Evolutionary Sequence of Spacetime and Intrinsic Spacetime and Associated Sequence of Geometries in Metric Force Fields I

O. Akindele Adekugbe Joseph [a*]

[a]*Department of Physics, Ondo State University of Science and Technology, Center for the Fundamental Theory and Unification, Okitipupa, Nigeria.*

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ABSTRACT

Two classes of three-dimensional metric spaces are identified. They are the conventional three-dimensional metric space and a new 'three-dimensional' absolute intrinsic metric space. Whereas an initial flat conventional proper metric space $\mathbb{E}'^3$ can transform into a curved three-dimensional Riemannian metric space $\mathbb{M}'^3$ without any of its dimension spanning the time dimension (or in the absence of the time dimension), in conventional Riemann geometry, an initial flat 'three-dimensional' absolute intrinsic metric space $\varnothing\hat{\mathbb{E}}^3$ (as a flat hyper-surface) along the horizontal, evolves into a curved 'three-dimensional' absolute intrinsic metric space $\varnothing\hat{\mathbb{M}}^3$, which is curved (as a curved hyper-surface) toward the absolute intrinsic metric time 'dimension' along the vertical, and it is identified as 'three-dimensional' absolute intrinsic Riemannian metric space.

Corresponding author: E-mail: cfs_ib@yahoo.com;

It invariantly projects a flat 'three-dimensional' absolute proper intrinsic metric space $\varnothing\mathbb{IE}'^3_{ab}$ along the horizontal, which is made manifested outwardly in flat 'three-dimensional' absolute proper metric space $\mathbb{IE}'^3_{ab}$, overlying it, both as flat hyper-surfaces along the horizontal. The flat conventional three-dimensional relative proper metric space $\mathbb{E}'^3$ and its underlying flat three-dimensional relative proper intrinsic metric space $\varnothing\mathbb{E}'^3$ remain unchanged. The observers are located in $\mathbb{E}'^3$. The projective $\varnothing\mathbb{IE}'^3_{ab}$ is imperceptibly embedded in $\varnothing\mathbb{E}'^3$ and $\mathbb{IE}'^3_{ab}$ in $\mathbb{E}'^3$. The corresponding absolute intrinsic metric time 'dimension' is not curved from its vertical position simultaneously with 'three-dimensional' absolute intrinsic metric space. The development of absolute intrinsic Riemannian geometry is commenced and the conclusion that the resulting geometry is more all-encompassing then the conventional Riemannian geometry on curved conventional metric space $\mathbb{M}'^3$ only is reached.

1 INTRODUCTION

There is perhaps no better place to start a fundamental theory of physics than a discourse of the underlying space(s) and geometry(ies). We have started this by isolating the flat four-dimensional proper metric spacetimes and their underlying flat two-dimensional proper intrinsic metric spacetimes of co-existing four symmetrical universes, referred to as positive (or our) universe, negative universe, positive time-universe and negative time-universe, in previous articles [1, 2, 3, 4].

Lorentz transformation and intrinsic Lorentz transformation (LT/∅LT) and their inverses are derived with a new set of affine spacetime/intrinsic affine spacetime diagrams on the flat proper metric spacetimes and the underlying flat proper intrinsic metric spacetimes in the pertinent four-world picture in those papers.

The four universes exhibit perfect symmetry of natural laws, which means that natural laws take on identical forms in the universes, as established in section 2 of [2] and section 2 of [3]. Perfect symmetry of state among the universes is shown in section 3 of [4], where it means that the physical appearances of the universes at any scale of observation are perfectly identical at all times.

Symmetry of state is shown to be guaranteed by the fact that the four members of every quartet of symmetry-partner particles or bodies in the four universes have perfectly identical magnitudes of masses, perfectly identical shapes and perfectly identical sizes, and that they are involved in perfectly identical relative motions at all times. The immutability of Lorentz invariance is also shown to be a consequence of perfect symmetry of state among the four universes in section 3 of [4].

The flat two-dimensional proper intrinsic metric spacetime $(\varnothing\rho', \varnothing c_s\varnothing t')$ that underlies the flat four-dimensional proper metric spacetime $(\Sigma', c_s t')$ in our universe, introduced as *ansatz* in the two-world picture in sub-section 4.4 of [1], is derived formally in the four-world picture in sub-section 1.2 of [4]. There is essentially no outstanding issue in [1, 2, 3, 4] that could prevent the description of the isolation of the four-world picture in those articles as having attained a close-form.

