mathematics

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A New Relativistic Model for Polyatomic Gases Interacting with an Electromagnetic Field

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Abstract: Maxwell's equations in materials are studied jointly with Euler equations using new knowledge recently appeared in the literature for polyatomic gases. For this purpose, a supplementary conservation law is imposed; one of the results is a restriction on the law linking the magnetic field in empty space and the electric field in materials to the densities of the 4-Lorentz force ν^α and its dual μ^α: These are the derivatives of a scalar function with respect to ν^α and μ^α, respectively. Obviously, two of Maxwell's equations are not evolutive (Gauss's magnetic and electric laws); to simplify this mathematical problem, a new symmetric hyperbolic set of equations is here presented which uses unconstrained variables and the solutions of the new set of equations, with initial conditions satisfying the constraints, restore the previous constrained set. This is also useful because it allows to maintain an overt covariance that would be lost if the constraints were considered from the beginning. This is also useful because in this way the whole set of equations becomes a symmetric hyperbolic system as usually in Extended Thermodynamics.

Keywords: Maxwell's equations; Extended Thermodynamics; polyatomic gases

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Citation: Pennisi, S.; Tchame, R.E.; Obounou, M. A New Relativistic Model for Polyatomic Gases Interacting with an Electromagnetic Field. *Mathematics* **2022**, *10*, 110. https://doi.org/10.3390/math10010110

Academic Editor: Rami Ahmad El-Nabulsi

Received: 22 October 2021
Accepted: 25 December 2021
Published: 30 December 2021

1. Introduction

Up to now it has been shown that Maxwell's Equations are compatible with a supplementary conservation law [1]; but this property was demonstrated only in the case of the empty space. Here we want to improve this result by applying it also in the case in which there is an interaction with a polyatomic gas. Now Maxwell's equations in materials must necessarily be coupled with the balance equations of this material and we begin to couple them with the Euler equations for polyatomic gases; hence the whole set of equations is:

$$\partial_\alpha V^\alpha = 0\,, \quad \partial_\alpha T^{\alpha\beta} = q\, k^\beta\,, \quad \partial_\alpha F^{\alpha\beta} = -J^\beta\,, \quad \partial_\alpha G^{\alpha\beta} = 0\,, \quad \partial_\alpha J^\alpha = 0\,, \tag{1}$$

where $U_\beta = \frac{1}{m\,n} V_\beta$, m is the particle mass, $n = \frac{1}{m\,c}\sqrt{V^\beta V_\beta}$ and c is the speed of light (hence $V^\alpha = m\,n\,U^\alpha$ and $U^\beta U_\beta = c^2$ follow). Furthermore, $T^{\alpha\beta}$ is the energy momentum tensor, q is the charge density, $q\,k^\beta = q\,\frac{1}{2}\,\eta^{\beta\epsilon\alpha\gamma}\,\frac{U_\epsilon}{c}\,G_{\alpha\gamma}$ is the Lorentz 4-force, J^β is the free current density and, in any fixed reference frame, the tensors $F^{\alpha\beta}$ and $G^{\alpha\beta}$ can be decomposed as follows:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & cD^1 & cD^2 & cD^3 \\ -cD^1 & 0 & H^3 & -H^2 \\ -cD^2 & -H^3 & 0 & H^1 \\ -cD^3 & H^2 & -H^1 & 0 \end{pmatrix}, \; G^{\alpha\beta} = \begin{pmatrix} 0 & cB^1 & cB^2 & cB^3 \\ -cB^1 & 0 & -E^3 & E^2 \\ -cB^2 & E^3 & 0 & -E^1 \\ -cB^3 & -E^2 & E^1 & 0 \end{pmatrix}, \tag{2}$$

For references on this subject, see for example [2–7] which contain only marginally the results of the present article (for example, Maxwell equations are not coupled with the equations for the material), or belong to another context such as general relativity, quantistic mechanics or the use of a Lagrangian function.

The Equation (1)$_{1,2}$ are Euler's equations and, when Maxwell's equations are not present, $T^{\alpha\beta}$ has the form

$$T^{\alpha\beta} = \frac{e}{c^2}\, U^\alpha U^\beta + p\, h\alpha\beta \quad \text{with} \quad h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}\, U^\alpha U^\beta \,, \tag{3}$$

where e is the energy, p is the pressure and $h^{\alpha\beta}$ is the projector into the 3-dimensional subspace orthogonal to U_α. Furthermore, e and p are constitutive functions of the absolute temperature T.

Now in the system (1) there are 14 independent equations, while the tensors that appear in it have 30 independent components; therefore only a part of these components can be assumed as independent variables. It follows that it is necessary to express a part of these components as functions of the rest; they are called constitutive functions and "the closure problem" deals with how to find them. To this end, we adopt well-known procedures which we now describe.

1.1. The Closure Problem in Extended Thermodynamics

As usual in Extended Thermodynamics (see, for example [8–11]), restrictions on these functions can be found by imposing the Entropy Principle which requires the existence of the entropy-entropy flux 4-vector h^α and of the entropy production Σ such that the following supplementary equation holds for each solution of the system (1)$_{1,2}$:

$$\partial_\alpha h^\alpha = \Sigma \geq 0\,. \tag{4}$$

This non-negative entropy production requirement is a binding condition because it must hold only for each solution of the system (1)$_{1,2}$. Its exploitation becomes easier if we use Liu's Theorem [12]; he showed that the requirement (4) for all solutions of the generic system $\partial_\alpha F^{\alpha A} = I^A$ is equivalent to assuming the existence of Lagrange multipliers λ_A such that the condition

$$\begin{aligned} &\partial_\alpha h^\alpha - \lambda_A\, \partial_\alpha\, F^{\alpha A} = 0\,,\, \Sigma = \lambda_A\, I^A \geq 0\,, \\ &\text{holds for every value (no more constrained) of the independent variables}\,. \end{aligned} \tag{5}$$

Subsequently, Dreyer in [13] introduced in the kinetic context the so-called Maximum Entropy Principle (MEP), i.e., to require that the generalized entropy

$$\rho\, s = h = h^\alpha U_\alpha = -k_B c\, U_\alpha \int_{\Re^3} \int_0^{+\infty} f \ln f\, p^\alpha \phi(\mathcal{I})\, d\vec{P}\, d\,\mathcal{I}$$

(with k_B the Boltzmann constant) has a maximum under the constraints $\partial_\alpha F^{\alpha A} = I^A$. This variational problem allows to find the expression of the distribution function f and the above λ_A are the associated Lagrange multipliers. In effect Dreyer worked on monoatomic gases, while the one above is the generalization of his functional to polyatomic gases, as reported in [14], page 427. However, we do not report further details on this aspect because they are not necessary for this article. We have said the above only to give a historical justification for the name "Lagrange multipliers" and because they will be needed when the present results will be updated to include dissipative phenomena.

Other important articles are [15–17] where it was found that:

- Equation (5)$_1$ can be written as $d\, h^\alpha - \lambda_A\, d\, F^{\alpha A} = 0\,,$
- The function h'^α (which they call 4-potential) can be defined by $h'^\alpha = -\, h^\alpha + \lambda_A\, F^{\alpha A}$ so that it follows $d\, h'^\alpha = F^{\alpha A}\, d\, \lambda_A$,
- If we change independent variables, from the original ones to the Lagrange multipliers λ_A, then we have $F^{\alpha A} = \frac{\partial\, h'^\alpha}{\partial\, \lambda_A}$ and the field equations $\partial_\alpha F^{\alpha A} = I^A$ become $\frac{\partial^2\, h'^\alpha}{\partial\, \lambda_A\, \partial\, \lambda_B}\, \partial_\alpha \lambda_B = I^A$. These equations are evidently symmetric so that, for their hyper-

bolicity in the time-like constant congruence ξ_α, it will be sufficient that the function $\xi_\alpha h'^\alpha$ is a convex function of the variables λ_A (Convexity requirement).

This methodology allows to express all the unknown functions present in the field equations in terms of the only function h'^α. Then you have to do the inverse of the aforementioned change of variables, from the Lagramge multipliers to the physical variables to have everything expressed in terms of the latter.

1.2. Application of the above Procedure to the Current Problem

Now, we want to apply this methodology to our problem. We therefore impose the existence of the supplementary conservation law (4) for all field Equation (1). Now, when Maxwell's equations are not present, this is surely the Entropy Principle; for Maxwell's equations there is a discussion among researchers on how to consider (4): still the entropy principle or an equation for energy? We do not want to express an opinion on this, so we simply call it a "supplementary conservation law", as in other articles in the literature. In fact, for what follows, it is not necessary to give it a precise name; we just want to take advantage of all its fine mathematical properties that we have described above and others present in the literature, such as the well-posedness of the Cauchy problem, the smooth dependence on initial values and so on.

So in our case we have the existence of 4-potential h'^α and the Lagrange multipliers which in our case we call λ, λ_β, ν_β, μ_β, ϑ. In this way, we have:

$$d\,h'^\alpha = V^\alpha\,d\,\lambda + T^{\alpha\beta}\,d\,\lambda_\beta + F^{\alpha\beta}\,d\,\nu_\beta + G^{\alpha\beta}\,d\,\mu_\beta + J^\alpha\,d\,\vartheta\,, \quad \Sigma = q\,k^\beta\,\lambda_\beta - J^\beta\,\nu_\beta \geq 0\,. \tag{6}$$

We will see in Sections 3 and 4 that we get the following expression for h'^α:

$$h'^\alpha = h_0\,\lambda^\alpha + \eta^{\alpha\beta\gamma\delta}\frac{\lambda_\beta}{\sqrt{G_{00}}}\,\nu_\gamma\mu_\delta\,, \tag{7}$$

where $\eta^{\alpha\beta\gamma\delta}$ is the 4-dimensional Levi-Civita symbol, $G_{00} = \lambda^\beta\lambda_\beta$ and h_0 is a function of G_{00}, $G_{11} = \mu_\alpha\mu^\alpha$, $G_{12} = \mu_\alpha\nu^\alpha$, $G_{22} = \nu_\alpha\nu^\alpha$, ϑ. We assume that μ_β, ν_β are not free but constrained by:

$$\lambda^\alpha\mu_\alpha = 0\,, \quad \lambda^\alpha\nu_\alpha = 0\,, \tag{8}$$

otherwise the number of independent equations would not equal the number of independent variables. We will also find that J^β is parallel to λ^β (see $(24)_2$) and $k^\beta = \nu^\beta$ (see last 3 lines of Section 3, below), so that Equation $(6)_2$ is satisfied with $\Sigma = 0$.

So we only need to know a scalar function $h_0(\lambda, G_{00}, G_{11}, G_{12}, G_{22}, \vartheta)$ to close the whole system. Its expression depends on the material that is considered and characterizes it. For example, we may define $h_0^M = h_0(\lambda, G_{00}, 0, 0, 0, 0)$ and $\tilde{h}_0 = h_0(\lambda, G_{00}, G_{11}, G_{12}, G_{22}, \vartheta) - h_0^M$ and (6) will give:

$$\begin{aligned}
&V^\alpha = V_M^\alpha + \frac{\partial\,\tilde{h}_0}{\partial\,\lambda}\lambda^\alpha\,, \quad J^\alpha = \left(\frac{\partial\,h_0^M}{\partial\,\vartheta} + \frac{\partial\,\tilde{h}_0}{\partial\,\vartheta}\right)\lambda^\alpha\,,\\
&T^{\alpha\beta} = T_M^{\alpha\beta} + 2\,\frac{\partial\,\tilde{h}_0}{\partial\,G_{00}}\,\lambda^\alpha\lambda^\beta + \tilde{h}_0\,g^{\alpha\beta} - \frac{\partial\,\tilde{h}_0}{\partial\,\nu_\alpha}\,\nu^\beta - \frac{\partial\,\tilde{h}_0}{\partial\,\mu_\alpha}\,\mu^\beta - \eta^{\alpha\theta\gamma\delta}\frac{h_\theta^\beta}{\sqrt{G_{00}}}\,\nu_\gamma\mu_\delta\,,\\
&F^{\alpha\gamma}\,h_\gamma^\beta = \eta^{\alpha\theta\beta\delta}\frac{\lambda_\theta}{\sqrt{G_{00}}}\,\mu_\delta\,,\; G^{\alpha\gamma}\,h_\gamma^\beta = \eta^{\alpha\theta\psi\beta}\frac{\lambda_\theta}{\sqrt{G_{00}}}\,\nu_\psi\,.
\end{aligned} \tag{9}$$

where

$$V_M^\alpha = \frac{\partial\,h_0^M}{\partial\,\lambda}\,, \quad T_M^{\alpha\beta} = 2\,\frac{\partial\,h_0^M}{\partial\,G_{00}}\,\lambda^\alpha\lambda^\beta + h_0^M\,g^{\alpha\beta}\,. \tag{10}$$

(see Equation (24) of Section 3) These expressions of V^α_M and $T^{\alpha\beta}_M$ are those obtained in the absence of the electromagnetic field and in Section 4 of [14] it was proved that for polyatomic gases they are (3) with:

$$
\begin{aligned}
& V^\alpha_M = \rho\, U^\alpha \,, \quad p = \frac{k_B c}{m}\, \frac{\rho}{\sqrt{G_{00}}} \,, \; e = \rho\, c^2\, \frac{\int_0^{+\infty} J^*_{2,2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right) \varphi(\mathcal{I})\, d\,\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \varphi(\mathcal{I})\, d\,\mathcal{I}} \\
& J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma \cosh s} \sinh^m s\, \cosh^n s\, d\,s \,, \; J^*_{m,n} = J_{m,n}\left[\gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)\right], \\
& \gamma = \frac{m\,c}{k_B}\, \sqrt{G_{00}} \,.
\end{aligned}
\tag{11}
$$

Here $\mathcal{I}$ indicates the internal energy of the molecule, due to its rotational and vibrational modes, and $\mathcal{I}$ is a measure of how polyatomic the gas is; in particular, for polytropic gases it is $\varphi(\mathcal{I}) = \mathcal{I}^a$ and monatomic gases are enclosed as a limiting case for a going to -1. More precisely, $a = \frac{D-5}{2}$ where D is relative to the degree of freedom of a molecule (The spatial dimension 3 plus the contribution of the internal degrees of freedom due to rotational and vibrational modes) . In the case of monatomic gas, we have $D = 3$. The expression $\varphi(\mathcal{I}) = \mathcal{I}^a$ is also classically valid (see the classic part of Equation (47) of [14]).

From (9) we see that the subsystem (in the sense of [18]) of our equations obtained by simply setting $\mu_\beta = 0$, $\nu_\beta = 0$, $\vartheta = 0$ and neglecting $(1)_{3-5}$ is that of polyatomic gases described in [14] for the part concerning the Euler's equations. It is true that [14] is now improved (see [19] for the classical case, while the relativistic case [20] is forthcoming), but these further developments do not change the part concerning the Euler's Equations which are here considered. We prefer to insert the present article in the framework of polyatomic gases because they are more general than the monoatomic gases and include it as a particular case. Moreover, polyatomic gases allows the formation of dipoles and also magnetization and polarization effects. As confirmation of the results here described, we will take in Section 2 their non relativistic limits and we will find that they become the same of [21] which were obtained there by working directly in the non relativistic framework.

Now the above reported equations are expressed in terms of the Lagraange multipliers as variables; so the last step remains to convert them in terms of physical variables. Let us see how to do this step in the simpler case of a weak electro-magnetic field.

