equivalently, $\mathrm{d}\varphi(x;.)$ is bounded (cf. Rudin [4]).

Now, put

$$\Phi(\xi) := \varphi(x+\xi) - \varphi(x) - \mathrm{d}\varphi(x;\xi), \quad \xi \in \overline{U_1} := -x + \overline{U_0}.$$

Since Φ is lower semi-continuous convex functional on $\overline{U_1}$ satisfying

$$\min_{\overline{U_1}} \Phi = \Phi(0) = 0,$$

the Fréshet differentiability at x is proved if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$\Phi(\xi) < \varepsilon, \quad \xi \in B(0;\delta). \quad (4.1)$$

To see this, fix any $\varepsilon > 0$. Put

$$C_1(\Phi;\lambda) := \left\{\xi \in \overline{U_1} : \Phi(\xi) \le \lambda\right\}.$$

Since $C_1(\Phi;n\varepsilon) \subset nC_1(\Phi;\varepsilon)$ and $\bigcup_{n\in\mathbf{N}} C_1(\Phi;n\varepsilon) = \overline{U_1}$, $\bigcup_{n\in\mathbf{N}} nC_1(\Phi;\varepsilon) = \overline{U_1}$. Thus, Baire category theorem implies that $C_1(\Phi;\varepsilon)$ has an inner point z_1. Take $\rho_1 > 0$ such that $B(z_1;\rho_1) \subset C_1(\Phi;\varepsilon)$.

If $0 \in B(z_1;\rho_1)$, then taking $\delta > 0$ such that $B(0,\delta) \subset B(z_1;\rho)$ implies (4.1). Hence, the proof is finished.

In the case where $0 \notin B(z_1;\rho)$, take the following closed cone.

$$K := \left\{\kappa\xi : \kappa \le 0, \xi \in \bar{B}(z_1;\rho_1)\right\}.$$

Then, taking $C_1(\Phi;\lambda) \cap K$ instead of $C_1(\Phi;\lambda)$ in the same discussion implies that there is an open ball $B(z_2;\rho_2) \subset C_1(\Phi;\varepsilon) \cap K$. Since $C_1(\Phi;\varepsilon)$ is convex, the convex hull

$$\left\{t\xi_1 + (1-t)\xi_2 : \xi_1 \in B(z_1,\rho_1), \xi_2 \in B(z_2,\rho_2), t \in [0,1]\right\}$$

is an open subset of $C_1(\Phi;\varepsilon)$, and 0 is included. Thus, Theorem 2.2 is proved.

5. Proof of Theorem 2.3

Suppose that the result is not true. Then, there are v_∞ and a sequence $\{v_k\}$ in U such that for some $\delta > 0$

$$v_k \to v_\infty, \quad \left\|\mathrm{d}\varphi(v_k) - \mathrm{d}\varphi(v_\infty)\right\| > 3\delta.$$

For each k, there is $w_k \in S(0,1)$ satisfying $\left\{\mathrm{d}\varphi(v_k) - \mathrm{d}\varphi(v_\infty)\right\}(w_k) > 3\delta$. Hence, for all $h > 0$,

$$\varphi(v_k + hw_k) \ge \varphi(v_k) + \mathrm{d}\varphi(v_k)(hw_k) \ge \varphi(v_k) + \mathrm{d}\varphi(v_\infty)(hw_k) + 3h\delta \tag{5.1}$$

where in the first inequality, the convexity of φ is used.

On the other hand, since φ is Fréchet differentiable at v_∞,

$$\varphi(v_k + hw_k) = \varphi(v_\infty) + \mathrm{d}\varphi(v_\infty)(v_k + hw_k - v_\infty) + o(v_k + hw_k - v_\infty),$$

$$\varphi(v_k) = \varphi(v_\infty) + \mathrm{d}\varphi(v_\infty)(v_k - v_\infty) + o(v_k - v_\infty).$$

Thus, (5.1) implies

$$o(v_k + hw_k - v_\infty) \ge o(v_k - v_\infty) + 3h\delta. \tag{5.2}$$

Take $h > 0$ such that

$$\left|o(\eta)\right| < \frac{\delta}{2}\|\eta\| \quad \text{if } \|\eta\| < 2h.$$

Then, taking k such that $\|v_k - v_\infty\| < h$ in (5.2) yields that $\delta > 2\delta$, which is a contradiction. Therefore, the aimed result is true.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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