Now, as discussed in section 4 of [4], the special theory of relativity/intrinsic special theory of relativity (SR/∅SR) operate on extended flat proper metric space-times/underlying extended flat proper intrinsic metric spacetimes of the four universes with the assumed absence of strong gravitational field. However, since SR/∅SR involve affine spacetime/intrinsic affine spacetime (or affine spacetime/intrinsic affine spacetime geometry) in each universe, SR/∅SR cannot alter the extended flat four-dimensional proper metric

spacetime/ extended flat proper intrinsic metric spacetime on which they operate with the assumed absence of strong gravitational field. It is the presence of a long-range metric force field, such as the gravitational field, that can change the extended flat proper metric spacetimes and its underlying extended flat two-dimensional proper intrinsic metric spacetimes to extended four-dimensional relativistic metric spacetimes and its underlying extended two-dimensional relativistic intrinsic metric spacetimes in all finite neighborhoods of the sources of symmetry-partner long-range metric force fields in the four universes.

The next natural step in the further development of the spaces and geometrical foundation for the theories of relativity and gravitation, in addition to the affine spacetime/intrinsic affine spacetime geometry for SR/∅SR in the four-world picture developed in [1, 2, 3, 4], is the development of the counterpart metric spacetime/intrinsic metric spacetime geometry, which will convert extended flat proper metric spacetimes and their underlying extended flat proper intrinsic metric spacetimes to extended relativistic metric spacetimes and their underlying extended relativistic intrinsic metric spacetimes in all finite neighborhoods of symmetry-partner long-range metric force fields in the four universes.

More often than not, there arises the need to adapt a subject from its sophisticated form in pure mathematics to an applicable form in an applied field. The reason being that, guided by logical and mathematical consistency only, a pure mathematical subject can be pursued to any level of generalization and sophistication.

In application, on the other hand, the requirement for mathematics to describe physical reality, that is, to model physical situations and concepts and to satisfy physical constraints, often leads to a lowering of the levels of sophistication and generalization of a mathematical subject in its applicable form.

It is therefore the responsibility of a physicist to marry the underlying concepts and constraints of a physical theory to the conceptual foundation of a mathematical subject to be applied and, in the process, as is often possible, evolve the applicable form of the mathematical subject. Sometimes the applicable form, having lost all sophistication in the process of putting on a physical or application face, bears only a crude resemblance to the original subject. However whatever beauty is lost in mathematics is usually gained in terms of ease of interpretation and transparency of connection to reality of the resulting physical theory.

Whenever an appropriate applicable form of a pure mathematical subject is not sought, or could not be found due to inability to link physical (or application) concepts with the concepts and principles of a mathematical subject, recourse to mathematical hypotheses and axioms becomes inevitable in propagating the physical theory. The resulting theory is often fraught with the problems of interpretation and dubiety of connection to reality in such circumstance.

One subject of pure mathematics that is of direct relevance to fundamental physics is Riemann geometry. Riemann geometry evolved from elementary differential geometry of surfaces in the Euclidean space by the usual mathematical processes. Albert Einstein advanced an infallible argument in support of curvature of the four-dimensional spacetime in the gravitational field [5] and applied Riemann geometry in an unaltered form to the problem of gravitation on curved spaetime.

On the other hand, the concepts of relative space, absolute space, absolutism and observers in physics are incorporated into Riemann geometry and an absolute intrinsic Riemann geometry on a certain curved 'three-dimensional' absolute intrinsic Riemannian metric space with absolute intrinsic sub-Riemannian metric tensor, is isolated in a long-range metric force field in general in this paper.

2   ON THE INCORPORATION OF THE TIME DIMENSION INTO RIEMANN GEOMETRY IN GENERAL RELATIVITY AND THE PRESENT CONTEXT

Friederich Bernhard Riemann in his famous lecture of June 10, 1854, at the Götingen University entitled, "On the Hypotheses Which Lie at the Foundation of Geometry", as translated in [6], evolved the geometry that is now named after him. With a prophetic vision, Riemann had raised issues during this lecture that would have far-reaching consequences in physics. For example, he wrote in the paper he presented at the lecture, "... the basis of the metric relation of a manifold must be sought outside the manifold in the binding forces that act upon it."

It would be a disservice to describe Riemann lesser than a precursor of the various metric theories of physics, with the general theory of relativity as the leading member. However the time dimension and the significant role it plays in linking Riemann geometry to physics, as developed by Albert Einstein, see pages 111 − 149 of [5], was unknown to Riemann. Riemann simply generalized Gauss's theory of surfaces in the Euclidean 3-space to general curved $n-$dimensional spaces (without time dimension), where points are characterized by n coordinates as

$$u^\nu = f^\nu(x^1, x^2, x^3, ..., x^n); \ \nu = 1, 2, 3, \ ... \ n \ . \tag{1}$$

The distance element ds between two indefinitely close points in this general n-dimensional curved space is

$$ds^2 = \sum_{\mu,\nu=1}^{n} g_{\mu\nu}(x^1, x^2, x^3, ..., x^n)dx^\mu dx^\nu \ , \tag{2}$$

where the metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu}(x^1, x^2, x^3, ..., x^n) \quad = \quad \sum_{\alpha=1}^{n} \frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\alpha}{\partial x^\nu} = \sum_{\alpha=1}^{n} \frac{\partial u^\alpha}{\partial x^\mu}\frac{\partial u^\alpha}{\partial x^\nu} \ . \tag{3}$$