1.3. A Simple Example of Inversion from the Lagrange Multipliers to Physical Variables

We consider the simple case of a homogeneous and isotropic medium with a weak electromagnetic field so that h_0 can be considered linear in G_{11} and G_{22} and the term with G_{12} is not present:

$$
\begin{aligned}
& h'^\alpha = \left(\frac{c\,\mu_0}{2}\, G_{11} + \frac{c\,\epsilon_0}{2}\, G_{22}\right) \frac{\lambda^\alpha}{\sqrt{G_{00}}} + \eta^{\alpha\beta\gamma\delta}\, \frac{\lambda_\beta}{\sqrt{G_{00}}}\, \nu_\gamma \mu_\delta - k_B\, c \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\, p^\alpha\, \varphi(\mathcal{I})\, d\,\mathcal{I}\, d\,\vec{P} \,, \\
& \text{with} \quad \chi = m\,\lambda + \left(1+\frac{\mathcal{I}}{m\,c^2}\right)\lambda_\beta p^\beta + \left(1+\frac{\mathcal{I}}{m\,c^2}\right)^2 \vartheta \,.
\end{aligned}
\tag{12}
$$

Here μ_0 and ϵ_0 are constants. If we call h'^α_1 the last term of $(12)_1$, we see that

$$
\lambda_\alpha\, \frac{\partial^2\, h'^\alpha_1}{\partial\, \lambda_A\, \partial\, \lambda_B} d\,\lambda_A\, d\,\lambda_B = -\frac{c}{k_b} \int_{\Re^3} \int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\, p^\alpha\, (d\,\chi)^2 \varphi(\mathcal{I})\, d\,\mathcal{I}\, d\,\vec{P} < 0 \,,
$$

therefore the convexity of this part of h'^α is satisfied; we will see that it also holds for the other side, at least for a weak electromagnetic field. Now the integrals can be calculated with a small modification to the one on page 422 of [14] and we find:

$$V^\alpha = m\,n\,U^\alpha\,,\; T^{\alpha\beta} = \frac{e}{c^2}\,U^\alpha U^\beta \,+\, p\,h^{\alpha\beta} \,-\, \frac{m\,c}{k_B\gamma}\left[\left(\frac{c\,\mu_0}{2}\,G_{11} + \frac{c\,\epsilon_0}{2}\,G_{22}\right)h^{\alpha\beta} \,+\, \eta^{\alpha\theta\gamma\delta}h^\beta_\theta\,\nu_\gamma\,\mu_\delta\right], \\ J^\alpha = q\,U^\alpha\,,\; F^{\alpha\beta} = 2\,\mu_0\,U^{[\alpha}\mu^{\beta]} + \frac{1}{c}\,\eta^{\alpha\phi\gamma\beta}\,U_\phi\,\nu_\gamma\,,\; G^{\alpha\beta} = 2\,\epsilon_0\,U^{[\alpha}\nu^{\beta]} + \frac{1}{c}\,\eta^{\alpha\phi\beta\gamma}\,U_\phi\,\mu_\gamma\,, \tag{13}$$

with

$$n = 4\,\pi\,m^3\,c^3 e^{-1-\frac{m\lambda}{k_B}}\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2} J_{2,1}(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}\,,\; U^\alpha = c\,\frac{\lambda^\alpha}{\sqrt{G_{00}}} \\ e = 4\,\pi\,m^4\,c^5 e^{-1-\frac{m\lambda}{k_B}}\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)J_{2,2}\,(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}\,, \\ p = \frac{4}{3}\,\pi\,m^4\,c^5 e^{-1-\frac{m\lambda}{k_B}}\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)J_{4,0}\,(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}\,, \\ q = 4\,\pi\,m^3\,c^3 e^{-1-\frac{m\lambda}{k_B}}\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)^2 J_{2,1}(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}\,, \tag{14}$$

where we called $\gamma = \frac{m\,c}{k_B}\sqrt{G_{00}}$ and in the last equation we used the identity $J_{2,3}(\gamma) - J_{4,1}(\gamma) = J_{2,1}(\gamma)$. Now, we can take λ from $(14)_1$ and replace it in $(14)_{2,3,4}$; we can also use the identity $\gamma\,J_{4,0}(\gamma) = 3\,J_{2,1}(\gamma)$ and get

$$\frac{e}{m\,n\,c^2} = \frac{\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)J_{2,2}\,(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}}{\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2} J_{2,1}\,(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}}\,,\; p = \frac{m\,n\,c^2}{\gamma}\,, \\ \frac{q}{n} = \frac{\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2}\left(1+\frac{\mathcal{I}}{m\,c^2}\right)^2 J_{2,1}\,(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}}{\int_0^{+\infty} e^{-\frac{\vartheta}{k_B}\left(1+\frac{\mathcal{I}}{mc^2}\right)^2} J_{2,1}\,(\gamma^*)\,\varphi(\mathcal{I})\,d\,\mathcal{I}}\,. \tag{15}$$

Moreover, by using the identities $J_{2,3}(\gamma) = J_{4,1}(\gamma) + J_{2,1}(\gamma)$, $J_{2,0}(\gamma) = -\,J_{4,0}(\gamma) + J_{2,2}(\gamma)$ and $(15)_1$, we see that $(15)_2$ can be expressed in terms of the energy e as:

$$\frac{q}{n} = \left(\frac{e}{m\,n\,c^2}\right)^2 - \frac{\partial}{\partial\,\gamma}\left(\frac{e}{m\,n\,c^2}\right) - \frac{3}{\gamma}\left(\frac{e}{m\,n\,c^2}\right) - \frac{3}{\gamma^2}\,. \tag{16}$$

Now $(15)_2$ can be used to desume the Lagrange multiplier ϑ and substitute in $(15)_1$ so obtaining $e = m\,n\,c^2\,\varepsilon(\gamma\,,\,\frac{q}{n})$. Therefore, we have changed variables from the Lagrange multipliers λ, λ_β, ϑ to the physical variables n, U^α, γ (or p), q. The closure depends on the function $\varepsilon(\gamma\,,\,\frac{q}{n})$. There remains the Lagrange multipliers ν_β, μ_β but these have already a physical meaning because, as we will see in Section 3, ν_β is the 4-force acting on an unitary charge and μ_β can be considered its dual:

$$\mu_\phi = -\,\frac{1}{2}\,\eta_{\phi\epsilon\alpha\gamma}\,F^{\alpha\gamma}\,\frac{U^\epsilon}{c}\,,\quad \nu_\phi = \frac{1}{2}\,\eta_{\phi\epsilon\alpha\gamma}\,G^{\alpha\gamma}\,\frac{U^\epsilon}{c}\,. \tag{17}$$

2. The Non Relativistic Limit

The same problem of the present article has been treated in [21] but following the non-relativistic formalism; now the relativistic context is clearly best suited to describe it, and this is the subject of the present article. However, as a validity test of the present model, it is useful to see if its non-relativistic limit provides the classical model in [21]. This will be proved in the present section.

So we start by taking the non-relativistic limit of Equations $(6)_1$ and (7). To achieve this goal we recall that from Equation (17) of [14] we have:

$$\lim_{c\to+\infty}\frac{V^0}{c}=F\,,\quad \lim_{c\to+\infty}\frac{T^{0i}}{c}=F^i\,,\quad \lim_{c\to+\infty}2\left(T^{00}-cV^0\right)=G^{ll}\,,\quad \lim_{c\to+\infty}V^k=F^k\,,\tag{18}$$
$$\lim_{c\to+\infty}2c\left(T^{k0}-cV^k\right)=G^{kll}\,,\quad \lim_{c\to+\infty}T^{ki}=T^{ki}\,,\quad \lim_{c\to+\infty}\left(\frac{h'^0}{c}\right)=h'^{Clas}\,,\ \lim_{c\to+\infty}h'^i=h'^{i\,Clas}\,.$$

These properties suggest us to define λ^{Clas}, h', λ_{ll} and v_i from

$$\lambda=\lambda^{Clas}-\lambda_0 c\,,\quad \frac{h_0\,\lambda^0}{c}=h'\,,\quad \lambda_\beta\equiv 2\,\lambda_{ll}\,\Gamma\,(c\,,\,v_i)\,,\tag{19}$$

where Γ is the Lorentz factor $\Gamma=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. In this way from (7) with $\alpha=0$, we get:

$$\frac{h'^0}{c}=h'+\eta^{0abc}\frac{\Gamma\,v_a}{c^2}\,\nu_b\mu_c\,,\quad\rightarrow\quad \lim_{c\to+\infty}\frac{h'^0}{c}=h'\,,$$

also in the present case. Similarly, from $(6)_1$ with $\alpha=0$, we get:

$$d\left(\frac{h'^0}{c}\right)=\frac{V^0}{c}\,d\,\lambda^{Clas}+2\left(T^{00}-cV^0\right)\frac{d\,\lambda_0}{2\,c}+\left(\frac{T^{0i}}{c}\right)d\,\lambda_i+\frac{F^{0i}}{c}\,d\,\nu_i+\frac{G^{0i}}{c}\,d\,\mu_i+q\,\frac{1}{\sqrt{G_{00}}}\,\lambda^0\,d\,\vartheta=$$
$$=\frac{V^0}{c}\,d\,\lambda^{Clas}+2\left(T^{00}-cV^0\right)d\,(\lambda_{ll}\,\Gamma)+\left(\frac{T^{0i}}{c}\right)d\,\lambda_i+\frac{F^{0i}}{c}\,d\,\nu_i+\frac{G^{0i}}{c}\,d\,\mu_i+q\,\Gamma\,d\,\vartheta\,,$$

where we have taken into account of $F^{00}=0$, $G^{00}=0$ and of $J^\alpha=q\,\frac{c}{\sqrt{G_{00}}}\,\lambda^\alpha$. The non-relativistic limit of this expression is:

$$d\,h'=F\,d\,\lambda^{Clas}+G^{kk}\,d\,\lambda_{ll}+F^i\,d\,\lambda_i+D^i\,d\,v_i+B^i\,d\,\mu_i+q\,d\,\vartheta\,,$$

such as in Equation $(10)_1$ of [21].

We now want to take the non-relativistic limit of Equations $(6)_1$ and (7) for $\alpha=k$. To this end, we first note that from the constraints (8) it follows $\mu_0=-\frac{\mu_i\lambda^i}{\lambda^0}=-\frac{\mu_i v^i}{c}$, $\nu_0=-\frac{\nu_i v^i}{c}$. Then, from (7) with $\alpha=k$, we get:

$$h'^k=h_0\,\lambda^k+\eta^{k0cd}\frac{\lambda_0}{\sqrt{G_{00}}}\,\nu_c\,\mu_d+\eta^{kb0d}\frac{\lambda_b}{\sqrt{G_{00}}}\,\nu_0\,\mu_d+\eta^{kbc0}\frac{\lambda_b}{\sqrt{G_{00}}}\,\nu_c\,\mu_0=$$
$$=\frac{c\,h'}{\lambda^0}\,\lambda^k+\eta^{k0cd}\,\Gamma\,\nu_c\,\mu_d-\eta^{kb0d}\,\Gamma\,v_b\,\frac{\nu_i v^i}{c}\,\mu_d-\eta^{kbc0}\,\Gamma\,v_b\,\nu_c\,\frac{\mu_i v^i}{c}=,$$
$$=h'\,v^k+\eta^{0kcd}\,\Gamma\,\nu_d\,\mu_c-\eta^{kb0d}\,\Gamma\,v_b\,\frac{\nu_i v^i}{c}\,\mu_d-\eta^{kbc0}\,\Gamma\,v_b\,\nu_c\,\frac{\mu_i v^i}{c}\,,$$

whose non relativistic limit is $h'^k=h'\,v^k+\eta^{0kcd}\,\mu_c\,\nu_d$ like in Equation (18) of [21] with $h_3=1$. (Note that from (19) we have $\lambda_k=2\,\lambda_{ll}\,v_k$ but, by raising a latin index, the result change sign for the present definition of the metric tensor $g_{\alpha\beta}=\text{diag}\,(1\,,\,-1\,,\,-1\,,\,-1)$ so that $\lambda^k=-2\,\lambda_{ll}\,v_k$ while in the classical context $v^k=v_k$).

Let us consider now $(6)_1$ for $\alpha=k$, i.e.,

$$
\begin{aligned}
d\,h'^k = V^k\,d\,\lambda + T^{k0}\,d\,\lambda_0 + T^{ki}\,d\,\lambda_i + F^{k0}\,d\,\nu_0 + F^{ki}\,d\,\nu_i + G^{k0}\,d\,\mu_0 + G^{ki}\,d\,\mu_i + J^k\,d\,\vartheta = \\
= V^k\,d\,\lambda^{Clas} + \left(T^{k0} - c\,V^k\right) d\,\lambda_0 + T^{ki}\,d\,\lambda_i - c\,D^k\,d\left(-\frac{\nu_i v^i}{c}\right) + F^{ki}\,d\,\nu_i - c\,B^k\,d\left(-\frac{\mu_i v^i}{c}\right) + \\
+ G^{ki}\,d\,\mu_i + J^k\,d\,\vartheta = V^k\,d\,\lambda^{Clas} + 2\,c\left(T^{k0} - c\,V^k\right) d\,(\lambda_{ll}\Gamma) + T^{ki}\,d\,\lambda_i + D^k\,d\left(\nu_i v^i\right) + F^{ki}\,d\,\nu_i + \\
+ B^k\,d\left(\mu_i v^i\right) + G^{ki}\,d\,\mu_i + q\,\Gamma\,v^k\,d\,\vartheta\,.
\end{aligned}
$$

The non relativistic limit of this expression is

$$
d\,h'^k = F^k\,d\,\lambda^{Clas} + G^{kll}\,d\,(\lambda_{ll}) + T^{ki}\,d\,\lambda_i + D^k\,d\left(\nu_i v^i\right) + F^{ki}\,d\,\nu_i + B^k\,d\left(\mu_i v^i\right) + G^{ki}\,d\,\mu_i + q\,v^k\,d\,\vartheta\,,
$$

like in Equation $(10)_2$ of [21]. We remark that here the presence of the terms $D^k\,d\,(\nu_i v^i) + B^k\,d\,(\mu_i v^i)$ is due to the fact that in the classical context the Equations $\partial_k B^k = 0$, $\partial_k D^k = q$ constitute differential constraints for the field equations, while in the relativistic context it is not possible to separate these differential constraints from the other equations without losing manifest covariance. In any case we are able here to overcome this problem by using constrained variables; but in this way the symmetric form of the field equations cannot be obtained. Additionally, this problem has been here completely overcome by considering an extended set of equations and of independent variables, which reduces to the previous one only by choosing the initial values satisfying the constraints on the independent variables.

Regarding the right hand side of $(1)_2$, we note that this equation for $\beta = i$ has $\partial_t F^i + \partial_k F^{ki} = q\,k^i$ as non-relativistic limit and this is the same of Equation $(1)_2$ supported by $(5)_1$ of [21]. The sum of Equation $(1)_1$ multiplied by $-\,c$ and of $(1)_2$ with $\beta = 0$ has to be multiplied by $2\,c$ before to take its non relativistic limit. In this way we obtain Equation $(1)_3$ of [21] if $\lim_{c\to+\infty} 2\,c\,k^0 = -\,2\,q\,v_i\,v^i$ (here too the minus sign is due to the choice of the metric tensor) and this is true because, from the constraint $U_\beta \nu^\beta = 0$ and the decomposition $U_\beta = \Gamma\,(c\,,\,v_i)$ we obtain exactly $2\,c\,k^0 = -\,2\,q\,v_i\,v^i$.

