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Graded Derived Equivalences

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Abstract: We consider the equivalences of derived categories of graded rings over different groups. A Morita type equivalence is established between two graded algebras with different group gradings. The results obtained here give a better understanding of the equivalences of derived categories of graded rings.

Keywords: derived categories; graded algebras; equivalences of categories

MSC: 16W50; 18C15

1. Introduction

In noncommunicative algebra and algebraic geometry, sometimes one has to establish some relations between two graded algebras over different groups, respectively; for example, a certain equivalence of some quotient categories of modules over graded rings with different group gradings (cf. [1,2]). Del Rio, in [3], established a Morita type equivalence between categories of graded modules over graded rings with different group gradings. However, a derived equivalence between graded rings with different group gradings is still missing. In this paper, we focus our attention on the derived equivalences of derived categories of graded rings with possible different group gradings.

Throughout this paper, G and Ω will denote two multiplicative groups. Let R and S be two ungraded rings. In [4], Rickard showed that R and S are derived as equivalent if and only if R is isomorphic to the endomorphism ring of a tilting complex over S. Let A and B be two G-graded rings. Then, the derived equivalence of A and B, which preserves gradings, follows easily from Rickard's theorem (cf. [5]). However, if the equivalence does not preserve gradings, or if A and B are graded rings over different groups, then the problem is subtle. To establish the equivalences of categories of graded modules over A and B, where A and B are graded rings over different groups, Del Rio introduced two functors between categories of graded modules in [3]: $-\otimes_A^{gr} P\colon \text{Gr-}A \to \text{Gr-}B$ and $(-)_*^P\colon \text{Gr-}B \to \text{Gr-}A$, for a bigraded A-B-bimodule P. The functor $-\otimes_A^{gr} P$ is left adjoint to $(-)_*^P$ and every pair of adjoint functors between Gr-A and Gr-B is of this form. We extend these functors to the derived categories of graded rings and then give a description of the equivalences of these derived categories.

For this purpose, we proceed as follows. We first review the basic facts on the categories of graded modules in Section 2. In Section 3, we define two derived functors $LF\colon D(\text{Gr-}A) \to D(\text{Gr-}B)$ and $RH\colon D(\text{Gr-}B) \to D(\text{Gr-}A)$. We prove that LF is left adjoint to RH. These derived functors will play a central role in the rest of the paper. Then, in Section 4, we give a description of derived equivalences of graded rings in Theorem 1. In the last part of the paper, we give the following applications of Theorem 1.

(1) We give a characterization of when the functors of derived categories of graded modules are graded functors;
(2) For a subgroup G' of G, let $A = \bigoplus_{g\in G} A_g$ and $A_{(G')} = \bigoplus_{g\in G'} A_g$ be two graded rings. We provide a characterization of when the two graded rings are derived as equivalent to each other;

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(3) We provide a characterization of when a derived category of graded modules is equivalent to a derived category of ungraded modules.

2. Notations and Preliminaries

Let G be a group. For a G-graded ring $A = \bigoplus_{g\in G} A_g$, Gr-$A$ will denote the category of right G-graded A-modules and A-Gr will denote the category of left G-graded A-modules. Mod-A will denote the category of right A-modules. Proj-A will denote the full subcategory of Gr-A containing all of the projective modules in Gr-A and proj-A will denote the full subcategory of Gr-A containing all of the finitely generated projective modules in Gr-A.

Given a right G-graded A-module $M = \bigoplus_{g\in G} M_g$, for $h \in G$, $(h)M$ will denote the right G-graded A-module whose component of degree g is M_{hg}, and Id_M will denote the identity morphism of M. For $m \in M$ and $g \in G$, we write m_g for the homogeneous component of m of degree g. Similarly, if $N \in A$-Gr and $n \in N$, we write ${}_g n$ for the homogenous component of n of degree g. For every $g \in G$, π_g^M will denote the map from M to M_g such that $\pi_g^M(m) = m_g$ for every $m \in M$ (cf. [6,7]).

Let A be a G-graded ring and B be an Ω-graded ring. A *bigraded* (A, B)-*bimodule* is an (A, B)-bimodule with a decomposition into a direct sum of additive subgroups $N = \bigoplus_{g\in G,\sigma\in\Omega} {}_g N_\sigma$ so that $A_g \cdot {}_h N_\sigma \cdot B_\tau \subseteq {}_{gh} N_{\sigma\tau}$ for $g, h \in G$ and $\sigma, \tau \in \Omega$,.

In [3], Del Rio constructed the graded tensor products of graded modules in the following way: Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring. Given $M \in$ Gr-A and $N \in A$-Gr, let $[M, N]$ be the subgroup of $M \otimes_A N$ generated by $\{m_g \otimes n - m \otimes {}_{g^{-1}}n \mid m \in M, n \in N, g \in G\}$. Then, the graded tensor product of M and N is the quotient additive group $M \otimes_A^{gr} N = (M \otimes_A N)/[M, N]$. For every $m \in M$, $n \in N$, $m \otimes^{gr} n$ will denote the class in $M \otimes_A^{gr} N$ containing $m \otimes n$. In particular, if B is an Ω-graded ring and N is a bigraded (A, B)-bimodule, then $M \otimes_A^{gr} N$ is a graded right B-module whose component of degree σ is $M \otimes_A^{gr} (N_\sigma)$, for $\sigma \in \Omega$. Thus, $- \otimes_A^{gr} N$ is a functor from Gr-A to Gr-B defined by

$$(- \otimes_A^{gr} N)(M) = M \otimes_A^{gr} N,$$

for $M \in$ Gr-A, and

$$(- \otimes_A^{gr} N)(f) = f \otimes_A^{gr} N,$$

for $f \in \mathrm{Hom}_{\text{Gr-}A}(M, L)$, $M, L \in$ Gr-A, where $f \otimes_A^{gr} N$ is a morphism in Gr-B from $M \otimes_A^{gr} N$ to $L \otimes_A^{gr} N$ defined by

$$(f \otimes_A^{gr} N)(m \otimes^{gr} n) = f(m) \otimes^{gr} n, \text{ for all } m \in M, n \in N.$$