Albert Einstein introduced the time dimension, $x^0 = c_s t$, into Riemann geometry in a direct manner somewhat. The usual notation ct for the time dimension is being replaced by $c_s t$ in this paper, having shown that ct is actually the time dimension with zero geodesic flow and re-denoted it by $c_s t$ in subsection 1.4 of [3]. Having added $c_s t$ to the three dimensions, x^1, x^2 and x^3, of the Euclidean 3-space, yielding the flat four-dimensional metric spacetime (the Minkowski space) in the special theory of relativity, see pages 37 − 65 of [5], he forwarded an argument that leads to the conclusion that the general principle of relativity is realizable on a curved four-dimensional spacetime continuum in the gravitational field. He thereby considered the four-dimensional spacetime as a Riemannian manifold in the gravitational field in the general theory of relativity, see pages 111 − 149 of [5] and chapter 3 of [7].

Albert Einstein applied Riemann geometry in an unaltered form on the proposed curved four-dimensional spacetime in the gravitational field, see pages 111 − 149 of [5] and chapter 5 of [7]. The only significant difference in Riemann geometry without time dimension (that is, manifolds of type $\mathbb{M}^p$) and Riemann geometry with time dimension (that is, manifolds of the type $\mathbb{M}^{p+q}$), is in the structure of the metric tensor. While the metric tensor is elliptical with signature $(++++)$ in a four-dimensional Riemann space (without time dimension), it is hyperbolic with signature $(+---)$ or $(---+)$ on a curved four-dimensional spactime. As a matter of fact, it is at the point of solving Einstein's field equations that K. Schwarzschild introduced the hyperbolic metric tensor, so that the metric tensor obtained could reduce to the Lorentzian metric tensor at infinity, see pages 185 − 186 of [7].

The important point to note in the foregoing is that, Albert Einstein introduced the time dimension into Riemann geometry by allowing the time dimension and the three dimensions of space to be curved at once (or simultaneously) to form a curved four-dimensional spacetime continuum with Riemannian metric tensor. He then applied Riemann geometry (for four-dimensional Riemann space without time dimension) in an unaltered form to the curved four-dimensional spacetime continuum thus obtained. This approach of introducing the time dimension into Riemann geometry by Albert Einstein has been referred to as direct approach above.

However, apart from the direct approach adopted by Einstein, there is another approach, which shall be referred to as indirect approach, toward the introduction of the time dimension into Riemann geometry. The indirect approach leads to a kind of Riemannian spacetime geometry that is different from the conventional Riemannian spacetime geometry of Einstein's direct approach.

3 ISOLATING TWO CLASS-ES OF 3-DIMENSIONAL RIEMANNIAN METRIC SPACES

Let us start by considering the proper Euclidean metric 3-space, denoted by Σ' in [1, 2, 3, 4], but which shall be denoted by $\mathbb{E}'^3$ in this paper, with dimensions, x'^1, x'^2 and x'^3, and the absolute time 'dimension' to be denoted by, $\hat{x}^0 \equiv \hat{c}_s\hat{t}$. The proper Euclidean 3-space and the absolute time 'dimension' constitutes the Galileo space $(\mathbb{E}'^3; \hat{c}_s\hat{t})$. Let us assume that due to a yet unspecified phenomenon, the proper Euclidean 3-space becomes a curved metric space, to be denoted by $\mathbb{M}^3$, within a region of the universal

3-space, while the absolute time coordinate remains not curved.

Let us give a graphical illustrations of the Galileo space $(\mathbb{E}'^3; \hat{c}_s\hat{t})$ and the curved metric 3-space - absolute time 'dimension' $(\mathbb{M}^3; \hat{c}_s\hat{t})$. In doing this, we shall consider $\mathbb{E}'^3$ as a hyper-surface, $\hat{c}_s\hat{t} =$ const., and represent it by a plane surface along the horizontal and the absolute time 'dimension' $\hat{c}_s\hat{t}$ by a vertical normal line to the hyper-surface, as illustrated in Fig. 1a.

In the case of the graphical representation of $(\mathbb{M}^3; \hat{c}_s\hat{t})$, there are two possibilities. The first is obtained by letting the hyper-surface $\mathbb{E}'^3$ along the horizontal in Fig. 1a to become a curved hyper-surface $\mathbb{M}^3$ still on the horizontal plane, so that none of the dimensions, x^1, x^2 and x^3 of $\mathbb{M}^3$, spans the absolute time 'dimension' $\hat{c}_s\hat{t}$ along the vertical, as illustrated in Fig. 1b.