We conclude this section by considering the dependence of h_0 on λ and G_{00}; we can assume without loss of generality, that it depends on these variables as composite functions of $\frac{1}{2c}\sqrt{G_{00}}$ and $\lambda + c\sqrt{G_{00}}$. From (19) It follows that

$$
\begin{aligned}
&\frac{1}{2\,c}\sqrt{G_{00}} = \lambda_{ll}\,, \\
&\lambda + c\sqrt{G_{00}} = \lambda^{Clas} - \lambda_0 c + 2\lambda_{ll}c^2 = \lambda^{Clas} - 2\lambda_{ll}c^2\,\Gamma + 2\lambda_{ll}c^2 = \lambda^{Clas} - 2\lambda_{ll}c^2\,\Gamma\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) = \\
&= \lambda^{Clas} - 2\lambda_{ll}c^2\,\Gamma\,\frac{\frac{v^2}{c^2}}{\left(1 + \sqrt{1 - \frac{v^2}{c^2}}\right)} \quad \text{whose non relativistic limit is} \quad \lambda^{Clas} - \lambda_{ll}\,v^2\,.
\end{aligned}
$$

This last one is the variable called $\hat{\mu}$ in [21].

Finally, we have $G_{11} = (\mu_0)^2 + \mu^i\mu_i$, $G_{12} = \mu_0\nu^0 + \mu^i\nu_i$, $G_{22} = (\nu_0)^2 + \nu^i\nu_i$ whose non-relativitic limits are $\mu^i\mu_i$, $\mu^i\nu_i$, $\nu^i\nu_i$, respectively, because, from the above found $\mu_0 = -\,\frac{\mu_i v^i}{c}$, $\nu_0 = -\,\frac{\nu_i v^i}{c}$, we have that the non relativitic limits of μ_0 and ν_0 are zero.

So also the dependence of h' on the scalar variables found in [21] has been recovered (There is only a change of sign from $\mu^i\mu_i$, $\mu^i\nu_i$, $\nu^i\nu_i$ to $-\mu^i\mu_i$, $-\mu^i\nu_i$, $-\nu^i\nu_i$, due to the choice of the metric tensor; however this does not effect the results).

3. Existence of a Supplementary Conservation Law

We will see here how, assuming the existence of an supplementary conservation law, we find strong restrictions on the generality of the unknown constitutive functions.

First we simply want to verify that (7) is a solution of $(6)_1$ in the independent variablesλ, λ_β, ν_β, μ_β, ϑ bound by Equation (8). From $(8)_1$ follows $\lambda_\beta \, d\,\mu^\beta + \mu^\beta \, d\,\lambda_\beta = 0$ from which we infer $\lambda_\beta \, d\,\mu^\beta = -\mu^\beta \, d\,\lambda_\beta$ which we will use in the third of the following steps:

$$d\,\mu^\alpha = g^\alpha_\beta \, d\,\mu^\beta = \left(-\,h^\alpha_\beta + \frac{\lambda^\alpha \lambda_\beta}{G_{00}} \right) d\,\mu^\beta = -\,h^\alpha_\beta \, d\,\mu^\beta - \frac{\lambda^\alpha}{G_{00}} \mu^\beta \, d\,\lambda_\beta \,. \tag{20}$$

With similar passages, from $(8)_2$ we obtain

$$d\,\nu^\alpha = -\,h^\alpha_\beta \, d\,\nu^\beta - \frac{\lambda^\alpha}{G_{00}} \nu^\beta \, d\,\lambda_\beta \,. \tag{21}$$

Substituting (7) into $(6)_1$ and using (20), (21), we get

$$\frac{\partial h_0}{\partial \lambda} \lambda^\alpha \, d\,\lambda + \frac{\partial h_0}{\partial \vartheta} \lambda^\alpha \, d\,\vartheta + \left(2 \frac{\partial h_0}{\partial G_{00}} \lambda^\alpha \lambda^\beta + h_0 \, g^{\alpha\beta} - \lambda^\alpha \frac{\partial h_0}{\partial \nu_\gamma} \frac{\lambda_\gamma}{G_{00}} \nu^\beta - \lambda^\alpha \frac{\partial h_0}{\partial \mu_\gamma} \frac{\lambda_\gamma}{G_{00}} \mu^\beta - \right.$$
$$\left. \eta^{\alpha\theta\gamma\delta} \frac{h^\beta_\theta}{\sqrt{G_{00}}} \nu_\gamma \mu_\delta \right) d\,\lambda_\beta + \left(\eta^{\alpha\theta\gamma\delta} \frac{\lambda_\theta}{\sqrt{G_{00}}} \mu_\delta - \lambda^\alpha \frac{\partial h_0}{\partial \nu_\gamma} \right) h^\beta_\gamma \, d\,\nu_\beta + \left(\eta^{\alpha\theta\psi\gamma} \frac{\lambda_\theta}{\sqrt{G_{00}}} \mu_\psi - \lambda^\alpha \frac{\partial h_0}{\partial \mu_\gamma} \right) h^\beta_\gamma \, d\,\mu_\beta =$$
$$= V^\alpha \, d\,\lambda + \left(T^{\alpha\beta} - F^{\alpha\gamma} \frac{\lambda_\gamma}{G_{00}} \frac{\lambda_\gamma}{G_{00}} \nu^\beta - G^{\alpha\gamma} \frac{\lambda_\gamma}{G_{00}} \frac{\lambda_\gamma}{G_{00}} \mu^\beta \right) d\,\lambda_\beta - F^{\alpha\gamma} \, h^\beta_\gamma \, d\,\nu_\beta - G^{\alpha\gamma} \, h^\beta_\gamma \, d\,\mu_\beta + J^\alpha \, d\,\vartheta \,.$$

This relation implies

$$\begin{aligned}
& V^\alpha = \frac{\partial h_0}{\partial \lambda} \lambda^\alpha \,, \quad J^\alpha = \frac{\partial h_0}{\partial \vartheta} \lambda^\alpha \,, \\
& T^{\alpha\beta} = F^{\alpha\gamma} \frac{\lambda_\gamma}{G_{00}} \nu^\beta + G^{\alpha\gamma} \frac{\lambda_\gamma}{G_{00}} \mu^\beta + 2 \frac{\partial h_0}{\partial G_{00}} \lambda^\alpha \lambda^\beta + h_0 \, g^{\alpha\beta} - \lambda^\alpha \frac{\partial h_0}{\partial \nu_\gamma} \frac{\lambda_\gamma}{G_{00}} \nu^\beta - \\
& \qquad \lambda^\alpha \frac{\partial h_0}{\partial \mu_\gamma} \frac{\lambda_\gamma}{G_{00}} \mu^\beta - \eta^{\alpha\theta\gamma\delta} \frac{h^\beta_\theta}{\sqrt{G_{00}}} \nu_\gamma \mu_\delta \,, \\
& F^{\alpha\gamma} \, h^\beta_\gamma = \eta^{\alpha\theta\beta\delta} \frac{\lambda_\theta}{\sqrt{G_{00}}} \mu_\delta + \lambda^\alpha \frac{\partial h_0}{\partial \nu_\gamma} h^\beta_\gamma \,, \; G^{\alpha\gamma} \, h^\beta_\gamma = \eta^{\alpha\theta\psi\beta} \frac{\lambda_\theta}{\sqrt{G_{00}}} \nu_\psi + \lambda^\alpha \frac{\partial h_0}{\partial \mu_\gamma} h^\beta_\gamma \,.
\end{aligned} \tag{22}$$

Now, we have

$$\lambda_\alpha F^{\alpha\gamma} \, h^\beta_\gamma = \lambda_\alpha F^{\alpha\gamma} \left(-\,g^\beta_\gamma + \frac{\lambda_\gamma \lambda^\beta}{G_{00}} \right) = -\,\lambda_\alpha F^{\alpha\beta} \,, \quad \text{and also} \quad \lambda_\alpha G^{\alpha\gamma} \, h^\beta_\gamma - \lambda_\alpha G^{\alpha\beta} \,.$$

So, by contracting $(22)_{4,5}$ with $\frac{\lambda_\alpha}{G_{00}}$, we get:

$$F^{\alpha\gamma} \frac{\lambda_\gamma}{G_{00}} = \frac{\partial h_0}{\partial \nu_\gamma} h^\alpha_\gamma = -\frac{\partial h_0}{\partial \nu_\alpha} \,, \quad G^{\alpha\gamma} \frac{\lambda_\gamma}{G_{00}} = \frac{\partial h_0}{\partial \mu_\gamma} h^\alpha_\gamma = -\frac{\partial h_0}{\partial \mu_\alpha} \,, \tag{23}$$

where in the second step we took into account that:

$$\frac{\partial h_0}{\partial \nu_\gamma} = \frac{\partial h_0}{\partial G_{12}} \mu^\gamma + 2 \frac{\partial h_0}{\partial G_{22}} \nu^\gamma \quad \to \quad \frac{\partial h_0}{\partial \nu_\gamma} \lambda_\gamma = 0 \,, \; \frac{\partial h_0}{\partial \nu_\gamma} h^\alpha_\gamma = -\frac{\partial h_0}{\partial \nu_\alpha} \,,$$

and similarly, $\frac{\partial h_0}{\partial \mu_\gamma} h^\alpha_\gamma = -\frac{\partial h_0}{\partial \mu_\alpha}$. Hence Equation (22) simplifies to:

$$V^\alpha = \frac{\partial h_0}{\partial \lambda}\lambda^\alpha\,, \quad J^\alpha = \frac{\partial h_0}{\partial \vartheta}\lambda^\alpha\,,$$
$$T^{\alpha\beta} = -\frac{\partial h_0}{\partial \nu_\alpha}\nu^\beta - \frac{\partial h_0}{\partial \mu_\alpha}\mu^\beta + 2\frac{\partial h_0}{\partial G_{00}}\lambda^\alpha\lambda^\beta + h_0\, g^{\alpha\beta} - \eta^{\alpha\theta\gamma\delta}\frac{h^\beta_\theta}{\sqrt{G_{00}}}\nu_\gamma\mu_\delta\,, \tag{24}$$
$$F^{\alpha\gamma}\, h^\beta_\gamma = \eta^{\alpha\theta\beta\delta}\frac{\lambda_\theta}{\sqrt{G_{00}}}\mu_\delta - \lambda^\alpha\frac{\partial h_0}{\partial \nu_\beta}\,,\; G^{\alpha\gamma}\, h^\beta_\gamma = \eta^{\alpha\theta\psi\beta}\frac{\lambda_\theta}{\sqrt{G_{00}}}\nu_\psi - \lambda^\alpha\frac{\partial h_0}{\partial \mu_\beta}\,.$$

Since Equation (24)$_{4,5}$ contracted with $\frac{\lambda_\alpha}{G_{00}}$ give (23), they can be replaced by their contractions with h^δ_α, that is:

$$h^\delta_\alpha F^{\alpha\gamma}\, h^\beta_\gamma = -\,\eta^{\delta\theta\beta\psi}\frac{\lambda_\theta}{\sqrt{G_{00}}}\mu_\delta\,,\; h^\delta_\alpha G^{\alpha\gamma}\, h^\beta_\gamma = -\eta^{\delta\theta\psi\beta}\frac{\lambda_\theta}{\sqrt{G_{00}}}\nu_\psi\,. \tag{25}$$

Now, Equations (23) and (25) fully determine $F^{\alpha\beta}$ and $G^{\alpha\beta}$, while V^α, J^α and $T^{\alpha\beta}$ are determined by (24)$_{1-3}$.

In particular, from (24)$_1$ we see that λ^α is parallel to $V^\alpha = m\, n\, U^\alpha$; therefore Equation (19)$_3$ now becomes $\lambda_\beta = 2\,\lambda_{II}\, U_\beta$ with $U_\beta \equiv \Gamma\,(c\,,\, v_i)$ which ensures that v_i is the 3-velocity of the fluid.

From Equation (24)$_2$ we see that $J^\alpha = q\, U^\alpha$ with $q = \frac{\partial}{\partial\vartheta}\left(\frac{\sqrt{G_{00}}}{c}\, h_0\right)$.

The Equation (25) can be contracted with $\eta_{\phi\epsilon\delta\beta}\,\frac{U^\epsilon}{c}$ and give the above reported (17), where the property $\eta_{\phi\epsilon\delta\beta}\,\eta^{\delta\theta\beta\psi}\frac{U_\theta}{c}\frac{U^\epsilon}{c} = -2\, h^{[\delta}_\phi h^{\psi]}_\delta = -2\, h^\psi_\phi$ was used. The result shows the physical meaning of the Lagrange multipliers μ_ϕ and ν_ϕ by relating them to $F^{\alpha\gamma}$, $G^{\alpha\gamma}$ and U^α.

In particular from (17), by using (2) and $U_\beta \equiv \Gamma\,(c\,,\, v_i)$, we obtain $\vec{\mu} = \Gamma\left(\vec{H} - \vec{v}\wedge\vec{D}\right)$ and $\vec{\nu} = \Gamma\left(\vec{E} + \vec{v}\wedge\vec{B}\right)$. Together with $\mu_\phi U^\phi = 0$, $\nu_\phi U^\phi = 0$, we thus obtain that ν_ϕ is the 4-force acting on a unit charge and μ_ϕ can be considered its dual.

4. An Extended Set of Field Equations with the Symmetric Hyperbolic Form

In the non-relativistic approach [21] we were able to find a set of field equations with the symmetric hyperbolic form; this was possible because we separated the differential constraints from (1)$_{3-5}$ and used them in this framework. In the current relativistic approach this is not possible without losing the manifest covariance. So we adopt a different strategy by considering an extended set of independent variables. Consequently, we will find the expressions (31)$_{3,4}$ for the tensors $F^{\alpha\gamma}$ and $G^{\alpha\gamma}$, which are certainly more elegant than (9)$_{4,5}$ and (17).

To this end, we define $G_{01} = \lambda^\alpha\,\nu_\alpha$, $G_{02} = \lambda^\alpha\,\mu_\alpha$. In other words we leave out the constraints (8) and we will see that considering them only as constraints on the initial manifold, then they will be satisfied even outside it simply as a consequence of the field equations.