Proposition 1 ([3], Lemma 1)**.** *Let $A = \bigoplus_{g\in G} A_g$ be a graded ring. Let $N \in A$-Gr. Given $g \in G$, there exists an isomorphism $\alpha\colon ((g)A) \otimes_A^{gr} N \to {}_g N$ such that, for $a \in ((g)A)_h$ and $n \in {}_l N$,*

$$\alpha(a \otimes^{gr} n) = \begin{cases} an \in {}_g N, & h = l^{-1}. \\ 0, & h \neq l^{-1}. \end{cases}$$

Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{\sigma\in\Omega} B_\sigma$ be two graded rings. Given a bigraded (A, B)-bimodule N and a graded right B-module L, consider $\mathrm{Hom}_{\text{Gr-}B}({}_g N, L)$ as a subset of $\mathrm{Hom}_{\text{Gr-}B}(N, L)$ for every $g \in G$. Then, $\bigoplus_{g\in G} \mathrm{Hom}_{\text{Gr-}B}({}_{g^{-1}} N, L)$ is a graded right A-module and $\bigoplus_{g\in G} \mathrm{Hom}_{\text{Gr-}B}({}_{g^{-1}} N, -)$ is a functor from Gr-B to Gr-A.

Proposition 2 ([3], Proposition 2)**.** *Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{\sigma\in\Omega} B_\sigma$ be two graded rings. Let N be a bigraded (A, B)-bimodule. Then, $- \otimes_A^{gr} N$ is the left adjoint of $\bigoplus_{g\in G} \mathrm{Hom}_{\text{Gr-}B}({}_{g^{-1}} N, -)$.*

3. Functors between Derived Categories of Graded Rings

Given an abelian category $\mathcal{A}$, $C(\mathcal{A})$ will denote the category of the complexes of $\mathcal{A}$, $K(\mathcal{A})$ will denote the homotopy category of $\mathcal{A}$ and $D(\mathcal{A})$ will denote the derived category of $\mathcal{A}$. Given a complex $P = (P^n, d^n)$ in $C(\mathcal{A})$, $P[i]$ will denote the complex whose n^{th} component is P^{n+i} and n^{th} differential is $(-1)^i \cdot d^{n+i}$.

Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{\sigma \in \Omega} B_\sigma$ be two graded rings. Given a complex $T = (T^i, d_T^i)$ of bigraded (A,B)-bimodules, $F = - \otimes_A^{gr} T \colon C(\text{Gr-}A) \to C(\text{Gr-}B)$ is a functor defined by

$$(FP)^n = \bigoplus_{i+j=n} P^i \otimes_A^{gr} T^j,$$

$$d_{FP}^n = \sum_{i+j=n} (d_P^i \otimes_A^{gr} \mathrm{Id}_{T^i} + (-1)^i \mathrm{Id}_{P^i} \otimes_A^{gr} d_T^j),$$

$$(F(f))^n = \sum_{i+j=n} f^i \otimes_A^{gr} \mathrm{Id}_{T^j},$$

for $P = (P^n, d_P^n), Q \in C(\text{Gr-}A)$ and $f = \{f^n\} \in \mathrm{Hom}_{C(\text{Gr-}A)}(P, Q)$.

Given $P_1 = (P_1^i, d_1^i), P_2 = (P_2^i, d_2^i) \in C(\text{Gr-}A)$ and $f = \{f^n\}, h = \{h^n\} \in \mathrm{Hom}_{C(\text{Gr-}A)}(P_1, P_2)$. If f and h are chain homotopic, that is, for every $n \in \mathbb{Z}$, there exists $s^n \in \mathrm{Hom}_{\text{Gr-}A}(P_1^n, P_2^{n-1})$, such that

$$f^n - h^n = d_2^{n-1} s^n + s^{n+1} d_1^n.$$

Let $r^n = \sum_{i+j=n} s^i \otimes_A^{gr} \mathrm{Id}_{T^j} \in \mathrm{Hom}_{\text{Gr-}B}((FP_1)^n, (FP_2)^{n-1})$; then, we have

$$\begin{aligned}
(Ff)^n - (Fh)^n &= \textstyle\sum_{i+j=n} (f^i - h^i) \otimes_A^{gr} \mathrm{Id}_{T^j} \\
&= \textstyle\sum_{i+j=n} (d_2^{i-1} s^i + s^{i+1} d_1^i) \otimes_A^{gr} \mathrm{Id}_{T^j} \\
&= \textstyle\sum_{i+j=n} (d_2^{i-1} s^i \otimes_A^{gr} \mathrm{Id}_{T^j} + (-1)^{i-1} s^i \otimes_A^{gr} d_T^j) \\
&\quad + \textstyle\sum_{i+j=n} (s^{i+1} d_1^i \otimes_A^{gr} \mathrm{Id}_{T^j} + (-1)^i s^i \otimes_A^{gr} d_T^j) \\
&= d_{FP_2}^{n-1} r^n + r^{n+1} d_{FP_1}^n.
\end{aligned}$$

Thus, f and h being homotopic in $C(\text{Gr-}A)$ implies Ff and Fh being homotopic in $C(\text{Gr-}B)$. Hence, $F = - \otimes_A^{gr} T$ induces a functor from $K(\text{Gr-}A)$ to $K(\text{Gr-}B)$ that will be denoted by the same symbol.