The dimensions of the curved space $\mathbb{M}^3$ span the dimensions of the proper Euclidean 3-space $\mathbb{E}'^3$ only. Actually the proper Euclidean 3-space $\mathbb{E}'^3$ has evolved into the curved space $\mathbb{M}^3$ within the region of 3-space being considered. Hence the proper Euclidean space does not exist along with $\mathbb{M}^3$ within the region. Nevertheless the curved metric space $\mathbb{M}^3$ is embedded in the global proper Euclidean 3-space $\mathbb{E}'^3$ and the coordinates x'^i of $\mathbb{E}'^3$ serve as cartesian coordinates for points on $\mathbb{M}^3$, while x^i are the coordinates of $\mathbb{M}^3$.

The second possibility (or case) is obtained by allowing the dimensions, x^1, x^2 and x^3, of the curved space $\mathbb{M}^3$ to span the absolute time 'dimension' $\hat{c}_s\hat{t}$ along the vertical solely, so that $\mathbb{M}^3$ is curved (as a hyper-surface) toward $\hat{c}_s\hat{t}$, as illustrated in Fig. 1c. Intermediate cases in which some dimensions of $\mathbb{M}^3$ span the absolute time 'dimension', while others do not, are actually possible. However such cases must be considered as generic forms of the second case illustrated in Fig. 1c.
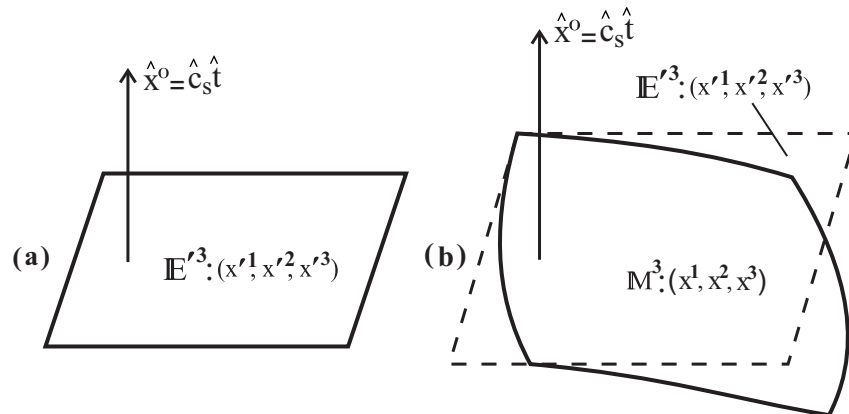
Fig. 1. (a) Graphical representation of the Galileo space and (b) the Euclidean 3-space $\mathbb{E}'^3$ of the Galileo space evolves into a curved 3-dimensional (Riemannian) metric space $\mathbb{M}^3$, such that none of the dimensions of $\mathbb{M}^3$ spans the absolute time 'dimension' along the vertical.
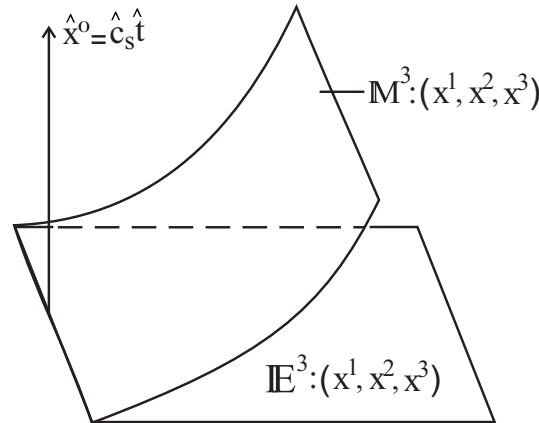

Fig. 1(c). The proper Euclidean 3-space $\mathbb{E}'^3$ of the Galileo space evolves into a curved 3-dimensional (Riemannian) metric space $\mathbb{M}^3$, such that the dimensions of $\mathbb{M}^3$ span the absolute time 'dimensions' $\hat{c}_s\hat{t}$ along the vertical solely. The curved space $\mathbb{M}^3$ (as a curved hyper-surface), projects a new Euclidean 3-space $\mathbb{E}^3$, with dimensions, x^1, x^2 and x^3, underneath itself as a flat hyper-surface along the horizontal.

Since a vacuum cannot be created along the horizontal, the curved space $\mathbb{M}^3$ will project a new flat hyper-surface — a new Euclidean 3-space — to be denoted by $\mathbb{E}^3$, with dimensions x^1, x^2 and x^3 along the horizontal, as shown in Fig. 1c. In other words, the curved space $\mathbb{M}^3$ will be underlay by its projective Euclidean 3-space $\mathbb{E}^3$ in this second case. The concept of underlying projective space does not arise in the first case (Fig. 1b), since the curved hyper-surface $\mathbb{M}^3$ lies along the horizontal in that case.

We shall now investigate the two cases of curved metric space formed from the Galileo space (of Fig. 1a) described above, in order to show the essential difference that may exist between them.