With this in mind, let us introduce four-vectors:

$$\begin{aligned} h'^\alpha &= h_0\lambda^\alpha + h_1\mu^\alpha + h_2\nu^\alpha + h_3\,\eta^{\alpha\phi\gamma\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}}\nu_\gamma\mu_\delta + G_{01}h'^\alpha_1 + G_{02}h'^\alpha_2\,, \\ \text{with}\quad h'^\alpha_1 &= \psi_0\lambda^\alpha + \psi_1\mu^\alpha + \psi_2\nu^\alpha\,,\quad h'^\alpha_2 = \theta_0\lambda^\alpha + \theta_1\mu^\alpha + \theta_2\nu^\alpha\,, \end{aligned} \tag{26}$$

where h_i, ψ_i, θ_i are functions of λ, ϑ, G_{00}, G_{01}, G_{02}, G_{11}, G_{12}, G_{22}. We look for these scalar coefficients and two additional ones X and Y such that:

$$\frac{\partial h'^{(\alpha}}{\partial \mu_{\beta)}} = Xg^{\alpha\beta} + G_{01}\frac{\partial h'^{(\alpha}_1}{\partial \mu_{\beta)}} + G_{02}\frac{\partial h'^{(\alpha}_2}{\partial \mu_{\beta)}}\,,\quad \frac{\partial h'^{(\alpha}}{\partial \nu_{\beta)}} = Yg^{\alpha\beta} + G_{01}\frac{\partial h'^{(\alpha}_1}{\partial \nu_{\beta)}} + G_{02}\frac{\partial h'^{(\alpha}_2}{\partial \nu_{\beta)}}\,.$$

In the reference frame where $\lambda^\alpha \equiv (\sqrt{G_{00}}\,, 0\,, 0\,, 0)$, $\mu^\alpha \equiv (\mu^0\,, \mu^1\,, 0\,, 0)$, $\nu^\alpha \equiv (\nu^0\,, \nu^1\,, \nu^2\,, 0)$, the components 33, 23, 13, 03 of the previous equations give:

$$X = h_1\,,\; Y = h_2\,,\; \frac{\partial\, h_3}{\partial\, G_{12}} = 0\,,\; \frac{\partial\, h_3}{\partial\, G_{22}} = 0\,,\; \frac{\partial\, h_3}{\partial\, G_{11}} = 0\,,\; \frac{\partial\, h_3}{\partial\, G_{01}} = 0\,,\; \frac{\partial\, h_3}{\partial\, G_{02}} = 0\,.$$

We see, in particular, that h_3 does not depend on μ_α and ν_α. From components 22, 12, 11 we obtain:

$$\frac{\partial\, h_2}{\partial\, G_{12}} = 0\,,\; \frac{\partial\, h_2}{\partial\, G_{22}} = 0\,,\; \frac{\partial\, h_1}{\partial\, G_{12}} + 2\,\frac{\partial\, h_2}{\partial\, G_{11}} = 0\,,\; \frac{\partial\, h_1}{\partial\, G_{22}} = 0\,,\; \frac{\partial\, h_1}{\partial\, G_{11}} = 0\,,\; \frac{\partial\, h_1}{\partial\, G_{12}} = 0\,. \tag{27}$$

From these results we see that h_1 and h_2 do not depend on G_{11}, G_{12} and G_{22}. Finally, components 00, 01, 02 give:

$$h_1'^{\alpha} = -\frac{\partial\, h_0}{\partial\, \mu_\alpha} - \frac{\partial\, h_1}{\partial\, G_{01}}\,\mu^\alpha - \frac{\partial\, h_2}{\partial\, G_{01}}\,\nu^\alpha\,, \quad h_2'^{\alpha} = -\frac{\partial\, h_0}{\partial\, \nu_\alpha} - \frac{\partial\, h_1}{\partial\, G_{02}}\,\mu^\alpha - \frac{\partial\, h_2}{\partial\, G_{02}}\,\nu^\alpha\,. \tag{28}$$

As a consequence of these results, we get:

$$\begin{aligned}
\frac{\partial\, h'^{\alpha}}{\partial\, \mu_\beta} &= 2\,\lambda^{[\alpha}\,\frac{\partial\, h_0}{\partial\, \mu_{\beta]}} + h_1 g^{\alpha\beta} + h_3\,\eta^{\alpha\phi\gamma\beta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\nu_\gamma + G_{01}\,\frac{\partial\, h_1'^{\alpha}}{\partial\, \mu_\beta} + G_{02}\,\frac{\partial\, h_2'^{\alpha}}{\partial\, \mu_\beta}\,,\\
\frac{\partial\, h'^{\alpha}}{\partial\, \nu_\beta} &= 2\,\lambda^{[\alpha}\,\frac{\partial\, h_0}{\partial\, \nu_{\beta]}} + h_2 g^{\alpha\beta} + h_3\,\eta^{\alpha\phi\beta\delta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\mu_\delta + G_{01}\,\frac{\partial\, h_1'^{\alpha}}{\partial\, \nu_\beta} + G_{02}\,\frac{\partial\, h_2'^{\alpha}}{\partial\, \nu_\beta}\,.
\end{aligned}$$

Now, we want that $\left(\frac{\partial\, h'^{\alpha}}{\partial\, \mu_\beta}\right)_{G_{00}=0\,, G_{02}=0} = F^{\alpha\beta}$ and $\left(\frac{\partial\, h'^{\alpha}}{\partial\, \nu_\beta}\right)_{G_{01}=0\,, G_{02}=0} = G^{\alpha\beta}$ which are skew-symmetric. This is only possible if $h_1 = 0$ and $h_2 = 0$. After that, (26) and (28) give:

$$h'^{\alpha} = h_0\lambda^\alpha + h_3\,\eta^{\alpha\phi\gamma\delta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\nu_\gamma\mu_\delta - G_{01}\,\frac{\partial\, h_0}{\partial\, \mu_\alpha} - G_{02}\,\frac{\partial\, h_0}{\partial\, \nu_\alpha}\,. \tag{29}$$

The function h_3 may depend on λ, ϑ and G_{00} but it is reasonable to simply assume that $h_3 = 1$. In this case (29), calculated in the physical case $G_{01} = 0$, $G_{02} = 0$ provides the above Equation (7).

The resulting field equations are $(1)_{1,2,5}$ with

$$\begin{aligned}
V^\alpha &= \frac{\partial\, h'^{\alpha}}{\partial\, \lambda} = \frac{\partial\, h_0}{\partial\, \lambda}\,\lambda^\alpha - G_{01}\,\frac{\partial^2\, h_0}{\partial\, \lambda\,\partial\, \mu_\alpha} - G_{02}\,\frac{\partial^2\, h_0}{\partial\, \lambda\,\partial\, \nu_\alpha}\,,\\
T^{\alpha\beta} &= \frac{\partial\, h'^{\alpha}}{\partial\, \lambda_\beta} = h_0 g^{\alpha\beta} + 2\,\frac{\partial\, h_0}{\partial\, G_{00}}\,\lambda^\alpha\lambda^\beta - \frac{1}{\sqrt{G_{00}}}\eta^{\alpha\phi\gamma\delta}h^\beta_\phi\nu_\gamma\mu_\delta - \mu^\beta\,\frac{\partial\, h_0}{\partial\, \mu_\alpha} - \nu^\beta\,\frac{\partial\, h_0}{\partial\, \nu_\alpha}\\
&\quad - 2\left(G_{01}\,\frac{\partial^2\, h_0}{\partial\, G_{00}\,\partial\, \mu_\alpha} + G_{02}\,\frac{\partial^2\, h_0}{\partial\, G_{00}\,\partial\, \nu_\alpha}\right)\lambda^\beta\,,\\
J^\alpha &= \frac{\partial\, h'^{\alpha}}{\partial\, \theta} = \frac{\partial\, h_0}{\partial\, \vartheta}\,\lambda^\alpha - G_{01}\,\frac{\partial^2\, h_0}{\partial\, \vartheta\,\partial\, \mu_\alpha} - G_{02}\,\frac{\partial^2\, h_0}{\partial\, \vartheta\,\partial\, \nu_\alpha}\,,
\end{aligned} \tag{30}$$

while $(1)_{3-4}$ are replaced by:

$$\begin{aligned}
\partial_\alpha\left(F^{\alpha\beta} - G_{01}\,\frac{\partial^2\, h_0}{\partial\, \mu_\alpha\,\partial\, \mu_\beta} - G_{02}\,\frac{\partial^2\, h_0}{\partial\, \nu_\alpha\,\partial\, \mu_\beta}\right) &= -J^\beta\,,\\
\partial_\alpha\left(G^{\alpha\beta} - G_{01}\,\frac{\partial^2\, h_0}{\partial\, \mu_\alpha\,\partial\, \nu_\beta} - G_{02}\,\frac{\partial^2\, h_0}{\partial\, \nu_\alpha\,\partial\, \nu_\beta}\right) &= 0\,,
\end{aligned} \tag{31}$$

$$\text{with} \quad F^{\alpha\beta} = 2\,\lambda^{[\alpha}\,\frac{\partial\,h_0}{\partial\,\mu_{\beta]}} + \eta^{\alpha\phi\gamma\beta}\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\nu_\gamma \quad , \quad G^{\alpha\beta} = 2\,\lambda^{[\alpha}\,\frac{\partial\,h_0}{\partial\,\nu_{\beta]}} + \eta^{\alpha\phi\beta\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\mu_\delta\,.$$

The last two of these equations are restrictions on the law linking the magnetic field in the empty space and the electric field in materials: Without the imposition of a supplementary conservation law, we would have that $F^{\alpha\beta}$ and $G^{\alpha\beta}$ are arbitrary skew-symmetric tensorial functions of ν_β and μ_β; here they are determined except for the scalar function h_0.

We now prove the above property, namely that $G_{01} = 0$ and $G_{02} = 0$ as long as they are null in the initial manifold. To this end, we consider (31) contracted with λ_β, that is,

$$\begin{aligned}
&-\frac{\partial^2\,h_0}{\partial\,\mu_\alpha\,\partial\,\mu_\beta}\lambda_\beta\,\partial_\alpha G_{01} - \frac{\partial^2\,h_0}{\partial\,\nu_\alpha\,\partial\,\mu_\beta}\lambda_\beta\,\partial_\alpha G_{02} = -\,\lambda_\beta\left(\partial_\alpha F^{\alpha\beta} + J^\beta\right) + \\
&\qquad + G_{01}\,\lambda_\beta\,\partial_\alpha\left(\frac{\partial^2\,h_0}{\partial\,\mu_\alpha\,\partial\,\mu_\beta}\right) + G_{02}\,\lambda_\beta\,\partial_\alpha\left(\frac{\partial^2\,h_0}{\partial\,\nu_\alpha\,\partial\,\mu_\beta}\right), \\
&-\frac{\partial^2\,h_0}{\partial\,\mu_\alpha\,\partial\,\nu_\beta}\lambda_\beta\,\partial_\alpha G_{01} - \frac{\partial^2\,h_0}{\partial\,\nu_\alpha\,\partial\,\nu_\beta}\lambda_\beta\,\partial_\alpha G_{02} = -\,\lambda_\beta\,\partial_\alpha G^{\alpha\beta} + G_{01}\,\lambda_\beta\,\partial_\alpha\left(\frac{\partial^2\,h_0}{\partial\,\mu_\alpha\,\partial\,\nu_\beta}\right) + \\
&\qquad + G_{02}\,\lambda_\beta\,\partial_\alpha\left(\frac{\partial^2\,h_0}{\partial\,\nu_\alpha\,\partial\,\nu_\beta}\right).
\end{aligned}$$

If we calculate here the coefficients of $\partial_\alpha G_{01}$, $\partial_\alpha G_{02}$ and the right-hand members in $G_{01} = 0$, $G_{02} = 0$, it becomes:

$$\begin{pmatrix} 2\,\frac{\partial\,h_0}{\partial\,G_{11}} & \frac{\partial\,h_0}{\partial\,G_{12}} \\ \frac{\partial\,h_0}{\partial\,G_{12}} & 2\,\frac{\partial\,h_0}{\partial\,G_{22}} \end{pmatrix} \begin{pmatrix} \lambda^\alpha\,\partial_\alpha G_{01} \\ \lambda^\alpha\,\partial_\alpha G_{02} \end{pmatrix} = \begin{pmatrix} \lambda_\beta\left(\partial_\alpha F^{\alpha\beta} + J^\beta\right) \\ \lambda_\beta\,\partial_\alpha G^{\alpha\beta} \end{pmatrix},$$

and we will demonstrate in Section 7 (as a consequence of the hyperbolicity requirement) that the coefficient matrix on the left has a positive determinant. From this fact it follows that, if $\bar{\vartheta}$, $\bar{\lambda}$, $\bar{\lambda}_\beta$, $\bar{\mu}_\beta$, $\bar{\nu}_\beta$, is the solution of the non-extended set (1), corresponding to the initial condition $\vartheta(0)$, $\lambda(0)$, $\lambda_\beta(0)$, $\mu_\beta(0)$, $\nu_\beta(0)$, then $\bar{\vartheta}$, $\bar{\lambda}$, $\bar{\lambda}_\beta$, $\bar{\mu}_\beta$, $\bar{\nu}_\beta$, $G_{01} = 0$, $G_{02} = 0$ is the solution of the extended set corresponding to the initial condition $\vartheta(0)$, $\lambda(0)$, $\lambda_\beta(0)$, $\mu_\beta(0)$, $\nu_\beta(0)$, $G_{01}(0) = 0$, $G_{02}(0) = 0$ and this completes our proof.

5. Wave Speeds for the above Field Equations

We aim here to calculate the speeds of the propagation waves. The characteristic equations corresponding to (30) and (31) are the following:

$$\begin{aligned}
&\varphi_\alpha\,d\left[\frac{\partial\,h_0}{\partial\,\lambda}\,\lambda^\alpha - G_{01}\,\frac{\partial^2\,h_0}{\partial\,\lambda\,\partial\,\mu_\alpha} - G_{02}\,\frac{\partial^2\,h_0}{\partial\,\lambda\,\partial\,\nu_\alpha}\right] = 0\,, \\
&\varphi_\alpha\,d\left[h_0\,g^{\alpha\beta} + 2\,\frac{\partial\,h_0}{\partial\,G_{00}}\,\lambda^\alpha\lambda^\beta - \frac{1}{\sqrt{G_{00}}}\eta^{\alpha\phi\gamma\delta}h^\beta_\phi\nu_\gamma\mu_\delta - \mu^\beta\,\frac{\partial\,h_0}{\partial\,\mu_\alpha} - \nu^\beta\,\frac{\partial\,h_0}{\partial\,\nu_\alpha}\right. \\
&\qquad\left. -\,2\left(G_{01}\,\frac{\partial^2\,h_0}{\partial\,G_{00}\,\partial\,\mu_\alpha} + G_{02}\,\frac{\partial^2\,h_0}{\partial\,G_{00}\,\partial\,\nu_\alpha}\right)\lambda^\beta\right] = 0\,, \\
&\varphi_\alpha\,d\left[\frac{\partial\,h_0}{\partial\,\vartheta}\,\lambda^\alpha - G_{01}\,\frac{\partial^2\,h_0}{\partial\,\vartheta\,\partial\,\mu_\alpha} - G_{02}\,\frac{\partial^2\,h_0}{\partial\,\vartheta\,\partial\,\nu_\alpha}\right] = 0\,, \\
&\varphi_\alpha\,d\left[2\,\lambda^{[\alpha}\,\frac{\partial\,h_0}{\partial\,\mu_{\beta]}} + \eta^{\alpha\phi\gamma\beta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\nu_\gamma - G_{01}\,\frac{\partial^2\,h_0}{\partial\,\mu_\alpha\,\partial\,\mu_\beta} - G_{02}\,\frac{\partial^2\,h_0}{\partial\,\nu_\alpha\,\partial\,\mu_\beta}\right] = 0\,, \\
&\varphi_\alpha\,d\left[2\,\lambda^{[\alpha}\,\frac{\partial\,h_0}{\partial\,\nu_{\beta]}} + \eta^{\alpha\phi\beta\delta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\mu_\delta - G_{01}\,\frac{\partial^2\,h_0}{\partial\,\mu_\alpha\,\partial\,\nu_\beta} - G_{02}\,\frac{\partial^2\,h_0}{\partial\,\nu_\alpha\,\partial\,\nu_\beta}\right] = 0\,,
\end{aligned} \tag{32}$$

$$\text{with}\quad \varphi_\alpha = n_\alpha - \frac{\mu}{c}\,\xi_\alpha\,,\quad \xi^\alpha \xi_\alpha = 1\,,\quad n^\alpha n_\alpha = -1\,,\quad \xi^\alpha n_\alpha = 0\,,$$

and the Eigenvalues μ corresponding to the Eigenvectors are the characteristic velocities.

Since in the physical case we have $G_{01} = 0$, $G_{02} = 0$, it is not restrictive to calculate the coefficients of the differentials in $G_{01} = 0$, $G_{02} = 0$; we will do this in the subsequent calculations, even without explicitly saying it.