Given $Q_1 = (Q_1^i, \delta_1^i), Q_2 = (Q_2^i, \delta_2^i) \in C(\text{Gr-}B)$, let $\mathrm{Hom}_{\text{Gr-}B}^\bullet(Q_1, Q_2)$ denote the complex whose n^{th} component is

$$\mathrm{Hom}_{\text{Gr-}B}^n(Q_1, Q_2) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\text{Gr-}B}(Q_1^i, Q_2^{i+n}),$$

and n^{th} differential is d^n, such that, for every $f = \{f^i\} \in \mathrm{Hom}_{\text{Gr-}B}^n(Q_1, Q_2)$ with $f^i \in \mathrm{Hom}_{\text{Gr-}B}(Q_1^i, Q_2^{i+n})$,

$$d^n f = \{(d^n f)^i\} \in \mathrm{Hom}_{\text{Gr-}B}^{n+1}(Q_1, Q_2),$$

where $(d^n f)^i = \delta_2^{i+n} \circ f^i - (-1)^n f^{i+1} \circ \delta_1^i \in \mathrm{Hom}_{\text{Gr-}B}(Q_1^i, Q_2^{i+n+1})$. In particular, if Q_1 is a complex of bigraded (A,B)-bimodules, then $\bigoplus_{g \in G} \mathrm{Hom}_{\text{Gr-}B}^\bullet({}_{g^{-1}}Q_1, Q_2)$ is a complex of graded right A-modules.

Therefore, for every bigraded (A,B)-bimodule T, we can define a functor $H = \bigoplus_{g \in G} \mathrm{Hom}_{\text{Gr-}B}^\bullet({}_{g^{-1}}T, -)$ from $C(\text{Gr-}B)$ to $C(\text{Gr-}A)$. Similarly, two morphisms f and h being homotopic in $C(\text{Gr-}B)$ implies Hf and Hh being homotopic in $C(\text{Gr-}A)$. Hence, H induces a functor from $K(\text{Gr-}B)$ to $K(\text{Gr-}A)$ that will be denoted by the same symbol.

Similar to Proposition 2, we have the following result.

Lemma 1. *Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{\sigma \in \Omega} B_\sigma$ be two graded rings. Let T be a complex of bigraded (A,B)-bimodules, then the functors*

$$F = - \otimes_A^{gr} T \colon K(\text{Gr-}A) \to K(\text{Gr-}B)$$

and

$$H = \bigoplus_{g\in G} (\mathrm{Hom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, -)) \colon K(\text{Gr-}B) \to K(\text{Gr-}A)$$

defined above are an adjoint pair.

A complex P in $K(\text{Gr-}A)$ is called a *homotopically projective complex* if, for every acyclic complex E in $K(\text{Gr-}A)$, $\mathrm{Hom}_{K(\text{Gr-}A)}(P, E) = 0$. Let $K_{\text{hproj}}(\text{Gr-}A)$ denote the full subcategory of $K(\text{Gr-}A)$ containing all of the homotopically projective complexes. Then, by ([8], Section 4.5), there exists an equivalent functor $p \colon D(\text{Gr-}A) \to K_{\text{hproj}}(\text{Gr-}A)$ such that, for $M \in D(\text{Gr-}A)$, $p(M)$ is a homotopically projective resolution of M. Consider p as a functor from $D(\text{Gr-}A)$ to $K(\text{Gr-}A)$; then, p is left adjoint to the quotient functor $Q_A \colon K(\text{Gr-}A) \to D(\text{Gr-}A)$.

Similarly, a complex I in $K(\text{Gr-}A)$ is called a *homotopically injective complex* if, for every acyclic complex E in $K(\text{Gr-}A)$, $\mathrm{Hom}_{K(\text{Gr-}A)}(E, I) = 0$. Let $K_{\text{hinj}}(\text{Gr-}B)$ denote the full subcategory of $K(\text{Gr-}B)$ containing all of the homotopically injective complexes. Then, there exists an equivalent functor $i \colon D(\text{Gr-}B) \to K_{\text{hinj}}(\text{Gr-}B)$ such that, for $M \in D(\text{Gr-}B)$, $i(M)$ is a homotopically injective resolution of M. Consider i as a functor from $D(\text{Gr-}B)$ to $K(\text{Gr-}B)$; then, i is right adjoint to the quotient functor $Q_B \colon K(\text{Gr-}B) \to D(\text{Gr-}B)$.

We can define the derived functors LF and RH by $LF = Q_B \circ F \circ p$ and $RH = Q_A \circ H \circ i$; then, LF is a functor from $D(\text{Gr-}A)$ to $D(\text{Gr-}B)$ and RH is a functor from $D(\text{Gr-}B)$ to $D(\text{Gr-}A)$ (cf. [9]).

Lemma 2. (LF, RH) *is an adjoint pair.*

Proof. Suppose $P \in D(\text{Gr-}A)$ and $Q \in D(\text{Gr-}B)$; then, there exist natural isomorphisms,

$$\begin{aligned} \mathrm{Hom}_{D(\text{Gr-}B)}(LF(P), Q) &\cong \mathrm{Hom}_{K(\text{Gr-}B)}(F(p(P)), iQ) \\ &\cong \mathrm{Hom}_{K(\text{Gr-}A)}(p(P), H(iQ)) \\ &\cong \mathrm{Hom}_{D(\text{Gr-}A)}(P, RH(Q)). \end{aligned}$$

Hence, (LF, RH) is an adjoint pair. □

4. The Main Theorem

Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{\sigma\in\Omega} B_\sigma$ be two graded rings. Let T be a complex of bigraded (A, B)-bimodules and Q be a complex of graded right B-modules. We have shown that $\bigoplus_{g\in G} \mathrm{Hom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, Q)$ is a complex of graded right A-modules. Similarly, we can consider $\mathrm{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}T, Q)$ as a subgroup of $\mathrm{Hom}_{D(\text{Gr-}B)}(T, Q)$ for every $g \in G$, since ${}_{g^{-1}}T$ is a subcomplex of T. Then, $\bigoplus_{g\in G} \mathrm{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}T, Q)$ is a subgroup of $\mathrm{Hom}_{D(\text{Gr-}B)}(T, Q)$. Thus, $\bigoplus_{g\in G} \mathrm{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}T, Q)$ has a graded right A-module structure.