Case I: Conventional Riemannian metric 3-space

The first case of curved metric space formed from the Galileo space, in which each dimension of the curved space $\mathbb{M}^3$ spans one, two or all the

dimensions of the proper Euclidean 3-space $\mathbb{E}'^3$ that evolved into it, and none spans the absolute time 'dimension' along the vertical, illustrated in Fig. 1b, is a conventional Riemannian metric 3-space. It is to be noted that two metric spaces namely, the proper Euclidean 3-space $\mathbb{E}'^3$ (with Euclidean metric tensor) and the curved metric space $\mathbb{M}^3$ (with Riemannian metric tensor), do not co-exist in Fig. 1b. This is so because $\mathbb{E}'^3$ has evolved into $\mathbb{M}^3$ within the given region of the universal proper Euclidean 3-space. The curved three-dimensional metric space, such as encountered in the Robertson-Walker metric tensor is an example.

Case II: A new kind of Riemannian metric 3-space

The curved space $\mathbb{M}^3$ in Fig. 1c has evolved from the proper (or classical) Euclidean 3-space $\mathbb{E}'^3$. Hence Eqs. (4) through (6) of conventional Riemann geometry are equally valid for the curved space $\mathbb{M}^3$ in Fig. 1c. We must simply let $n = 3$ in them to have

$$x'^\nu = f^\nu(x^1, x^2, x^3); \ \nu = 1, 2, 3 , \qquad (4)$$

where x'^ν are the coordinates of the three-dimensional proper (or classical) Euclidean space $\mathbb{E}'^3$ of the Galileo space that evolved into $\mathbb{M}^3$, but which still serve as the cartesian coordinates for points on the curved space $\mathbb{M}^3$, and x^ν are the coordinates of $\mathbb{M}^3$. The distance element is given on $\mathbb{M}^3$ as

$$ds^2 = \sum_{\mu,\nu=1}^{3} g_{\mu\nu}(x^1, x^2, x^3) dx^\mu dx^\nu \qquad (5)$$

where

$$g_{\mu\nu}(x^1, x^2, x^3) = \sum_{\alpha=1}^{3} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\alpha}{\partial x^\nu} = \sum_{\alpha=1}^{3} \frac{\partial u^\alpha}{\partial x^\mu} \frac{\partial u^\alpha}{\partial x^\nu} .$$
$$(6)$$

Apart from Eqs. (4) through (6) of conventional Riemann geometry on the curved metric space $\mathbb{M}^3$ in Fig. 1c, there is a necessary further step to be taken, which consists in obtaining the projection of the Riemannian metric space $\mathbb{M}^3$ into the horizontal to obtain the underlying new Euclidean 3-space $\mathbb{E}^3$ in that figure.

Since we have identified the first case of a curved space that evolved from the Galileo space, illustrated in Fig. 1b as a conventional Riemannian metric space of type $\mathbb{M}^p; \ p = 3$, there is nothing new to know about it. We shall therefore proceed to investigate the second case illustrated in Fig. 1c. The second case shall undergo extensive modification with further development in this paper. We shall be led, in a consistent manner, to the identification of certain curved 'three-dimensional' absolute intrinsic metric space for it, instead of the physical (or relative) 3-space in Fig. 1c.

4 ISOLATING ABSOLUTE INTRINSIC RIEMANNIAN METRIC SPACE AND ABSOLUTE INTRINSIC RIEMANN GEOMETRY

Now two observers located at two distinct positions P_1 and P_2 in the Riemannian metric space $\mathbb{M}^3$ in Fig. 1b or 1c are located at positions of different Riemannian curvatures K_1 and K_2 respectively, where K_1 and K_2 are determined relative to the reference Euclidean space $\mathbb{E}'^3$. These observers will therefore observe different curvatures K_{31} and K_{32} respectively, of a third position P_3 on the curved space $\mathbb{M}^3$. Consequently these observers will observe different metric tensors and construct different Riemann geometries for the third position.

Since observers within the region of space being considered are necessarily located on the curved space $\mathbb{M}^3$ in the first case (Fig. 1b), there is no way of resolving the problem of the non-uniqueness of Riemann geometry derived by observers located at different positions in a Riemannian metric space discussed in the preceding paragraph in the first case. On the other hand, Riemann geometry of the curved space $\mathbb{M}^3$ can be formulated uniquely with respect to observers located at different positions in the underlying Euclidean space $\mathbb{E}^3$ in the second case (Fig. 1c), as explained below.
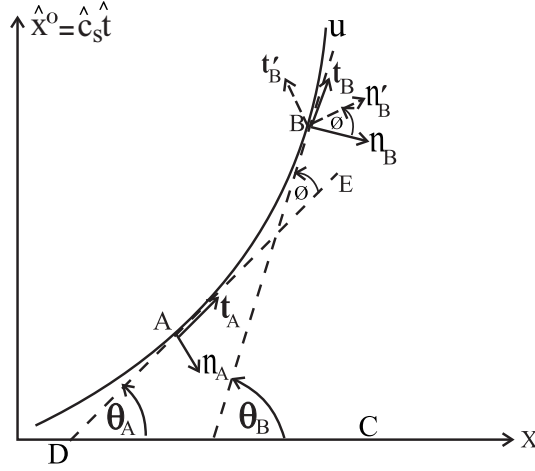
Fig. 2. A one-dimensional metric space curving onto the absolute time 'dimension' along the vertical, projects a straight line one-dimensional metric space along the horizontal.