First of all, we note that an Eigenvalue is:

$$\frac{\mu}{c} = \frac{n^\alpha \lambda_\alpha}{\xi^\alpha \lambda_\alpha}\,,\quad \text{i.e.,}\quad \varphi^\alpha \lambda_\alpha = 0\,. \tag{33}$$

In fact, for every pair of values $d\,\lambda$, $d\,\vartheta$ constrained only by:

$$\frac{\partial\, h_0}{\partial\, \lambda}\, d\,\lambda + \frac{\partial\, h_0}{\partial\, \vartheta}\, d\,\vartheta = 0\,,$$

the derivatives of this relation also hold with respect to λ, G_{00}, G_{11}, G_{12}, G_{22}, ϑ; this fact makes it easy to verify that $d\,\lambda$, $d\,\vartheta$, $d\,\lambda^\alpha = 0$, $d\,\mu^\alpha = 0$, $d\,\nu^\alpha = 0$ is an Eigenvector of the system (32) corresponding to the Eigenvalue (33). This Eigenvalue has at least multiplicity 1. In particular cases its multiplicity can be greater than 1. For example,

If $\varphi_\alpha \mu^\alpha = 0\,,\ \varphi_\alpha \nu^\alpha = 0\,,\ \eta^{\alpha\phi\gamma\delta}\varphi_\alpha \lambda_\phi \mu_\gamma \nu_\delta \neq 0\,,$ therefore, for any value of $d\,\lambda\,,\ d\,\vartheta\,,\ X$ constrained only by $\displaystyle \frac{\partial\, h_0}{\partial\, \lambda}\, d\,\lambda + \frac{\partial\, h_0}{\partial\, \vartheta}\, d\,\vartheta + 2\,X\,G_{00}\,\frac{\partial\, h_0}{\partial\, G_{00}} = 0\,,$

we get an Eigenvector with $d\,\lambda^\alpha = X\,\lambda^\alpha$, $d\,\mu^\alpha = 0$, $d\,\nu^\alpha = 0$. So in this case the Eigenvalue $\varphi^\alpha \lambda_\alpha = 0$ has multiplicity 2.

We note that this Eigenvalue is present also without the electromagnetic field (and, consequently, also without the variable ϑ); in fact, in this case, we have only the Equation $(32)_{1,2}$ which now reduce to:

$$\frac{\partial\, h_0}{\partial\, \lambda}\,\varphi_\alpha\, d\,\lambda^\alpha = 0\,,\quad \varphi^\beta\left(\frac{\partial\, h_0}{\partial\, \lambda}\, d\,\lambda + 2\,\frac{\partial\, h_0}{\partial\, G_{00}}\,\lambda^\gamma\, d\,\lambda_\gamma\right) = 0\,,$$

because $\varphi^\alpha \lambda_\alpha = 0$. So, now we have the 5 unknowns subject only to the two conditions $\varphi_\alpha\, d\,\lambda^\alpha = 0$, $\frac{\partial\, h_0}{\partial\, \lambda}\, d\,\lambda + 2\,\frac{\partial\, h_0}{\partial\, G_{00}}\,\lambda^\gamma\, d\,\lambda_\gamma = 0$. It follows that the Eigenvalue (33) has multiplicity 3 in this case.

We note also that, in the reference frame where $\xi^\alpha \equiv (1\,,0\,,0\,,0)$ and with the decomposition $\lambda^\alpha = \sqrt{G_{00}}\,\Gamma(v)\left(1\,,\frac{v^i}{c}\right)$, the Eigenvalue (33) becomes $\mu = \vec{v}\cdot\vec{n}$, as in the classical case [21].

For the research of other wave velocities, it is preferred for simplicity to consider the particular case

$$h_0 = h_0^*(\lambda\,,\vartheta\,,G_{00}) + \frac{c}{2\sqrt{G_{00}}}\,(\mu_0\,G_{11} + \epsilon_0\,G_{22})\,, \tag{34}$$

with μ_0 and ϵ_0 constants. This case is important because, by executing its non relativistic limit as in Section 2, we obtain that the classical expression of h' which corresponds to it is equal to that in Equation (29) of [21] with $h_3 = 1$, $h^* = \lim_{c\to+\infty}\frac{\sqrt{G_{00}}}{c}\,h_0^*$. So we can recognize that (34) is the expression of h_0 in an homogeneous and isotropic media. With this expression, Equation (32) becomes:

$$
\begin{aligned}
&\varphi_\alpha\, d\left(\frac{\partial\, h_0^*}{\partial\, \lambda}\, \lambda^\alpha\right) = 0\,, \quad \varphi_\alpha\, d\left(\frac{\partial\, h_0^*}{\partial\, \vartheta}\, \lambda^\alpha\right) = 0\,, \\
&\varphi^\beta\, d\left[h_0^* + \frac{c}{2\sqrt{G_{00}}}\,(\mu_0\, G_{11} + \epsilon_0\, G_{22})\right] + \\
&+ \varphi_\alpha\, d\left[2\,\frac{\partial\, h_0^*}{\partial\, G_{00}}\,\lambda^\alpha\lambda^\beta - \frac{c}{2\sqrt{G_{00}}}\,(\mu_0\, G_{11} + \epsilon_0\, G_{22})\,\frac{\lambda^\alpha\lambda^\beta}{G_{00}} - \frac{1}{\sqrt{G_{00}}}\eta^{\alpha\phi\gamma\delta}h_\phi^\beta\nu_\gamma\mu_\delta\right. \\
&\qquad \left. - \frac{c}{\sqrt{G_{00}}}\left(\mu_0\,\mu^\alpha\mu^\beta + \epsilon_0\,\nu^\alpha\nu^\beta\right)\right] + \frac{c}{\sqrt{G_{00}}}\,\frac{\lambda^\beta}{G_{00}}(\mu_0\,\varphi_\alpha\mu^\alpha\, d\, G_{01} + \epsilon_0\,\varphi_\alpha\nu^\alpha\, d\, G_{02}) = 0\,, \\
&\varphi_\alpha\, d\left[\frac{2\, c\,\mu_0}{\sqrt{G_{00}}}\,\lambda^{[\alpha}\,\mu^{\beta]} + \eta^{\alpha\phi\gamma\beta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\nu_\gamma\right] - \frac{c\,\mu_0}{\sqrt{G_{00}}}\,\varphi^\beta\, d\, G_{01} = 0\,, \\
&\varphi_\alpha\, d\left[\frac{2\, c\,\epsilon_0}{\sqrt{G_{00}}}\,\lambda^{[\alpha}\,\nu^{\beta]} + \eta^{\alpha\phi\beta\delta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\mu_\delta\right] - \frac{c\,\epsilon_0}{\sqrt{G_{00}}}\,\varphi^\beta\, d\, G_{02} = 0\,.
\end{aligned}
\tag{35}
$$

Returning to the Eigenvalue $\varphi^\alpha\lambda_\alpha = 0$, we now see that $(35)_{1,2}$ are equivalent to $\varphi_\alpha\, d\,\lambda^\alpha = 0$, $(35)_{4,5}$ contracted by φ_β give $d\, G_{01} = 0, d\, G_{02} = 0$ (It is not possible that $\varphi^\beta\varphi_\beta = 0$, otherwise we would have $n_\alpha = \frac{\mu}{c}\,\xi_\alpha$ followed by $-1 = \left(\frac{\mu}{c}\right)^2$). After that, Equation $(35)_{4,5}$ contracted by λ_β give the expressions of $\varphi_\alpha\, d\,\mu^\alpha$ and of $\varphi_\alpha\, d\,\nu^\alpha$, respectively. The same equations, contracted by h_β^θ give the expressions of $h_\delta^\gamma\, d\,\nu_\gamma$ and of $h_\delta^\gamma\, d\,\mu_\gamma$, respectively. Using also the result $d\, G_{01} = 0, d\, G_{02} = 0$, we obtain the following expressions:

$$
\begin{aligned}
d\,\mu^\beta &= \left\{\left(\varphi_\theta\varphi^\theta\right)^{-1}\left[\frac{2\, c\,\epsilon_0}{G_{00}}\,(\varphi_\epsilon\nu^\epsilon)\eta^{\mu\delta\beta\gamma}\lambda_\delta\varphi_\mu - \frac{1}{2\, c\,\mu_0 G_{00}}\,\varphi_\epsilon\eta^{\epsilon\delta\mu\gamma}\lambda_\delta\nu_\mu\varphi^\beta\right] - \frac{\lambda^\beta}{G_{00}}\,\mu^\gamma\right\} d\,\lambda_\gamma\,, \\
d\,\nu^\beta &= \left\{\left(\varphi_\theta\varphi^\theta\right)^{-1}\left[\frac{-2\, c\,\mu_0}{G_{00}}\,(\varphi_\epsilon\mu^\epsilon)\eta^{\psi\delta\beta\gamma}\lambda_\delta\varphi_\psi + \frac{1}{2\, c\,\epsilon_0 G_{00}}\,\varphi_\epsilon\eta^{\epsilon\delta\psi\gamma}\lambda_\delta\mu_\psi\varphi^\beta\right] - \frac{\lambda^\beta}{G_{00}}\,\nu^\gamma\right\} d\,\lambda_\gamma\,.
\end{aligned}
$$

In the calculations we have used the identies reported here in the Appendix A. It now remains to replace these partial results in $(35)_3$, which now reduces to:

$$
\varphi^\beta\, d\left[h_0^* + c\,\frac{\mu_0\, G_{11} + \epsilon_0\, G_{22}}{2\sqrt{G_{00}}}\right] - \varphi_\alpha\, d\left[\frac{1}{\sqrt{G_{00}}}\eta^{\alpha\phi\gamma\delta}h_\phi^\beta\nu_\gamma\mu_\delta + \frac{c}{\sqrt{G_{00}}}\left(\mu_0\,\mu^\alpha\mu^\beta + \epsilon_0\,\nu^\alpha\nu^\beta\right)\right] = 0\,.
$$

This is equivalent to its contractions with λ_β, φ_β and with the tensor $h_\beta^\theta + \frac{\varphi_\beta\varphi^\theta}{\varphi_\psi\varphi^\psi}$, that is

$$
\varphi_\alpha\,\frac{c}{\sqrt{G_{00}}}\left(\mu_0\,\mu^\alpha\mu^\beta + \epsilon_0\,\nu^\alpha\nu^\beta\right) d\,\lambda_\beta = 0\,, \tag{36}
$$

$$
\begin{aligned}
&d\left[h_0^* + c\,\frac{\mu_0\, G_{11} + \epsilon_0\, G_{22}}{2\sqrt{G_{00}}}\right] - \frac{\varphi_\alpha\varphi_\beta}{\varphi_\psi\varphi^\psi}\, d\left[\frac{1}{\sqrt{G_{00}}}\eta^{\alpha\phi\gamma\delta}h_\phi^\beta\nu_\gamma\mu_\delta + \frac{c}{\sqrt{G_{00}}}\left(\mu_0\,\mu^\alpha\mu^\beta + \epsilon_0\,\nu^\alpha\nu^\beta\right)\right] = 0\,, \\
&\left(h_\beta^\theta + \frac{\varphi_\beta\varphi^\theta}{\varphi_\psi\varphi^\psi}\right)\varphi_\alpha\, d\left[\frac{1}{\sqrt{G_{00}}}\eta^{\alpha\phi\gamma\delta}h_\phi^\beta\nu_\gamma\mu_\delta + \frac{c}{\sqrt{G_{00}}}\left(\mu_0\,\mu^\alpha\mu^\beta + \epsilon_0\,\nu^\alpha\nu^\beta\right)\right] = 0\,.
\end{aligned}
$$

We have taken into account here that $\varphi_\alpha\,\eta^{\alpha\phi\gamma\delta}h_\phi^\beta\nu_\gamma\mu_\delta = 0$ because in the reference frame Σ with $\frac{\lambda_\beta}{\sqrt{G_{00}}} \equiv (1\,,0\,,0\,,0)$, $\varphi_\alpha \equiv (0\,,\varphi_1\,,0\,,0)$ all indices of $\eta^{\alpha\phi\gamma\delta}$ are different from 0. By calculating all the differentials in $(36)_3$ and, after that, by substituting there the previous expressions of $d\,\mu^\beta, d\,\nu^\beta$, it becomes

$$
-\left(h_\beta^\theta + \frac{\varphi_\beta\varphi^\theta}{\varphi_\psi\varphi^\psi}\right)\varphi_\alpha\,\frac{c}{G_{00}}\left(\mu_0\,\mu^\alpha\mu^\beta + \epsilon_0\,\nu^\alpha\nu^\beta\right)\frac{\lambda^\gamma}{\sqrt{G_{00}}}\, d\,\lambda_\gamma + \frac{1}{2\, G_{00}}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,\varphi_\alpha\,\eta^{\phi\alpha\psi\beta}\nu_\psi\mu_\beta h^{\theta\gamma} d\,\lambda_\gamma = 0\,. \tag{37}
$$

It is easier to demonstrate the equivalence of $(36)_3$ and (37) in the above mentioned reference frame Σ. The conclusions of these calculations are as follows:

- If $\eta^{\phi\alpha\psi\beta}\lambda_\phi\varphi_\alpha\nu_\psi\mu_\beta \neq 0$, $(\varphi_\alpha\mu^\alpha)^2 + (\varphi_\alpha\mu^\alpha)^2 \neq 0$, hence the Eigenvalue $\varphi_\alpha\lambda^\alpha = 0$ has multiplicity 1. Indeed, we can infer $h^{\theta\gamma}d\,\lambda_\gamma$ from (37) and replace in $(36)_1$ which now becomes

$$
\begin{aligned}
&-2c^2\left(\eta^{\phi\alpha\psi\beta}\lambda_\phi\varphi_\alpha\nu_\psi\mu_\beta\right)^{-1} S\,\frac{\lambda^\gamma}{\sqrt{G_{00}}}d\,\lambda_\gamma = 0 \quad \text{with} \\
&S = \left(\mu_0\,\varphi_\alpha\mu^\alpha\mu^\beta + \epsilon_0\,\varphi_\alpha\nu^\alpha\nu^\beta\right)\left(h_{\beta\theta} + \frac{\varphi_\beta\varphi_\theta}{\varphi_\psi\varphi^\psi}\right)\left(\mu_0\,\varphi_{\alpha'}\mu^{\alpha'}\mu^\theta + \epsilon_0\,\varphi_{\alpha'}\nu^{\alpha'}\nu^\theta\right).
\end{aligned}
\tag{38}
$$