Let per(Gr-A) be the full subcategory of $D(\text{Gr-}A)$ containing all of the complexes that are quasi-isomorphic to the complexes in $K^b(\text{proj-}A)$.

Lemma 3 ([10], Section 8.1.3)**.** *Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring; then, a full triangulated subcategory of $D(\text{Gr-}A)$ equals $D(\text{Gr-}A)$ if and only if it contains $(g)A$ for all $g \in G$ and is closed under forming infinite direct sums.*

Now, we can characterize the equivalence of derived categories of graded modules, which does not preserve gradings.

Theorem 1. *Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{\sigma\in\Omega} B_\sigma$ be two graded rings. Given a complex T of bigraded (A, B)-bimodules, let F denote the functor $- \otimes^{gr}_A T$ from $K(\text{Gr-}A)$ to $K(\text{Gr-}B)$ and let H denote the functor $\bigoplus_{g\in G}(\mathrm{Hom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, -))$ from $K(\text{Gr-}B)$ to $K(\text{Gr-}A)$. The following conditions are equivalent.*

(i) $LF\colon D(\text{Gr-}A) \to D(\text{Gr-}B)$ *is an equivalence of triangulated categories;*
(ii) $LF\colon \text{per}(\text{Gr-}A) \to \text{per}(\text{Gr-}B)$ *is an equivalence of triangulated categories;*
(iii) *The object* $T = \bigoplus_{g\in G}\, {}_gT$ *satisfies*
 (a) *For every* $h \in G$, $(h)A$ *is isomorphic to* $\bigoplus_{g\in G} \text{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}T, {}_hT)$ *as graded right* A*-modules and* $\text{Hom}_{D(\text{Gr-}B)}({}_gT, {}_hT[n]) = 0$ *for all* $g, h \in G$ *and all* $n \neq 0$*;*
 (b) *For every* $g \in G$, ${}_gT \in \text{per}(\text{Gr-}B)$*;*
 (c) *The smallest full triangulated subcategory of* $D(\text{Gr-}B)$ *containing* $\{{}_gT\}_{g\in G}$ *and closed under forming direct summands equals* $\text{per}(\text{Gr-}B)$.

Proof. (i) ⇒ (ii) Suppose $P \in D(\text{Gr-}A)$; then, by ([11], Proposition 6.3), $P \in \text{per}(\text{Gr-}A)$ if and only if the functor $\text{Hom}_{D(\text{Gr-}A)}(P, -)$ commutes with infinite direct sums. Thus, the condition that $LF\colon D(\text{Gr-}A) \to D(\text{Gr-}B)$ is an equivalence of triangulated categories implies $LF\colon \text{per}(\text{Gr-}A) \to \text{per}(\text{Gr-}B)$ is an equivalence of triangulated categories.

(ii) ⇒ (iii) If $LF\colon \text{per}(\text{Gr-}A) \to \text{per}(\text{Gr-}B)$ is an equivalence, then, for every $g, h \in G$ and every $n \in \mathbb{Z}$, we have the following natural isomorphisms,

$$\begin{aligned} & \text{Hom}_{D(\text{Gr-}A)}((g)A, ((h)A)[n]) \\ \cong\ & \text{Hom}_{D(\text{Gr-}B)}(LF((g)A), LF(((h)A)[n])) \\ \cong\ & \text{Hom}_{D(\text{Gr-}B)}(((g)A) \otimes_A^{gr} T, (((h)A)[n]) \otimes_A^{gr} T) \\ \cong\ & \text{Hom}_{D(\text{Gr-}B)}({}_gT, ({}_hT)[n]). \end{aligned}$$

Thus, for every $g, h \in G$ and every $n \neq 0$, $\text{Hom}_{D(\text{Gr-}B)}({}_gT, ({}_hT)[n]) = 0$, and we have the following isomorphisms of right graded A-modules for every $h \in G$,

$$\begin{aligned} (h)A &\cong \textstyle\bigoplus_{g\in G} A_{hg} \\ &\cong \textstyle\bigoplus_{g\in G} \text{Hom}_{D(\text{Gr-}A)}((g^{-1})A, (h)A) \\ &\cong \textstyle\bigoplus_{g\in G} \text{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}A \otimes_A^{gr} T, {}_hA \otimes_A^{gr} T) \\ &\cong \textstyle\bigoplus_{g\in G} \text{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}T, {}_hT), \end{aligned}$$

where the third isomorphism being the graded A-module morphism follows from the definition of LF and the last isomorphism being the graded A-module morphism follows from Lemma 1.

Condition (b) holds since LF carries $\text{per}(\text{Gr-}A)$ to $\text{per}(\text{Gr-}B)$. Condition (c) holds since the smallest full triangulated subcategory of $D(\text{Gr-}A)$ containing $\{(g)A\}_{g\in G}$ and closed under forming direct summands equals $\text{per}(\text{Gr-}A)$.

(iii) ⇒ (i) Let $\theta = RH \circ LF$. Since (LF, RH) is an adjoint pair, LF is fully faithful if and only if the adjunction morphism $\varphi_P\colon P \to \theta(P)$ is an isomorphism for all $P \in D(\text{Gr-}A)$. Let $\mathcal{U}$ be the full subcategory of $D(\text{Gr-}A)$ containing all of the objects on which φ are isomorphisms. By condition (iii) (a), we have the following isomorphisms in $D(\text{Gr-}A)$, for every $h \in G$,

$$\begin{aligned} \theta((h)A) &= \textstyle\bigoplus_{g\in G} \text{RHom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, (h)A \otimes_A^{gr} T) \\ &\cong \textstyle\bigoplus_{g\in G} \text{RHom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, {}_hT) \\ &\cong \textstyle\bigoplus_{g\in G} \text{Hom}_{D(\text{Gr-}B)}({}_{g^{-1}}T, {}_hT) \\ &\cong (h)A. \end{aligned}$$

Therefore, $\mathcal{U}$ contains $(h)A$ for all $h \in G$.