Let us consider a curved one-dimensional space to constitute plane curve u, which is curved onto the absolute time 'dimension' and underneath which lies a straight line one-dimensional space x along the horizontal (which the curved space u projects along the horizontal), as illustrated in Fig. 2. The curve u and its projection x shall be taken to be one-dimensional metric spaces.

The curvatures k_A and k_B at points A and B respectively of the one-dimensional curved metric space u are given by definition, see chapter one of [6] as

$$\frac{d\theta}{du}|_A = \frac{dt_A}{du} = k_A \qquad (7a)$$

and

$$\frac{d\theta}{du}|_B = \frac{dt_B}{du} = k_B . \qquad (7b)$$

The angle θ is measured relative to the one-dimensional straight line metric space x along the horizontal in Fig. 2. The t_A and n_A are the unit tangent and unit normal vectors to the curve u at point A, with respect to 'one-dimensional observers' located along x. It can thus be said that the curvatures k_A and k_B at points A and B respectively, of the curve u are valid relative to 'one-dimensional observer' at point C that can be anywhere in the dimension x along the horizontal.

Now let us consider the curvature of u at point B relative to a 'one-dimensional observer' at point A on the curve u. The projective one-dimensional metric space x along the horizontal, on which the 'one-dimensional observer' at point C is located, must be replaced by the tangent DE to the curve u on which the 'one-dimensional observer' at point A is located. The curvature of u must be defined in terms of a different angle ϕ measured relative to the line DE with respect to the 'one-dimensional observer' at A. Hence the curvature k_{BA} of point B relative to point A of the curve u is given as

$$\frac{d\phi}{du}|_B = \frac{dt'_B}{du} = k_{BA} , \qquad (8)$$

where t'_B and n'_B are the unit tangent vector and unit normal vector to the curve u with respect to the 'one-dimensional observer' at A, which correspond to t_B and n_B respectively (shown in Fig. 2) with respect to 'one-dimensional observer' located anywhere along x.

We find from the above that the curvature at a given point on a plane curve u — a one-dimensional metric space — on the vertical x-$\hat{c}_s\hat{t}$-plane, depends on the position of the 'one-dimensional observer' located along the curve u, but is the same relative to 'one-dimensional observers' located at different positions in the one-dimensional straight line metric space x, which the one-dimensional curved metric space

u projects along the horizontal. The curvatures k_A and k_B of Eqs. (7a) and (7b) are valid relative to a 'one-dimensional observer' located at point C that may be anywhere along the one-dimensional space x. Hence the position C of such 'observer' does not appear as a label on k_A and k_B. On the other hand, the position A of the 'observer' located along the curve u appears as a label on the curvature k_{BA} at position B of the curve u in Eq. (8).

Now the curve u in Fig. 2 is a one-dimensional Riemannian metric space $\mathbb{M}^1$, as mentioned above. It is a member of the second case of Riemannian metric spaces illustrated in Fig. 1c, which can be generated from the Galileo space of Fig. 1a. Figure 2 and the discussion on it above can be generalized to the case of the 3-dimensional metric space $\mathbb{M}^3$ (with dimensions u^1, u^2 and u^3), which is curved toward the absolute time 'dimensions' $\hat{c}_s \hat{t}$, and which is curved relative to its projective 3-dimensional Euclidean space $\mathbb{E}^3$ (with dimensions x^1, x^2 and x^3), which is also a metric space in Fig. 1c, re-illustrated as Fig. 3.

One observes that there are two co-existing metric spaces of different metric tensors namely, the curved space $\mathbb{M}^3$ with Riemannian metric tensor and the underlying flat space $\mathbb{E}^3$ with Euclidean metric tensor in Fig. 3. However only singular metric spaces are known in Riemann geometry. This paradox raised by Fig. 3 shall be resolved with further development of this article. The first class of Riemannian metric spaces $\mathbb{M}^3$ illustrated in Fig. 1b, which evolves from the proper (or classical) Euclidean 3-space, does not raise the paradox raised by Fig. 1c or Fig. 3, since the curved hyper-surface $\mathbb{M}^3$ lies along the horizontal, thereby precluding any projective space in Fig. 1b of conventional Riemannian metric space. There is no duality of metric spaces in the first case (or in conventional Riemann geometry).

Now the Riemannian curvature K_{BC}, and hence the metric tensor $g_{ik}^{(BC)}$ at point B on $\mathbb{M}^3$ are the same for different positions C (or for different 3-observers or different 'frames') in the underlying Euclidean space $\mathbb{E}^3$. Thus the label C of the position of the 3-observer in $\mathbb{E}^3$ is redundant and does not have to appear on the curvature and metric tensor at any point on the curved space $\mathbb{M}^3$. In other words, $g_{ik}^{(B)}$ and $g_{ik}^{(A)}$ are the unique or invariant metric tensors at points B and A respectively on $\mathbb{M}^3$, with respect to 3-observers located at different positions in $\mathbb{E}^3$ in Fig. 3.