  We now have $S \neq 0$, otherwise in the above frame Σ we would have $\mu_0\,\varphi_\alpha\mu^\alpha\mu^2 + \epsilon_0\,\varphi_\alpha\nu^\alpha\nu^2 = 0$, $\mu_0\,\varphi_\alpha\mu^\alpha\mu^3 + \epsilon_0\,\varphi_\alpha\nu^\alpha\nu^3 = 0$ which is a system in the 2 unknowns $\mu_0\,\varphi_\alpha\mu^\alpha$ and $\epsilon_0\,\varphi_\alpha\nu^\alpha$ whose determinant of the coefficients is $\mu^2\nu^3 - \mu^3\nu^2 = -\left(\varphi_1\sqrt{G_{00}}\right)^{-1}$ $\eta^{\phi\alpha\psi\beta}\lambda_\phi\varphi_\alpha\nu_\psi\mu_\beta \neq 0$. Then the system would give $\varphi_\alpha\mu^\alpha = 0$ and $\varphi_\alpha\nu^\alpha = 0$ against the hypothesis. So our equation gives $\frac{\lambda^\gamma}{\sqrt{G_{00}}}d\,\lambda_\gamma = 0$ which, replaced in (37) gives $d\,\lambda_\gamma = 0$. So there remain the free unknowns $d\,\lambda$, $d\,\vartheta$ constrained by $(36)_2$.
- If $\eta^{\phi\alpha\psi\beta}\lambda_\phi\varphi_\alpha\nu_\psi\mu_\beta \neq 0$, $\varphi_\alpha\mu^\alpha = 0$, $\varphi_\alpha\mu^\alpha = 0$, then the Eigenvalue $\varphi_\alpha\lambda^\alpha = 0$ has multiplicity 2. Indeed, we can repeat the the previous steps and get (38). However, now $S = 0$ so that there remain the free unknowns $d\,\lambda$, $d\,\vartheta$, $\lambda^\gamma\,d\,\lambda_\gamma$ constrained only by $(36)_2$.
- If $\eta^{\phi\alpha\psi\beta}\lambda_\phi\varphi_\alpha\nu_\psi\mu_\beta = 0$, $\left(h^\theta_\beta + \frac{\varphi_\beta\varphi^\theta}{\varphi_\psi\varphi^\psi}\right)\left(\mu_0\,\varphi_\alpha\mu^\alpha\mu^\beta + \epsilon_0\,\varphi_\alpha\nu^\alpha\nu^\beta\right) \neq 0$, then the Eigenvalue $\varphi_\alpha\lambda^\alpha = 0$ has multiplicity 2. Indeed, in this case (37) returns $\lambda^\gamma\,d\,\lambda_\gamma = 0$; then the 6 free unknowns remain $d\,\lambda$, $d\,\vartheta$, $d\,\lambda_\gamma$ constrained only by the scalar conditions $\lambda^\gamma\,d\,\lambda_\gamma = 0$, $\varphi^\gamma\,d\,\lambda_\gamma = 0$, $(36)_{1,2}$.
- If $\eta^{\phi\alpha\psi\beta}\lambda_\phi\varphi_\alpha\nu_\psi\mu_\beta = 0$, $\left(h^\theta_\beta + \frac{\varphi_\beta\varphi^\theta}{\varphi_\psi\varphi^\psi}\right)\left(\mu_0\,\varphi_\alpha\mu^\alpha\mu^\beta + \epsilon_0\,\varphi_\alpha\nu^\alpha\nu^\beta\right) = 0$, hence the Eigenvalue $\varphi_\alpha\lambda^\alpha = 0$ has multiplicity 4. Indeed, in this case (37) and $(36)_1$ are identities; then the 6 free unknowns remain $d\,\lambda$, $d\,\vartheta$, $d\,\lambda_\gamma$ costrained only by the scalar conditions $\varphi^\gamma\,d\,\lambda_\gamma = 0$ and $(36)_2$. We note that this is the situation if the electromagnetic field is not present, except that we do not have the free unknown $d\,\vartheta$ so that the multiplicity is 3.

For other Eigenvalues, we first note that $h^{\alpha\beta}\varphi_\beta \neq 0$, otherwise we would have $\varphi_\alpha = \frac{\lambda^\alpha}{G_{00}}\lambda^\beta\varphi_\beta$ from which it follows $-1 + \frac{\mu^2}{c^2} = \varphi_\alpha\varphi^\alpha = \frac{1}{G_{00}}\left(\lambda^\beta\varphi_\beta\right)^2 > 0$ against the fact that $\mu^2 \leq c^2$. This fact allows us to define

$$
H^{\alpha\beta} = h^{\alpha\beta} - \frac{(h^{\alpha\gamma}\varphi_\gamma)(h^{\beta\delta}\varphi_\delta)}{h^{\mu\nu}\varphi_\mu\varphi_\nu},
$$

which is the projector into the 2-dimensional subspace orthogonal to λ_α and to φ_α. After that, any equation $X^\beta = 0$ is equivalent to the system $\lambda_\beta X^\beta = 0$, $\varphi_\beta X^\beta = 0$, $H_{\alpha\beta}X^\beta = 0$.

By contracting $(35)_{4,5}$ with λ_β, φ_β, $H_{\theta\beta}$, they become:

$$\begin{aligned}
&\varphi_\alpha d\,\mu^\alpha = \left(\frac{-1}{c\,\mu_0\,G_{00}}\eta^{\alpha\phi\gamma\beta}\varphi_\alpha\,h^\delta_\phi\,\nu_\gamma\,\lambda_\beta - \frac{\lambda_\beta\varphi^\beta}{G_{00}}\,\mu^\delta\right)d\,\lambda_\delta\,, \quad \varphi_\beta\varphi^\beta\,d\,G_{01} = 0\,,\\
&H_{\alpha\beta}\,\varphi_\alpha\left[\frac{-c\,\mu_0}{G_{00}\sqrt{G_{00}}}\lambda^\alpha\mu^\beta\lambda^\gamma\,d\,\lambda_\gamma + \frac{c\,\mu_0}{\sqrt{G_{00}}}\left(\lambda^\alpha\,d\,\mu^\beta + \mu^\beta\,d\,\lambda^\alpha - \mu^\alpha\,d\,\lambda^\beta\right) - \right.\\
&\qquad\left.\eta^{\alpha\phi\gamma\beta}h^\delta_\phi\,\frac{\nu_\gamma}{\sqrt{G_{00}}}\,d\,\lambda_\delta + \eta^{\alpha\phi\gamma\beta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,d\,\nu_\gamma\right] = 0\,,\\
&\varphi_\alpha d\,\nu^\alpha = \left(\frac{1}{c\,\epsilon_0\,G_{00}}\eta^{\alpha\phi\gamma\beta}\varphi_\alpha\,h^\delta_\phi\,\mu_\gamma\,\lambda_\beta - \frac{\lambda_\beta\varphi^\beta}{G_{00}}\,\mu^\delta\right)d\,\lambda_\delta\,, \quad \varphi_\beta\varphi^\beta\,d\,G_{02} = 0\,,\\
&H_{\alpha\beta}\,\varphi_\alpha\left[\frac{-c\,\epsilon_0}{G_{00}\sqrt{G_{00}}}\lambda^\alpha\nu^\beta\lambda^\gamma\,d\,\lambda_\gamma + \frac{c\,\epsilon_0}{\sqrt{G_{00}}}\left(\lambda^\alpha\,d\,\nu^\beta + \nu^\beta\,d\,\lambda^\alpha - \nu^\alpha\,d\,\lambda^\beta\right) + \right.\\
&\qquad\left.+\,\eta^{\alpha\phi\gamma\beta}h^\delta_\phi\,\frac{\mu_\gamma}{\sqrt{G_{00}}}\,d\,\lambda_\delta - \eta^{\alpha\phi\gamma\beta}\,\frac{\lambda_\phi}{\sqrt{G_{00}}}\,d\,\mu_\gamma\right] = 0\,.
\end{aligned} \tag{39}$$

By using the identity:

$$H^\theta_\beta\,\varphi_\alpha\,\eta^{\alpha\phi\gamma\beta}\lambda_\phi = \left[h^\theta_\beta - \frac{h^{\theta\delta}\varphi_\delta}{h^{\mu\nu}\varphi_\mu\varphi_\nu}\left(-g_{\beta\psi} + \frac{\lambda_\beta\lambda_\psi}{G_{00}}\right)\varphi^\psi\right]\varphi_\alpha\,\eta^{\alpha\phi\gamma\beta}\lambda_\phi = -\varphi_\alpha\,\eta^{\alpha\phi\gamma\theta}\lambda_\phi\,,$$

from Equation (39)$_{3,6}$ we desume:

$$\begin{aligned}
&H^\theta_\beta\,d\,\mu^\beta = \left(\varphi_\psi\lambda^\psi\right)^{-1}\left[h^{\mu\nu}\varphi_\mu\,d\,\lambda_\nu H^\theta_\beta\,\mu^\beta + \varphi_\alpha\mu^\alpha H^\theta_\beta\,d\,\lambda^\beta + \frac{1}{c\,\mu_0}\eta^{\alpha\phi\gamma\theta}\varphi_\alpha\,\lambda_\phi\,d\,\nu_\gamma\right] -\\
&\qquad\frac{1}{c\,\mu_0\,G_{00}}H^\theta_\beta\eta^{\alpha\delta\gamma\beta}\lambda_\alpha\,\nu_\gamma\,d\,\lambda_\delta\,,\\
&H^\theta_\beta\,d\,\nu^\beta = \left(\varphi_\psi\lambda^\psi\right)^{-1}\left[h^{\mu\nu}\varphi_\mu\,d\,\lambda_\nu H^\theta_\beta\,\nu^\beta + \varphi_\alpha\nu^\alpha H^\theta_\beta\,d\,\lambda^\beta - \frac{1}{c\,\epsilon_0}\eta^{\alpha\phi\gamma\theta}\varphi_\alpha\,\lambda_\phi\,d\,\mu_\gamma\right] +\\
&\qquad+\frac{1}{c\,\epsilon_0\,G_{00}}H^\theta_\beta\eta^{\alpha\delta\gamma\beta}\lambda_\alpha\,\mu_\gamma\,d\,\lambda_\delta\,.
\end{aligned} \tag{40}$$

Now in (40)$_2$ the term $\eta^{\alpha\phi\gamma\theta}\varphi_\alpha\,\lambda_\phi\,d\,\mu_\gamma$ can be written as $-\,\eta^{\alpha\phi\gamma'\theta}\varphi_\alpha\,\lambda_\phi H^\gamma_{\gamma'}\,d\,\mu_\gamma$ and we can use $H^\gamma_{\gamma'}\,d\,\mu_\gamma$ from (40)$_1$; in this way (40)$_2$ becomes:

$$\begin{aligned}
&H^\theta_\beta\,d\,\nu^\beta\left(1 - \frac{h^{\mu\nu}\varphi_\mu\varphi_\nu}{\left(\varphi_\psi\lambda^\psi\right)^2}\,\frac{G_{00}}{c^2\epsilon_0\,\mu_0}\right) = \underline{\left(\varphi_\psi\lambda^\psi\right)^{-1}\left[\left(h^{\mu\nu}\varphi_\mu\,d\,\lambda_\nu\right)H^\theta_\beta\,\nu^\beta + \varphi_\alpha\nu^\alpha H^\theta_\beta\,d\,\lambda^\beta\right]} +\\
&+\frac{1}{c\,\epsilon_0\,G_{00}}H^\theta_\beta\eta^{\alpha\delta\gamma\beta}\lambda_\alpha\,\mu_\gamma\,d\,\lambda_\delta + \frac{1}{c\,\epsilon_0\,\left(\varphi_\psi\lambda^\psi\right)^2}\,\eta^{\alpha'\phi\gamma'\theta}\varphi_{\alpha'}\,\lambda_\phi\left[\left(h^{\mu\nu}\varphi_\mu\,d\,\lambda_\nu\right)H_{\gamma'\beta}\,\mu^\beta + \varphi_\alpha\mu^\alpha H_{\gamma'\beta}\,d\,\lambda^\beta\right] -\\
&\qquad\underline{\frac{1}{c^2\,\epsilon_0\,\mu_0\,G_{00}\left(\varphi_\psi\lambda^\psi\right)}\,\eta^{\alpha'\phi\gamma'\theta}\varphi_{\alpha'}\,\lambda_\phi H_{\gamma'\beta}\,\eta^{\alpha\delta\gamma\beta}\lambda_\alpha\,\nu_\gamma\,d\,\lambda_\delta}\,.
\end{aligned}$$

Here the underlined terms are equal to $\dfrac{-2}{\varphi_\psi\lambda^\psi}\left(1 - \dfrac{1}{c^2\,\epsilon_0\,\mu_0}\right)\nu_{[\gamma}\,d\,\lambda_{\mu]}\,h^{\gamma\delta}\,\varphi_\delta\,H^{\mu\theta}\,,$

and the remaining terms are equal to $\dfrac{-1}{c\,\epsilon_0}\,\dfrac{\varphi^\mu\,\varphi_\mu}{\left(\varphi_\psi\lambda^\psi\right)^2}\,\eta^{\alpha\phi\beta\gamma}\lambda_\alpha\,H^\theta_\phi\,\mu_\beta\,d\,\lambda_\gamma\,,$

as it can be seen more easily in the reference frame Σ where $\frac{\lambda_\beta}{\sqrt{G_{00}}} \equiv (1\,,0\,,0\,,0)$, $\varphi_\alpha \equiv (\varphi_0\,,\varphi_1\,,0\,,0)$. Using these properties, the above result can be written as:

$$H^\theta_\beta \, d\,\nu^\beta \left(1 - \frac{h^{\mu\nu}\varphi_\mu\varphi_\nu}{(\varphi_\psi\lambda^\psi)^2}\,\frac{G_{00}}{c^2\epsilon_0\,\mu_0}\right) = \\ = \frac{-2}{\varphi_\psi\lambda^\psi}\left(1 - \frac{1}{c^2\,\epsilon_0\,\mu_0}\right)\nu_{[\gamma}\,d\,\lambda_{\mu]}\,h^{\gamma\delta}\,\varphi_\delta\,H^{\mu\theta} - \frac{1}{c\,\epsilon_0}\,\frac{\varphi^\mu\,\varphi_\mu}{(\varphi_\psi\lambda^\psi)^2}\,\eta^{\alpha\phi\beta\gamma}\lambda_\alpha\,H^\theta_\phi\,\mu_\beta\,d\,\lambda_\gamma\,. \tag{41}$$

Now let us look for two coefficients X and Y and see if $d\,\mu^\alpha = X\,\varphi^\alpha$, $d\,\nu^\alpha = Y\,\varphi^\alpha$, $d\,\lambda = 0$, $d\,\vartheta = 0$, $d\,\lambda^\alpha = 0$ is a solution of the system in the case $\varphi^\alpha\varphi_\alpha = 0$. Substituting in (35) we obtain that they are identically satisfied. So X and Y remain free unknowns and we can say that $\varphi^\alpha\varphi_\alpha = 0$ gives Eigenvalues with multiplicity 2; these Eigenvalues are $\mu = \pm\,c$.

6. The Vlasov Equation

It is useful to compare some of the present results with those of refs. [22–25] which were obtained in the context of monoatomic gases. They considered the Vlasov Equation [22] multiplied by the rest particle mass, i.e.,

$$p^\alpha\partial_\alpha f - \frac{q}{2\,n\,c}\,\eta^{\alpha\beta\gamma\delta}G_{\gamma\delta}\;p_\alpha\frac{\partial\,f}{\partial\,p^\beta} = 0\,, \tag{42}$$

(We have only substituted $\frac{q}{n}$ for the electron charge and $-\frac{1}{2c}\,\eta^{\alpha\beta\gamma\delta}G_{\gamma\delta}$ to their $F^{\alpha\beta}$ because their article dealt with the effects of Maxwell's equations on matter but only as an external field; this fact allowed them to use Maxwell equations in the empty space where $F^{\alpha\beta}$ and $G^{\alpha\beta}$ are each the dual of the other; this is not true in the present more general context and we have to use the appropriate field). Now, for polyatomic gases (see [14,26]), the distribution function is:

$$f = e^{-1-\frac{1}{K_B}\left[m\,\lambda + \left(1+\frac{\mathcal{I}}{m\,c^2}\right)p^\mu\lambda_\mu\right]}\,. \tag{43}$$

However, (42) has been derived in the context of monoatomic gases where (43) reduces to $f = e^{-1-\frac{1}{K_B}\left[m\,\lambda + p^\mu\lambda_\mu\right]}$, so that (42) becomes:

$$p^\alpha\partial_\alpha f + f\,\frac{q}{2\,n\,c\,k_B}\,\lambda_\beta\,\eta^{\alpha\beta\gamma\delta}G_{\gamma\delta}\;p_\alpha = 0\,. \tag{44}$$

It is reasonable (as we will see later) to assume (44) also for polyatomic gases, but with f given by (43).