Suppose $\mathcal{U}$ contains X, Y and $X \to Y \to Z \to X[1]$ is a distinguished triangle in $D(\text{Gr-}A)$; then, $\theta(X) \to \theta(Y) \to \theta(Z) \to \theta(X[1])$ is a distinguished triangle since LF and RH are triangulated functors. Thus, we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \varphi_Z} & & \downarrow{\scriptstyle \varphi_{X[1]}} \\ \theta(X) & \xrightarrow{\theta(u)} & \theta(Y) & \xrightarrow{\theta(v)} & \theta(Z) & \xrightarrow{\theta(w)} & \theta(X[1]). \end{array}$$

Since φ_X, φ_Y and $\varphi_{X[1]}$ are isomorphisms, we have φ_Z as an isomorphism. Then, $\mathcal{U}$ contains Z. Thus, $\mathcal{U}$ is a triangulated subcategory of $D(\text{Gr-}A)$.

Suppose I is an infinite set and $P_i \in \mathcal{U}$ for all $i \in I$. Then,

$$\begin{aligned} \theta(\textstyle\bigoplus_{i\in I} P_i) &= \textstyle\bigoplus_{g\in G} \text{RHom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, (\bigoplus_{i\in I} P_i) \otimes^{gr}_A T) \\ &\cong \textstyle\bigoplus_{i\in I} (\bigoplus_{g\in G} \text{RHom}^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, P_i \otimes^{gr}_A T)) \\ &= \textstyle\bigoplus_{i\in I} \theta(P_i). \end{aligned}$$

Therefore, $\mathcal{U}$ is closed under infinite direct sums. By Lemma 3, $\mathcal{U}$ is equal to $D(\text{Gr-}A)$. Thus, LF is fully faithful.

Let ImLF be the triangulated subcategory of $D(\text{Gr-}B)$ such that each object M of ImLF is isomorphic to $LF(N)$ for some $N \in D(\text{Gr-}A)$. Then, ImLF is closed under infinite direct sums, since D(Gr-A) is closed under infinite direct sums. For every $g \in G$, $LF((g)A) \cong {}_gT$; therefore, ${}_gT \in \text{Im}LF$. Then, per(Gr-B) is a subcategory of ImLF and, in particular, $(\sigma)B \in \text{Im}LF$ for all $\sigma \in \Omega$. Thus, by Lemma 3, ImLF is equal to $D(\text{Gr-}B)$ and, then, LF is a dense functor. Hence, $LF\colon D(\text{Gr-}A) \to D(\text{Gr-}B)$ is an equivalence of triangulated categories. □

Remark 1. *We remark that the derived equivalences of graded algebras over different groups are subtle. For example, let A be a finite dimensional algebra. Let T_A be a finite dimensional tilting module; that is, T_A has a projective dimension not larger than 1, $\text{Ext}^i_A(T,T) = 0$ for $i \neq 0$, there is an exact sequence $0 \to A \to T_1 \to \cdots \to T_n \to 0$ for some $n \geq 1$ and T_i is a direct summand of a direct sum of copies of T for all $i = 1,\ldots,n$ (see ([10], Definition 3.11)). Let B be a $\mathbb{Z}_2$-graded algebra with $B_0 = A$ and $B_1 = 0$. Let $\mathcal{T} = \mathcal{T}' \oplus \mathcal{T}''$, where $\mathcal{T}'_0 = T$ and $\mathcal{T}'_1 = 0$, and $\mathcal{T}''_0 = 0$ and $\mathcal{T}''_1 = T$. Then, $\mathcal{T}$ is a right $\mathbb{Z}_2$-graded B-module satisfying the conditions in Theorem 1 (iii). Let $C = \text{End}_A(T)$. Then, the $\mathbb{Z}_2$-graded algebra B is derived as equivalent to $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.*

5. Applications

Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring. For every $g \in G$, we can define a g-suspension functor $S^A_g\colon \text{Gr-}A \to \text{Gr-}A$ by, for all $P \in \text{Gr-}A$,

$$S^A_g(P) = (g)P,$$

and for all $f \in \text{Hom}_{\text{Gr-}A}(P,Q)$,

$$S^A_g(f) = f.$$

Given $M = (M^n, d^n)$, $N = (N^n, \delta^n) \in K(\text{Gr-}A)$, $f \in \text{Hom}_{K(\text{Gr-}A)}(M,N)$, let $S^A_g(M)$ denote the complex $((g)M^n, d^n)$ and let $(S^A_g(f))^n = f^n$. Then, S^A_g is a functor from $K(\text{Gr-}A)$(resp. $D(\text{Gr-}A)$) to $K(\text{Gr-}A)$(resp. $D(\text{Gr-}A)$).

A functor F from $K(\text{Gr-}A)$(resp. $D(\text{Gr-}A)$) to $K(\text{Gr-}B)$(resp. $D(\text{Gr-}B)$) is said to be a *graded functor* if, for all $g \in G$, $F \circ S^A_g$ is naturally isomorphic to $S^B_g \circ F$.

Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be two G-graded rings. Given a graded (A,B)-bimodule P and a graded right A-module Q, $Q \otimes_A P$ is a G-graded right B-module by putting $(Q \otimes_A P)_g$, $g \in G$, equal to the additive subgroup of $Q \otimes_A P$ generated by elements

$q \otimes p$ with $q \in Q_h, p \in P_l$ such that $hl = g$. Then, $- \otimes_A P$ is a graded functor from Gr-A to Gr-B.