On the other hand, the curvature and metric tensor at a given point on the curved space $\mathbb{M}^3$, relative to an observer at another point on $\mathbb{M}^3$, depends on the position of the observer in Fig. 3. Thus the curvature K_{BA} and the metric tensor $g_{ik}^{(BA)}$ of point B on $\mathbb{M}^3$, relative to an observer at position A on $\mathbb{M}^3$ in Fig. 3, contains the position A of the observer as a label.
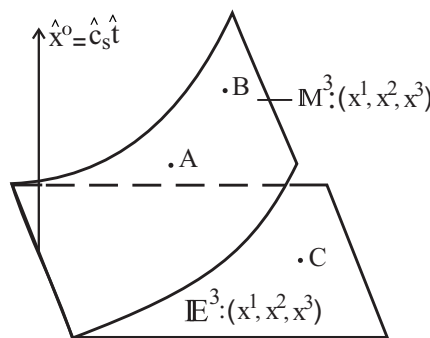

Fig. 3. A 3-dimensional Riemannian metric space curving onto the absolute time 'dimension' along the vertical (as a curved hyper-surface) and its underlying projective Euclidean 3-space (as a flat hyper-surface) along the horizontal.

We shall sometimes refer to the 3-observers in the underlying projective Euclidean space $\mathbb{E}^3$ as Euclidean observers, while observers at different positions on the curved (or Riemann) space $\mathbb{M}^3$ shall be referred to as Riemannian observers. The foregoing paragraph simply states that the metric tensor at any given point on the curved metric manifold $\mathbb{M}^3$ is the same with respect to all Euclidean observers (or all 'frames') in the underlying Euclidean space $\mathbb{E}^3$, but depends on the location (or the local 'frame') of a Riemannian observer. We shall be concerned with the Riemann geometry of the curved metric manifold $\mathbb{M}^3$ in the second case illustrated in Fig. 1c or Fig. 3, relative to Euclidean observers mainly.

The curved manifold $\mathbb{M}^3$ possesses a unique Riemannian metric tensor $g_{ik}^{(B)}$ at point B relative to all observers in the underlying Euclidean 3-space $\mathbb{E}^3$ (or relative to all Euclidean observers). Hence Euclidean observers will formulate Riemann geometry on $\mathbb{M}^3$ in Fig. 1c or Fig. 3 in the context of conventional Riemann geometry by writing general coordinate transformations like system (4), which shall be re-written as follows because of a certain point to be made.

$$x'^{\nu} = f^{\nu}(u^1, u^2, u^3); \ \nu = 1, 2, 3, \qquad (9a)$$

hence,

$$g_{ik}(u^1, u^2, u^3) = \sum_{\alpha=1}^{3} \frac{\partial f^{\alpha}}{\partial u^{\mu}} \frac{\partial f^{\alpha}}{\partial u^{\nu}} = \sum_{\alpha=1}^{3} \frac{\partial x'^{\alpha}}{\partial u^{\mu}} \frac{\partial x'^{\alpha}}{\partial u^{\nu}} . \qquad (9b)$$

The point to note is that x'^{ν} are the coordinates of the original proper (or classical) Euclidean 3-space $\mathbb{E}'^3$ in Fig. 1a, which evolved into the curved space $\mathbb{M}^3$ in Fig. 1c or Fig. 3, while u^{ν} are the coordinates of $\mathbb{M}^3$. The Euclidean observers in $\mathbb{E}^3$ will then write a unique Gaussian line element at point B on $\mathbb{M}^3$ as

$$ds^2 = \hat{c}_s^2 d\hat{t}^2 - \sum_{i,k=1}^{3} g_{ik}^{(B)}(u^1, u^2, u^3) du^i du^k ;$$

$$(\text{w.r.t. } 3 - \text{observers in } \mathbb{E}^3) . \qquad (10)$$

The Euclidean 3-observers will construct Riemann geometry in the context of conventional Riemann geometry (Eqs. (9b) and (10)), uniquely on the curved manifold $\mathbb{M}^3$ in terms of

coordinates, u^1, u^2 and u^3, of $\mathbb{M}^3$. They will also derive the projection of $\mathbb{M}^3$ into the horizontal to form the Euclidean space $\mathbb{E}^3$, as shall be done shortly.