If we multiply (44) by $m\,c\,\varphi(\mathcal{I})$ and then integrate in $d\,\vec{P}\,d\,\mathcal{I}$, we get:

$$\partial_\alpha V^\alpha_M + \frac{q}{2\,n\,c\,k_B}\,\lambda_\beta\,\eta^{\alpha\beta\gamma\delta}G_{\gamma\delta}\,V_{M\alpha} = 0\,,\text{with} \quad V^\alpha_M = m\,c\int_{\Re^3}\int_0^{+\infty} f\,p^\alpha\,\varphi(\mathcal{I})\,d\,\vec{P}\,d\,\mathcal{I}\,.$$

However, λ_β is parallel to U_β ($\lambda_\beta = \frac{U_\beta}{T}$) so that this equation reduces to $\partial_\alpha V^\alpha_M = 0$, i.e., the usual mass conservation law.

If we multiply (44) by $p^\beta\,c\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)\varphi(\mathcal{I})$ and then integrate in $d\,\vec{P}\,d\,\mathcal{I}$, we obtain:

$$\partial_\alpha T^{\alpha\beta}_M = -\frac{q}{2\,n\,c\,k_B}\,\lambda_\mu\,\eta^{\alpha\mu\gamma\delta}G_{\gamma\delta}\,T^\beta_{M\alpha}\,,\text{with} \quad T^{\alpha\beta}_M = c\int_{\Re^3}\int_0^{+\infty} f\,p^\alpha\,p^\beta\left(1 + \frac{\mathcal{I}}{m\,c^2}\right)\varphi(\mathcal{I})\,d\,\vec{P}\,d\,\mathcal{I}\,, \tag{45}$$

as in [23–25]. The right hand side of Equation (45) is:

$$\frac{q\,p}{2\,n\,c\,k_B\,T}\,U_\mu\,\eta^{\beta\mu\gamma\delta}G_{\gamma\delta} = \frac{q}{2\,c}\,U_\mu\,\eta^{\beta\mu\gamma\delta}G_{\gamma\delta} = q\,\nu^\beta\,,$$

where in the last step we used (17)$_2$. So we found the right hand side of (1)$_2$. This confirms the above choice of Equation (44) together with (43). If we had chosen (42) together with (43), then the right-hand side of Equation (45) was $B_9\, q\, \nu^\beta$ with

$$B_9 = \frac{\int_0^{+\infty} J^*_{2,1} \left(1 + \frac{\mathcal{I}}{m\, c^2}\right) \varphi(\mathcal{I})\, d\,\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\, \varphi(\mathcal{I})\, d\,\mathcal{I}}\,.$$

In this case the right hand side of Equation (45) is not the Lorentz force, but only proportional to it through the coefficient B_9 which is 1 for monoatomic gases and also in the non relativistic limit of polyatomic gases.

7. The Hyperbolicity Requirement

In the previous sections we have seen how the balance equations consisting of the Euler Equations for the material and the Maxwell Equations in that material can be written in symmetrical form. To be sure that this set of equations is hyperbolic, it remains to be seen whether it also satisfies the convexity of h'^α with respect to its variables (see Section 1.2) Using the multi-index notation X_A to denote the Lagrange multipliers ϑ, λ, λ_β, μ_β, ν_β, this means that the quadratic form

$$Q = \lambda_\alpha \frac{\partial^2\, h'^\alpha}{\partial\, X_A\, \partial\, X_B}\, d\, X_A\, d\, X_B = \lambda_\alpha\, d \left(\frac{\partial\, h'^\alpha}{\partial\, X_A}\right) d\, X_A\,,$$

is negative definite in the variables $d\, X_A$. Let us impose this condition when $d\, X_A = 0$ except for $d\,\mu_\beta$, $d\,\nu_\beta$ and use (29). Moreover, since G_{01} and G_{02} have no physical meaning and were introduced here only as a mathematical tool to have a symmetric system of equations, we can assume without loss of generality that h_0 does not depend on G_{01} and G_{02} and, furthermore, that $G_{11} = h^{\alpha\beta}\mu_\alpha\mu_\beta$, $G_{12} = h^{\alpha\beta}\mu_\alpha\nu_\beta$, $G_{22} = h^{\alpha\beta}\nu_\alpha\nu_\beta$, with $h^{\alpha\beta} = -\, g^{\alpha\beta} + \frac{\lambda^\alpha\lambda^\beta}{G_{00}}$. The second term in the expression (29) of h'^α gives no contribution because it is orthogonal to λ_α and in the above expression of Q there is a contraction with λ_α. The first term in the expression (29) of h'^α is $h_0\lambda^\alpha$ and it gives to Q the contribution

$$G_{00}\left[d\left(\frac{\partial\, h_0}{\partial\,\mu_\beta}\right) d\,\mu_\beta + d\left(\frac{\partial\, h_0}{\partial\,\nu_\beta}\right) d\,\nu_\beta\right] =$$
$$= G_{00}\left[d\left(2\,\frac{\partial\, h_0}{\partial\, G_{11}}\,\mu_\gamma h^{\gamma\beta} + \frac{\partial\, h_0}{\partial\, G_{12}}\,\nu_\gamma h^{\gamma\beta}\right) d\,\mu_\beta + d\left(\frac{\partial\, h_0}{\partial\, G_{12}}\,\mu_\gamma h^{\gamma\beta} + 2\,\frac{\partial\, h_0}{\partial\, G_{22}}\,\nu_\gamma h^{\gamma\beta}\right) d\,\nu_\beta\right].$$

By performing the calculations in the reference frame where $\lambda^\alpha \equiv (\lambda^0\,, 0\,, 0\,, 0)$, $\mu^\alpha \equiv (\mu^0\,, \mu^1\,, 0\,, 0)$, $\nu^\alpha \equiv (\nu^0\,, \nu^1\,, \nu^2\,, 0)$, we obtain that this contribution becomes equal to $Q_1 + Q_2$ with

$$Q_1 = -\, G_{00}\left[2\,\frac{\partial\, h_0}{\partial\, G_{11}}(d\,\mu_3)^2 + 2\,\frac{\partial\, h_0}{\partial\, G_{12}}(d\,\mu_3)(d\,\nu_3) + 2\,\frac{\partial\, h_0}{\partial\, G_{22}}(d\,\nu_3)^2\right],$$

$$Q_2 = G_{00}\left[a_{11}\,(d\,\mu_1)^2 + 2a_{12}\, d\,\mu_1\, d\,\nu_1 + 2a_{13}\, d\,\mu_1\, d\,\mu_2 + 2a_{14}\, d\,\mu_1\, d\,\nu_2 + a_{22}\,(d\,\nu_1)^2 + \right.$$
$$\left. + 2a_{23}\, d\,\nu_1\, d\,\mu_2 + 2a_{24}\, d\,\nu_1\, d\,\nu_2 + a_{33}\,(d\,\mu_2)^2 + 2a_{34}\, d\,\mu_2\, d\,\nu_2 + a_{44}\,(d\,\nu_2)^2\right],$$

with

$$a_{11} = 4\,\frac{\partial^2\, h_0}{\partial (G_{11})^2}\,(\mu_1)^2 + 4\,\frac{\partial^2\, h_0}{\partial\, G_{11} \partial\, G_{12}}\,\mu^1\nu^1 + \frac{\partial^2\, h_0}{\partial (G_{12})^2}\,(\nu_1)^2 - 2\,\frac{\partial\, h_0}{\partial\, G_{11}}\,,$$

$$a_{12} = 2\,\frac{\partial^2 h_0}{\partial\, G_{11}\partial\, G_{12}}\,(\mu_1)^2 + 4\,\frac{\partial^2 h_0}{\partial\, G_{11}\partial\, G_{22}}\mu^1\nu^1 + \frac{\partial^2 h_0}{\partial(G_{12})^2}\mu^1\nu^1 + 2\,\frac{\partial^2 h_0}{\partial\, G_{22}\partial\, G_{12}}\,(\nu_1)^2 - \frac{\partial\, h_0}{\partial\, G_{12}}\,,$$

$$a_{13} = 2\,\frac{\partial^2 h_0}{\partial\, G_{11}\partial\, G_{12}}\,\mu^1\nu^2 + \frac{\partial^2 h_0}{\partial(G_{12})^2}\nu^1\nu^2\,, \quad a_{14} = 4\,\frac{\partial^2 h_0}{\partial\, G_{11}\partial\, G_{22}}\,\mu^1\nu^2 + 2\,\frac{\partial^2 h_0}{\partial G_{12}\partial G_{22}}\nu^1\nu^2\,,$$

$$a_{22} = 4\,\frac{\partial^2 h_0}{\partial(G_{22})^2}\,(\nu_1)^2 + 4\,\frac{\partial^2 h_0}{\partial\, G_{22}\partial\, G_{12}}\mu^1\nu^1 + \frac{\partial^2 h_0}{\partial(G_{12})^2}\,(\mu_1)^2 - 2\,\frac{\partial\, h_0}{\partial\, G_{22}}\,,$$

$$a_{23} = 2\,\frac{\partial^2 h_0}{\partial\, G_{22}\partial\, G_{12}}\,\nu^1\nu^2 + \frac{\partial^2 h_0}{\partial(G_{12})^2}\mu^1\nu^2\,, \quad a_{24} = 4\,\frac{\partial^2 h_0}{\partial(G_{22})^2}\,\nu^1\nu^2 + 2\,\frac{\partial^2 h_0}{\partial G_{12}\partial G_{22}}\mu^1\nu^2\,,$$

$$a_{33} = \frac{\partial^2 h_0}{\partial(G_{12})^2}\,(\nu_2)^2 - 2\,\frac{\partial\, h_0}{\partial\, G_{11}}\,, \quad a_{34} = 2\,\frac{\partial^2 h_0}{\partial\, G_{22}\partial\, G_{12}}\,(\nu_2)^2 - \frac{\partial\, h_0}{\partial\, G_{12}}\,,$$

$$a_{44} = 4\,\frac{\partial^2 h_0}{\partial(G_{22})^2}\,(\nu_2)^2 - 2\,\frac{\partial\, h_0}{\partial\, G_{22}}\,.$$

Finally, we compute the contribution to Q of the last two terms in the expression (29) of h'^{α}; it is

$$-d\left[\frac{\partial}{\partial\mu_\beta}\left(2(G_{01})^2\frac{\partial\, h_0}{\partial\, G_{11}} + G_{01}G_{02}\,\frac{\partial\, h_0}{\partial\, G_{12}}\right)\right]d\,\mu_\beta - d\left[\frac{\partial}{\partial\nu_\beta}\left(2(G_{01})^2\frac{\partial\, h_0}{\partial\, G_{11}} + G_{01}G_{02}\,\frac{\partial\, h_0}{\partial\, G_{12}}\right)\right]d\,\nu_\beta$$
$$-d\left[\frac{\partial}{\partial\mu_\beta}\left(2(G_{02})^2\frac{\partial\, h_0}{\partial\, G_{22}} + G_{01}G_{02}\,\frac{\partial\, h_0}{\partial\, G_{12}}\right)\right]d\,\mu_\beta - d\left[\frac{\partial}{\partial\nu_\beta}\left(2(G_{02})^2\frac{\partial\, h_0}{\partial\, G_{22}} + G_{01}G_{02}\,\frac{\partial\, h_0}{\partial\, G_{12}}\right)\right]d\,\nu_\beta\,.$$

Now, we want to calculate the coefficients of the differentials in $G_{01} = 0$, $G_{02} = 0$; then the terms of the expression above where $(G_{01})^2$, $G_{01}G_{02}$, $(G_{02})^2$ are not derivated with respect to μ_β or ν_β give zero contribution. Consequently, of the above quadratic form remains

$$-d\left(4\,G_{01}\,\lambda^\beta\frac{\partial\, h_0}{\partial\, G_{11}} + G_{02}\,\lambda^\beta\frac{\partial\, h_0}{\partial\, G_{12}}\right)d\,\mu_\beta - d\left(G_{01}\,\lambda^\beta\frac{\partial\, h_0}{\partial\, G_{12}}\right)d\,\nu_\beta$$
$$-d\left(G_{02}\,\lambda^\beta\frac{\partial\, h_0}{\partial\, G_{12}}\right)d\,\mu_\beta - d\left(4\,G_{02}\,\lambda^\beta\frac{\partial\, h_0}{\partial\, G_{22}} + G_{01}\,\lambda^\beta\frac{\partial\, h_0}{\partial\, G_{12}}\right)d\,\nu_\beta\,.$$

Here too the terms in which G_{01} and G_{02} are not differentiated give the zero contribution zero and, moreover, $\lambda^\beta\,d\,\mu_\beta = d\,G_{01}$, $\lambda^\beta\,d\,\nu_\beta = d\,G_{02}$. So the contribution to Q of the last two terms in the expression (29) of h'^{α} is

$$Q_3 = -\,4\,\frac{\partial\, h_0}{\partial\, G_{11}}(d\,G_{01})^2\, - 4\,\frac{\partial\, h_0}{\partial\, G_{22}}\,(d\,G_{02})^2\, - 4\,\frac{\partial\, h_0}{\partial\, G_{12}}\,d\,G_{01}\,d\,G_{02}\,,$$

and $Q = Q_1 + Q_2 + Q_3$. Since they depend on distinct variables, each of them must be negative defined. In particular, this is true for Q_1 if and only if

$$\frac{\partial\, h_0}{\partial\, G_{11}} > 0 \quad , \quad \begin{vmatrix} 2\,\frac{\partial\, h_0}{\partial\, G_{11}} & \frac{\partial\, h_0}{\partial\, G_{12}} \\ \frac{\partial\, h_0}{\partial\, G_{12}} & 2\,\frac{\partial\, h_0}{\partial\, G_{22}} \end{vmatrix} > 0\,, \tag{46}$$

and we have used the second of these properties in the previous sections.

We see that also Q_3 is negative defined as a consequence of (46). Consequently, our choice to use an extended set of independent variables did not imply further conditions.

As for Q_2, it is negative defined if the fourth-order matrix (a_{ij}) is negative defined. Although this condition is mathematically a bit complex, we have seen that it is equivalent to saying that the function h_0 is convex function of μ_α and ν_α.