Let $A = \bigoplus_{g \in G} A_g$ and $A' = \bigoplus_{g \in G} A'_g$ be two G-graded rings. Given a graded (A, A')-bimodule $M = \bigoplus_{g \in G} M_g$, we define a bigraded (A, A')-bimodule $\hat{M} = \bigoplus_{g,h \in G} {}_gM_h$ in the following way: For $g, h \in G$, let ${}_g\hat{M}_h = M_{gh}$. The multiplication of A and A' on $\hat{M}$ is the same as those on M.

Lemma 4 ([3], Lemma 8)**.** *Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be two graded rings. Let P be a graded (A, B)-bimodule. Then, $- \otimes_A P$ is naturally isomorphic to $- \otimes_A^{gr} \hat{P}$. Thus, $- \otimes_A^{gr} \hat{P}$ is a graded functor.*

Let $T = (T^i, d^i)$ be a complex of graded (A, B)-bimodules. For every complex $M = (M^i, \delta^i)$, let

$$(M \otimes_A P)^n = \bigoplus_{i+j=n} M^i \otimes_A T^j,$$

$$d^n_{M \otimes_A P} = \sum_{i+j=n} (d^i \otimes \mathrm{Id}_{T^j} + (-1)^i \mathrm{Id}_{M^i} \otimes \delta^j).$$

Then, $- \otimes_A T$ and $- \otimes_A^{gr} \hat{T}$ are graded functors from $K(\text{Gr-}A)$(resp. $D(\text{Gr-}A)$) to $K(\text{Gr-}B)$(resp. $D(\text{Gr-}B)$).

In [3], Del Rio characterized the graded equivalences of categories of graded modules. We give a similar result concerning homotopy categories.

Theorem 2. *Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be two graded rings. Let $T = (T^n, d^n)$ be a complex of bigraded (A, B)-bimodules. Then, $F = - \otimes_A^{gr} T$ is a functor from $K(\text{Gr-}A)$ to $K(\text{Gr-}B)$ and $H = \bigoplus_{g \in G}(Hom^{\bullet}_{\text{Gr-}B}({}_{g^{-1}}T, -))$ is a functor from $K(\text{Gr-}B)$ to $K(\text{Gr-}A)$. The following conditions are equivalent.*

(i) F is a graded functor;
(ii) H is a graded functor;
(iii) There exists a complex of graded (A, B)-bimodule P such that $T \cong \hat{P}$ in $K(\text{Gr-}B)$.

Proof. (i) ⇒ (ii) Since (F, H) and $(S^A_g, S^B_{g^{-1}})$ are adjoint pairs, we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{K(\text{Gr-}B)}(F(S^A_g(M)), N) &\cong \mathrm{Hom}_{K(\text{Gr-}A)}(S^A_g(M), H(N)) \\ &\cong \mathrm{Hom}_{K(\text{Gr-}A)}(M, S^B_{g^{-1}}(H(N)))\end{aligned}$$

for all $M \in K(\text{Gr-}A)$, $N \in K(\text{Gr-}B)$ and $g \in G$. Thus, $(F \circ S^A_g, S^B_{g^{-1}} \circ H)$ is an adjoint pair.

Similarly, we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{K(\text{Gr-}B)}(S^A_g(F(M)), N) &\cong \mathrm{Hom}_{K(\text{Gr-}A)}(F(M), S^B_{g^{-1}}(N)) \\ &\cong \mathrm{Hom}_{K(\text{Gr-}A)}(M, H(S^B_{g^{-1}}(N)))\end{aligned}$$

for all $M \in K(\text{Gr-}A)$, $N \in K(\text{Gr-}B)$ and $g \in G$. Then, $(S^A_g \circ F, H \circ S^B_{g^{-1}})$ is an adjoint pair. If F is a graded functor, then, for every $g \in G$, $F \circ S^A_g$ is naturally isomorphic to $S^A_g \circ F$, and then $S^B_{g^{-1}} \circ H$ is naturally isomorphic to $H \circ S^B_{g^{-1}}$. Thus, H is a graded functor.

(ii) ⇒ (i) This is similar to the case (i) ⇒ (ii).

(i) ⇒ (iii) Since F is a graded functor, $F \circ S^A_g$ is naturally isomorphic to $S^A_g \circ F$ for every $g \in G$. For every $g \in G$, $F \circ S^A_g(A) \cong {}_gT$ and $S^A_g \circ F(A) \cong (g)_eT$. Then, there exists an isomorphism $\phi_g \colon {}_gT \to (g)_eT$ in $K(\text{Gr-}B)$; therefore,

$$\phi^n_g \colon ({}_gT)^n \to ((g)_eT)^n$$

is a morphism in Gr-B for every n. Given $a \in A_{hg^{-1}}$ for some $g, h \in G$, let $\rho(a)$ be the morphism from the complex $_gT$ to the complex $_hT$ such that, for all $t \in (_gT)^n$,

$$(\rho(a))^n\colon (_gT)^n \to (_hT)^n, (\rho(a))^n(t) = at.$$

Let μ_g^n denote the natural bijection from $(_eT)^n$ to $((g)_eT)^n$. Let

$$(\bar{\rho}(a))^n = (\mu_h^n)^{-1} \circ \phi_h^n \circ (\rho(a))^n \circ (\phi_g^n)^{-1} \circ \mu_g^n\colon (_eT)^n \to (_eT)^n,$$

then, $(\bar{\rho}(a))^n$ is a map from $(_eT)^n$ to $(_eT)^n$. Since ϕ_g^n and ϕ_h^n are morphisms in Gr-B, $(\bar{\rho}(a))^n((_eT_l)^n) \subseteq (_eT_{hg^{-1}l})^n$ for all $l \in G$. Then, $(_eT)^n = \bigoplus_{l \in G}(_eT_l)^n$ has a left G-graded A-module structure by

$$a \cdot t = (\bar{\rho}(a))^n(t)$$

for all $a \in A$ and all $t \in (_eT)^n$.