To date, we have imposed that Q is negative defined, but only when $d\,X_A = 0$ except for $d\,\mu_\beta$, $d\,\nu_\beta$. This has yielded some important results; they are useful for dealing more easily with the general case and we find, after many but direct calculations, that

$$Q = Q_1 + Q_2 + Q_3 + G_{00}\frac{\partial^2 h_0}{\partial \lambda^2}\,d\,(\lambda)^2 + 2\,G_{00}\frac{\partial^2 h_0}{\partial \lambda\,\partial \vartheta}\,d\,\lambda\,d\,\vartheta + 2\sqrt{G_{00}}\left(2\,G_{00}\frac{\partial^2 h_0}{\partial \lambda\,\partial G_{00}} + \right.$$
$$\left. + \frac{\partial h_0}{\partial \lambda}\right) d\,\lambda\,d\,\lambda_0 + 2G_{00}\left(2\,\frac{\partial^2 h_0}{\partial \lambda\,\partial G_{11}}\mu^1 + \frac{\partial^2 h_0}{\partial \lambda\,\partial G_{12}}\nu^1\right) d\,\lambda\,d\,\mu_1 + 2G_{00}\frac{\partial^2 h_0}{\partial \lambda\,\partial G_{12}}\nu^2\,d\,\lambda\,d\,\mu_2 +$$
$$+2G_{00}\left(2\,\frac{\partial^2 h_0}{\partial \lambda\,\partial G_{22}}\nu^1 + \frac{\partial^2 h_0}{\partial \lambda\,\partial G_{12}}\mu^1\right) d\,\lambda\,d\,\nu_1 + 4G_{00}\frac{\partial^2 h_0}{\partial \lambda\,\partial G_{22}}\nu^2\,d\,\lambda\,d\,\nu_2 + G_{00}\frac{\partial^2 h_0}{\partial \vartheta^2}\,d\,(\vartheta)^2 +$$
$$+2\sqrt{G_{00}}\left(2\,G_{00}\frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{00}} + \frac{\partial h_0}{\partial \vartheta}\right) d\,\vartheta\,d\,\lambda_0 + 2G_{00}\left(2\,\frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{11}}\mu^1 + \frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{12}}\nu^1\right) d\,\vartheta\,d\,\mu_1 +$$
$$+2G_{00}\frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{12}}\nu^2\,d\,\vartheta\,d\,\mu_2 + 2G_{00}\left(2\,\frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{22}}\nu^1 + \frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{12}}\mu^1\right) d\,\vartheta\,d\,\nu_1 +$$
$$+4G_{00}\frac{\partial^2 h_0}{\partial \vartheta\,\partial G_{22}}\nu^2\,d\,\vartheta\,d\,\nu_2 + 2G_{00}\left(2\,\frac{\partial^2 h_0}{\partial \vartheta\,\partial (G_{00})^2}\,G_{00} + 3\,\frac{\partial h_0}{\partial\,G_{00}}\right) d\,(\lambda_0)^2$$

$$-2\mu_1\nu_2\,d\,\lambda_0\,d\,\lambda_3 - 2\sqrt{G_{00}}\left[\left(4\,\frac{\partial^2 h_0}{\partial G_{00}\,\partial G_{11}}\mu^1 + 2\,\frac{\partial^2 h_0}{\partial G_{00}\,\partial G_{12}}\nu^1\right) G_{00} + 2\,\frac{\partial h_0}{\partial G_{11}}\mu^1 + \frac{\partial h_0}{\partial G_{12}}\nu^1\right]\cdot$$
$$\cdot\,d\,\lambda_0\,d\,\mu_1 - 2\sqrt{G_{00}}\left[\left(4\,\frac{\partial^2 h_0}{\partial G_{00}\,\partial G_{22}}\nu^1 + 2\,\frac{\partial^2 h_0}{\partial G_{00}\,\partial G_{12}}\mu^1\right) G_{00} + 2\,\frac{\partial h_0}{\partial G_{22}}\nu^1 + \frac{\partial h_0}{\partial G_{12}}\mu^1\right] d\,\lambda_0\,d\,\nu_1$$
$$-\,2\sqrt{G_{00}}\left(2\,\frac{\partial^2 h_0}{\partial G_{00}\,\partial G_{12}}\,G_{00} + \frac{\partial h_0}{\partial G_{12}}\right)\nu^2\,d\,\lambda_0\,d\,\mu_2$$
$$-\,4\sqrt{G_{00}}\left(2\,\frac{\partial^2 h_0}{\partial G_{00}\,\partial G_{22}}\,G_{00} + \frac{\partial h_0}{\partial G_{22}}\right)\nu^2\,d\,\lambda_0\,d\,\nu_2 +$$
$$\left[-2\,\frac{\partial h_0}{\partial G_{00}}\,G_{00} + 2\,\frac{\partial h_0}{\partial G_{11}}(\mu_1)^2 + 2\,\frac{\partial h_0}{\partial G_{22}}(\nu_1)^2 + 2\,\frac{\partial h_0}{\partial G_{12}}\,\mu_1\,\nu_1\right] d\,(\lambda_1)^2 +$$

$$+2\left(2\,\frac{\partial h_0}{\partial G_{22}}\nu_1 + \frac{\partial h_0}{\partial G_{12}}\mu_1\right)\nu_2\,d\,\lambda_1\,d\,\lambda_2 - 2\nu_2\,d\,\lambda_1\,d\,\mu_3 + 2\left[-\,\frac{\partial h_0}{\partial G_{00}}\,G_{00} + \frac{\partial h_0}{\partial G_{22}}(\nu_2)^2\right] d\,(\lambda_2)^2 +$$
$$+2\nu_1\,d\,\lambda_2\,d\,\mu_3 + 2\mu_1\,d\,\lambda_2\,d\,\nu_3 - 2\frac{\partial h_0}{\partial G_{00}}\,G_{00}\,d\,(\lambda_3)^2 + 2\nu_2\,d\,\lambda_3\,d\,\mu_1 - 2\nu_1\,d\,\lambda_3\,d\,\mu_2 - 2\mu_1\,d\,\lambda_3\,d\,\nu_2\,,$$

where Q_1, Q_2, Q_3 have the above expressions.

In conclusion, we see that the function h_0 is not arbitrary but must satisfy the conditions (46) (which were useful at the end of Section 4), it must be a convex function of μ_α and ν_α, and the above expression of Q must be negative defined.

As a simple case, let us consider that of a homogeneous and isotropic medium, that is, the expression (12). We have already seen that the last term in this equation is a convex function; so it remains to be seen that the first 2 terms also give a convex contribution. So, let us consider

$$h'^{\alpha} = \left(\frac{c\,\mu_0}{2}\,G_{11} + \frac{c\,\epsilon_0}{2}\,G_{22}\right)\frac{\lambda^\alpha}{\sqrt{G_{00}}} + \eta^{\alpha\beta\gamma\delta}\,\frac{\lambda_\beta}{\sqrt{G_{00}}}\,\nu_\gamma\mu_\delta\,.$$

In the corresponding expression of Q we can calculate the coefficients of the differentials in $\mu^\alpha = 0$, $\nu^\alpha = 0$ (for the hypothesis of a weak electromagnetic field), so that there remains

$$Q = c\,\mu_0\,\sqrt{G_{00}}\,d\,\mu^\beta\,d\,\mu_\beta + c\,\epsilon_0\,\sqrt{G_{00}}\,d\,\nu^\beta\,d\,\nu_\beta < 0\,.$$

The reason behind this sign is that $d\,\mu^\beta\,d\,\mu_\beta = d\,\mu_\alpha\,d\,\mu_\beta\,g^{\alpha\beta} = -\,d\,\mu_\alpha\,d\,\mu_\beta\,h^{\alpha\beta} + \frac{(U^\alpha d\,\mu_\alpha)^2}{c^2}$. However, $U^\alpha d\,\mu_\alpha = d\,(U^\alpha\,\mu_\alpha) - \mu^\alpha d\,U_\alpha = -\,\mu^\alpha d\,U_\alpha$. Since we are calculating the coefficients of the differentials in $\mu^\alpha = 0$, $\nu^\alpha = 0$, it follows $U^\alpha d\,\mu_\alpha = 0$ and $d\,\mu^\beta\,d\,\mu_\beta = -\,d\,\mu_\alpha\,d\,\mu_\beta\,h^{\alpha\beta} < 0$. The same thing can be said for $d\,\nu^\beta\,d\,\nu_\beta$ thus completing the proof of the convexity.

8. Conclusions

We found a restriction on the law linking the electromagnetic tensors $F^{\alpha\beta}$ and $G^{\alpha\beta}$ to the 4-force ν_β and its dual μ_β (which are some components of $F^{\alpha\beta}$ and $G^{\alpha\beta}$). Now these skew-symmetric tensors are determined except for the scalar function h_0. This result was achieved by imposing a supplementary conservation law. This further law also made it possible to globally obtain a symmetric system of partial differential equations which is also hyperbolic if h_0 satisfies the convexity condition. Furthermore, the non-relativistic limit of the present results gives those already known in the literature that have been derived directly in the non-relativistic context. The present model can be used in a future article to treat the case where dissipative effects are present, i.e., not limited to Euler Equations for the material but with further balance equations. Furthermore, it can be implemented considering also multi-component gas mixtures such as the one considered in [27]. Regarding this last article, it must be said that Maxwell's Equations were not imposed at the beginning but obtained at the end as a result; unfortunately, they are not Maxwell's Equations in matter, but only those in empty space. So also in this respect further investigation is needed.

Author Contributions: S.P., R.E.T. and M.O. were fully involved in: substantial conception and design of the paper; drafting the article and revising it critically for important intellectual content; final approval of the version to be published. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We tank T. Ruggeri, in the University of Bologna for useful discussions on this topic. We also thank two anonimous referees whose suggestions helped us a lot to improve the presentation of the article and inspired us further to further investigate.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Identies Holding for the 4-Dimensional Levi-Civita Symbol

The 4-dimensional Levi-Civita symbol is defined as

$$\eta^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if} \quad \alpha\beta\gamma\delta \quad \text{is an even permutation of} \quad 0123 \\ -1 & \text{if} \quad \alpha\beta\gamma\delta \quad \text{is an odd permutation of} \quad 0123 \\ 0 & \text{if} \quad \alpha\beta\gamma\delta \quad \text{is not a permutation of} \quad 0123 \end{cases}.$$

Now, we have that

$$\eta_{0123} = \eta^{\alpha\beta\gamma\delta} g_{\alpha 0}\; g_{\beta 1}\; g_{\gamma 2}\; g_{\delta 3} = -\,\eta^{0123}\,.$$

It follows that $\eta_{\alpha\beta\gamma\delta} = -\,\eta^{\alpha\beta\gamma\delta}$, i.e.,

$$\eta_{\alpha\beta\gamma\delta} = \begin{cases} -1 & \text{if} \quad \alpha\beta\gamma\delta \quad \text{is an even permutation of} \quad 0123 \\ 1 & \text{if} \quad \alpha\beta\gamma\delta \quad \text{is an odd permutation of} \quad 0123 \\ 0 & \text{if} \quad \alpha\beta\gamma\delta \quad \text{is not a permutation of} \quad 0123 \end{cases} .$$

We now want to prove the following identity

$$\eta^{\alpha\phi\beta\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}} \quad \eta_{\nu\psi\gamma\delta}\frac{\lambda^\psi}{\sqrt{G_{00}}} = -2\,h_\nu^{[\alpha}\,h_\gamma^{\beta]}\,. \tag{A1}$$

In fact, in the reference frame where $\frac{\lambda_\phi}{\sqrt{G_{00}}} \equiv (1\,,\,0\,,\,0\,,\,0)$, the left hand side of (A1) equals

$$\eta^{\alpha 0\beta\delta}\,\eta_{\nu 0\gamma\delta} = \eta^{0\alpha\beta\delta}\,\eta_{0\nu\gamma\delta} = -2\,h_\nu^{[\alpha}\,h_\gamma^{\beta]}\,.$$

To prove the last step, we note that both sides are skew-symmetric with respect to $\alpha\beta$ and with respect to $\nu\gamma$; then just prove the result for $\alpha\beta = 12$ and $\nu\gamma = 12$. In this case the above relationship becomes

$$\eta^{0123}\,\eta_{0123} = -\,h_1^1\,h_2^2 + h_1^2\,h_2^1 = -\,1\,,$$

and this is an identity for the above.
Another identity which has been used in the main text of this article is the following

$$\eta^{\alpha\phi\beta\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}} \quad \eta_{\nu\psi\gamma\delta}\,\nu^\gamma h^{\psi\theta} = 2\,h^{\theta[\alpha}\,\nu^{\beta]}\,\frac{\lambda_\nu}{\sqrt{G_{00}}}\,. \tag{A2}$$

To prove it, we note that its left hand side can be written as

$$\begin{aligned} &-\,\eta^{\alpha'\phi\beta'\delta'}\frac{\lambda_\phi}{\sqrt{G_{00}}}(-\,g^\alpha_{\alpha'})\left(-\,g^\beta_{\beta'}\right)\left(-\,g^\delta_{\delta'}\right) \quad \eta_{\nu\psi\gamma\delta}\,\nu^\gamma h^{\psi\theta} = \\ &= -\,\eta^{\alpha'\phi\beta'\delta'}\frac{\lambda_\phi}{\sqrt{G_{00}}}h^\alpha_{\alpha'}h^\beta_{\beta'}h^\delta_{\delta'} \quad \eta_{\nu'\psi\gamma'\delta}\left(-\,g^{\gamma'}_\epsilon\right)\nu^\epsilon\left(-\,g^{\nu'}_\nu\right)h^{\psi\theta} = \\ &= -\,\eta^{\alpha'\phi\beta'\delta'}\frac{\lambda_\phi}{\sqrt{G_{00}}}h^\alpha_{\alpha'}h^\beta_{\beta'}h^\delta_{\delta'} \quad \eta_{\nu'\psi\gamma'\delta}\,h^{\gamma'}_\epsilon\;\nu^\epsilon\left(h^{\nu'}_\nu - \frac{\lambda^{\nu'}\lambda_\nu}{G_{00}}\right)h^{\psi\theta}\,. \end{aligned}$$

However, we have $\eta_{\nu'\psi\gamma'\delta}\,h^{\nu'}_\nu\;h^{\psi\theta}h^{\gamma'}_\epsilon\;h^{\delta'}_\delta = 0$; so we can continue the previous steps and find

$$\begin{aligned} \eta^{\alpha\phi\beta\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}} \quad \eta_{\nu\psi\gamma\delta}\,\nu^\gamma h^{\psi\theta} &= \eta^{\alpha'\phi\beta'\delta'}\frac{\lambda_\phi}{\sqrt{G_{00}}}h^\alpha_{\alpha'}h^\beta_{\beta'}h^\delta_{\delta'} \quad \eta_{\nu'\psi\gamma'\delta}\,h^{\gamma'}_\epsilon\;\nu^\epsilon\,\frac{\lambda^{\nu'}}{\sqrt{G_{00}}}\,\lambda_\nu h^{\psi\theta}\frac{1}{\sqrt{G_{00}}} = \\ &= -\,\eta^{\alpha\phi\beta\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}} \quad \eta_{\nu'\psi\gamma'\delta}\,h^{\gamma'}_\epsilon\;\nu^\epsilon\,\frac{\lambda^{\nu'}}{\sqrt{G_{00}}}\,\lambda_\nu h^{\psi\theta}\frac{1}{\sqrt{G_{00}}} \overset{*}{=} \\ = \eta^{\alpha\phi\beta\delta}\frac{\lambda_\phi}{\sqrt{G_{00}}} \quad \eta_{\psi\nu'\gamma'\delta}\,h^{\gamma'}_\epsilon\;\nu^\epsilon\,\frac{\lambda^{\nu'}}{\sqrt{G_{00}}}\,\lambda_\nu h^{\psi\theta}\frac{1}{\sqrt{G_{00}}} &\overset{**}{=} -2\,h_\psi^{[\alpha}\,h_{\gamma'}^{\beta]}\,h^{\gamma'}_\epsilon\;\nu^\epsilon\,\lambda_\nu\,h^{\psi\theta}\frac{1}{\sqrt{G_{00}}} = \\ &= -2\,h^{\theta[\alpha}\,h_\epsilon^{\beta]}\,\nu^\epsilon\,\lambda_\nu\,\frac{1}{\sqrt{G_{00}}} = 2\,h^{\theta[\alpha}\,\nu^{\beta]}\,\frac{\lambda_\nu}{\sqrt{G_{00}}}\,, \end{aligned}$$

where in the step marked with $\overset{*}{=}$ we changed the order of the indexes $\nu'\psi$ and in the step marked with $\overset{**}{=}$ we used (A1). This completes our proof.

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