Given $g \in G$ and $a \in A$, it is clear that μ_g^n commutes with both d^i and right B-module actions on $(_eT)^n$. The map ϕ_g^n commutes with both d^i and right B-module actions on $(_eT)^n$ since ϕ_g is an isomorphism in $K(\text{Gr-}B)$. The map $(\rho(a))^n$ commutes with both d^i and right B-module actions on $(_eT)^n$ since T is a complex of bigraded (A, B)-bimodules. Therefore, $(\bar{\rho}(a))^n$ commutes with both d^i and right B-module actions on $(_eT)^n$. Thus, $(_eT)^n$ has a graded (A, B)-bimodule structure and $_eT$ is a complex of graded (A, B)-bimodules.

Let $P = {}_eT$. Since $(g)P = (g)_eT \cong {}_gT$ in $K(\text{Gr-}B)$, we have $\hat{P} = \bigoplus_{g \in G} (g)_eT \cong \bigoplus_{g \in G} {}_gT = T$ in $K(\text{Gr-}B)$.

(iii) ⇒ (i) If there exists a complex of graded (A, B)-bimodule P such that $T \cong \hat{P}$ in $K(\text{Gr-}B)$, then F is natural isomorphic to $- \otimes_A P$. Thus, F is a graded functor. □

Recall that U_A denotes the forgetful functor from Gr-A to Mod-A for a graded ring A.

Corollary 1. *Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be two graded rings. Let T be a complex of bigraded (A, B)-bimodules. Assume that $F = - \otimes_A^{gr} T\colon K(\text{Gr-}A) \to K(\text{Gr-}B)$ is a graded functor. The following conditions are equivalent.*

(i) *$LF\colon D(\text{Gr-}A) \to D(\text{Gr-}B)$ is an equivalence of triangulated categories;*

(ii) *There exists a complex P of graded (A, B)-bimodules such that $L\bar{F} = L(- \otimes_A P)\colon D(\text{Mod-}A) \to D(\text{Mod-}B)$ is an equivalence of triangulated categories and $U_B \circ LF$ is naturally isomorphic to $L\bar{F} \circ U_A$.*

Proof. (i) ⇒ (ii) By Theorem 2, there exists a complex of graded (A, B)-bimodules $P = {}_eT$ such that $T \cong \hat{P}$ in $K(\text{Gr-}B)$. Since LF is an equivalence of triangulated categories, we have $P \in \text{per}(\text{Gr-}B)$; that is, P is quasi-isomorphic to a complex P' in $K^b(\text{proj-}B)$. Consider P and P' as complexes of right B-modules; then, P is quasi-isomorphic to a bounded complex of finitely generated projective right B-modules P'. Thus, $P \in \text{per}(\text{Mod-}B)$.

Suppose $\mathcal{U}$ is the smallest full triangulated subcategory of $D(\text{Mod-}B)$ containing P and closed under forming direct summands. Since $(g)P = (g)_eT \cong {}_gT$ in $D(\text{Gr-}B)$ for every $g \in G$, we have $\text{per}(\text{Gr-}B)$ as the smallest full triangulated subcategory of $D(\text{Gr-}B)$ containing $\{(g)P\}_{g \in G}$ and closed under forming direct summands. Suppose $X \to Y \to Z \to X[1]$ is a distinguished triangle in $D(\text{Gr-}B)$; then, $U_B(X) \to U_B(Y) \to U_B(Z) \to U_B(X[1])$ is a distinguished triangle in $D(\text{Mod-}B)$. Therefore, for every $M \in \text{per}(\text{Gr-}B)$, $U_B(M) \in \mathcal{U}$. Then, $B \in \mathcal{U}$ since $B \in \text{per}(\text{Gr-}B)$. However, the smallest full triangulated subcategory of $D(\text{Mod-}B)$ containing B and closed under forming direct summands equals $\text{per}(\text{Mod-}B)$ and $P \in \text{per}(\text{Mod-}B)$. Then, $\mathcal{U}$ equals $\text{per}(\text{Mod-}B)$.

Since $P \in \mathrm{per}(\text{Mod-}B)$,

$$\begin{array}{rcl} \mathrm{Hom}_{D(\text{Mod-}B)}(P,P) & = & \mathrm{Hom}_{D(\text{Mod-}B)}({}_eT, {}_eT) \\ & \cong & \bigoplus_{g\in G} \mathrm{Hom}_{D(\text{Gr-}B)}({}_eT, (g)_eT) \\ & \cong & \bigoplus_{g\in G} \mathrm{Hom}_{D(\text{Gr-}B)}({}_eT, {}_gT) \\ & \cong & \bigoplus_{g\in G} A_g \\ & \cong & A. \end{array}$$

Similarly, $\mathrm{Hom}_{D(\text{Mod-}B)}(P, P[n]) = 0$ for all $n \neq 0$. By ([10], Section 8.1.4), $L\bar{F}$ is an equivalence of triangulated categories. Since F is natural isomorphic to $-\otimes_A P\colon K(\text{Gr-}A) \to K(\text{Gr-}B)$, we have natural isomorphisms

$$U_B \circ LF \cong U_B \circ L\bar{F} \cong L\bar{F} \circ U_A.$$

(ii) $\Rightarrow$ (i) By Theorem 2, $F \cong -\otimes_A P \cong -\otimes_A^{gr} \hat{P}$. Then, by Theorem 1, LF is an equivalence of triangulated categories. □

Let $A = \bigoplus_{g\in G} A_g$ be a graded ring. Let $P = \bigoplus_{g\in G} P_g$ be a graded right A-module and $Q = \bigoplus_{g\in G} {}_gQ$ be a graded left A-module. Let G' be a subgroup of G. Then, $A_{(G')}$ will denote the G'-graded ring $\bigoplus_{g\in G'} A_g$, $P_{(G')}$ will denote the graded right $A_{(G')}$-module $\bigoplus_{g\in G'} P_g$ and ${}_{(G')}Q$ will denote the graded left $A_{(G')}$-module $\bigoplus_{g\in G'} {}_gQ$.

Corollary 2. *Let $A = \bigoplus_{g\in G} A_g$ be a graded ring. Let G' be a subgroup of G. Let $B = \bigoplus_{g\in G'} B_g$ be the ring $A_{(G')}$ with $B_g = A_g$ for all $g \in G'$. Let $T = {}_{(G')}\hat{A} \cong \bigoplus_{g\in G'} (g)A$. The following conditions are equivalent.*

(i) $LF = L(-\otimes_B^{gr} T)\colon D(\text{Gr-}B) \to D(\text{Gr-}A)$ is an equivalence of triangulated categories;
(ii) The smallest full triangulated subcategory of $D(\text{Gr-}A)$ containing $\{(g)A \,|\, g \in G'\}$ and closed under forming direct summands equals $\mathrm{per}(\text{Gr-}A)$;
(iii) $\{(g)A \,|\, g \in G'\}$ is a set of generators in Gr-A;
(iv) $-\otimes_B^{gr} T\colon \text{Gr-}B \to \text{Gr-}A$ is an equivalence of categories of graded modules.

Proof. (i) $\Leftrightarrow$ (ii) is a consequence of Theorem 1 since, for every $g \in G'$, $LF((g)A) \cong (g)A$.
(i) $\Rightarrow$ (iii) Since LF is an equivalence,

$$RH = \bigoplus_{g\in G'} \mathrm{RHom}^{\bullet}_{\text{Gr-}A}({}_{g^{-1}}T, -)\colon D(\text{Gr-}A) \to D(\text{Gr-}B)$$

is an equivalence of triangulated categories and $LF \circ RH$ is naturally isomorphic to the identity functor of $D(\text{Gr-}A)$. Then, for every $h \in G$,

$$\begin{array}{rcl} RH((h)A) & = & \bigoplus_{g\in G'} \mathrm{RHom}^{\bullet}_{\text{Gr-}A}({}_{g^{-1}}T, (h)A) \\ & \cong & \bigoplus_{g\in G'} \mathrm{Hom}^{\bullet}_{\text{Gr-}A}((g^{-1})A, (h)A) \\ & \cong & \bigoplus_{g\in G'} A_{hg} \\ & = & ((h)A)_{(G')}. \end{array}$$

Therefore,

$$\begin{array}{rcl} (h)A & \cong & LF \circ RH((h)A) \\ & \cong & LF(((h)A)_{(G')}) \\ & \cong & ((h)A)_{(G')} \otimes_B^{gr} T. \end{array}$$

Then, we have $(h)A \cong ((h)A)_{(G')} \otimes_B^{gr} T$ as two modules in Gr-A. Since $\{(g)B \,|\, g \in G'\}$ is a set of generators in Gr-B, there exists an epimorphism for some $g_i \in G'$,

$$f\colon \bigoplus_{i\in I} (g_i)B \to ((h)A)_{(G')}.$$

Therefore,

$$f \otimes_B^{gr} T \colon \bigoplus_{i \in I} (g_i)B \otimes_B^{gr} T \to ((h)A)_{(G')} \otimes_B^{gr} T$$

is an epimorphism in Gr-A. Since $(g_i)B \otimes_B^{gr} T \cong {}_{g_i}T \cong (g_i)A$ for every $g_i \in G'$, we have $\{(g)A | g \in G'\}$ generates $(h)A$ in Gr-A. Thus, $\{(g)A \,|\, g \in G'\}$ is a set of generators in Gr-A.

(iii) ⇒ (iv) by ([3], Corollary 11).

(iv) ⇒ (i) is trivial. □

Corollary 3. *Let $A = \bigoplus_{g \in G} A_g$ be a graded ring. The following conditions are equivalent.*

(i) *There is an ungraded ring B and a complex T of bigraded (B, A)-bimodules, where B is considered as a graded ring with trivial grading, such that $L(- \otimes_B^{gr} T) \colon D(\text{Mod-}B) \to D(\text{Gr-}A)$ is an equivalence of triangulated categories;*

(ii) *There exists a finite subset G' of G, such that the smallest full triangulated subcategory of $D(\text{Gr-}A)$ containing $\{(g)A | g \in G'\}$ and closed under forming direct summands equals $\text{per}(\text{Gr-}A)$.*

Proof. (i) ⇒ (ii) By Theorem 1, $T \in \text{per}(\text{Gr-}A)$. Then, there exists $P \in D^b(\text{proj-}A)$, such that $T \cong P$ in $D(\text{Gr-}A)$. Since P is a bounded complex of finitely generated modules, there exists a finite subset G' of G such that $\{(g)A | g \in G'\}$ generates P^i for all $i \in \mathbb{Z}$. Since the smallest full triangulated subcategory of $D(\text{Gr-}A)$ containing P and closed under forming direct summands equals $\text{per}(\text{Gr-}A)$, we have the smallest full triangulated subcategory of $D(\text{Gr-}A)$ containing $\{(g)A | g \in G'\}$ and closed under forming direct summands being equal to $\text{per}(\text{Gr-}A)$.

(ii) ⇒ (i) Let $T = \bigoplus_{g \in G'} (g)A \in \text{Gr-}A$. Let $B = \text{Hom}_{D(\text{Gr-}A)}(T, T)$. Then, T is a bigraded (B, A)-bimodule by considering B trivially graded by e. By Theorem 1, $L(- \otimes_B^{gr} T) \colon D(\text{Mod-}B) \to D(\text{Gr-}A)$ is an equivalence of triangulated categories. □

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