Now let us change local coordinate set from (u^1, u^2, u^3) of one local frame to another local coordinate set (v^1, v^2, v^3) of another local frame at the same position B on the curved manifold $\mathbb{M}^3$ (in Fig. 3), in Eq. (10) to have the following

$$d\tilde{s}^2 = \hat{c}_s^2 d\hat{t}^2 - \sum_{i,k=1}^{3} \tilde{g}_{ik}^{(B)}(v^1, v^2, v^3) dv^i dv^k ;$$

$$(\text{w.r.t } 3 - \text{observers in } \mathbb{E}^3) . \qquad (11)$$

The line element is invariant with re-parametrization (or with change of local coordinate set). By applying this between equations (10) and (11) we have the following

$$g_{ik}^{(B)}(u^1, u^2, u^3) du^i du^k = \tilde{g}_{ik}^{(B)}(v^1, v^2, v^3) dv^i dv^k$$

Hence

$$\tilde{g}_{ik}^{(B)} = g_{ik}^{(B)} \frac{\partial u^i}{\partial v^i} \frac{\partial u^k}{\partial v^k} \qquad (12)$$

Now the Riemannian curvature K_B at point B on the manifold $\mathbb{M}^3$ relative to the underlying Euclidean 3-space $\mathbb{E}^3$ in Fig. 3 is the same for all local frames at that point. This is so because all local frames lie on the curved hyper-surface $\mathbb{M}^3$ at the given point B and thereby possess the same unique curvature K_B relative to $\mathbb{E}^3$ as the curved hyper-surface $\mathbb{M}^3$ itself. It follows then that the metric tensor at point B on $\mathbb{M}^3$ is unchanged as one changes from the local frame (u^1, u^2, u^3) to the local frame (v^1, v^2, v^3) at this point, with respect to 3-observers at different positions (or in different 'frames') in the underlying Euclidean 3-space $\mathbb{E}^3$. In other words, $\tilde{g}_{ik}^{(B)}(v^1, v^2, v^3)$ is the same as $g_{ik}^{(B)}(u^1, u^2, u^3)$, with respect to 3-observers at different positions (or in different frames) in the underlying Euclidean 3-space $\mathbb{E}^3$.

The foregoing paragraph states a significant difference between Riemann geometry of a curved metric space $\mathbb{M}^3$ of the second case, in which the curved metric space (as a curved hyper-surface), lies above its projective Euclidean space $\mathbb{E}^3$ (as a flat hyper-surface along the horizontal) in which the observers are located, illustrated in Fig. 1c or Fig. 3, and the conventional Riemann geometry of the first

case in which the curved metric space $\mathbb{M}^3$ is embedded in the global Euclidean 3-space $\mathbb{E}'^3$, as illustrated in Fig. 1b. There is no projective Euclidean space in the first case, and observers are necessarily located on the curved metric space $\mathbb{M}^3$ within the region covered by $\mathbb{M}^3$.

The significant difference between Riemann geometries for the two cases is that both the line element and metric tensor are invariant with re-parametrization ($ds^2 = d\tilde{s}^2$ and $g_{ik} = \tilde{g}_{ik}$), in the second case (of Fig. 1c or Fig. 3), while the line element is invariant but the metric tensor transforms as Eq. (12) with re-parametrization in the first case (of Fig. 1b). Riemann geometry for the first case (of Fig. 1b) is obviously the conventional Riemann geometry, as identified earlier.

The necessary invariance with re-parametrization of both the metric tensor and the line element in the second case of a curved metric space, which lies above its projective Euclidean space in which the observers are located (in Fig. 3), allows us to write the following from Eq. (12)

$$\tilde{g}_{ik}^{(B)} = g_{ik}^{(B)} \frac{\partial u^i}{\partial v^i} \frac{\partial u^k}{\partial v^k} = g_{ik}^{(B)} \ , \qquad (13)$$

hence,

$$\frac{\partial u^i}{\partial v^i} \frac{\partial u^k}{\partial v^k} = \delta_{ik} \ . \qquad (14)$$

Equation (14) is valid for every pair of local coordinate sets (or local 'frames') at any given point on the the curved manifold $\mathbb{M}^3$ relative to observers located at different positions in the underlying Euclidean 3-space $\mathbb{E}^3$, in the Riemann geometry of the second case illustrated in Fig. 1c or Fig. 3. It simply states that all local coordinate sets at a given point on the curved manifold $\mathbb{M}^3$ are identical with respect to 3-observers in the underlying projective Euclidean 3-space $\mathbb{E}^3$, and this is true at every point of $\mathbb{M}^3$, in the Riemann geometry of the second case.

It follows from the foregoing that all local coordinate sets, (u^1, u^2, u^3), (v^1, v^2, v^3), (w^1, w^2, w^3), etc, at any point on the curved manifold $\mathbb{M}^3$ are identical to a singular local coordinate set with coordinates to be denoted

by (ξ^1, ξ^2, ξ^3), with respect to all observers in the underlying Euclidean space $\mathbb{E}^3$. Thus natural laws formulated in terms of the singular local coordinate set (ξ^1, ξ^2, ξ^3) at any position on $\mathbb{M}^3$ are valid in terms of every local coordinate sets, (u^1, u^2, u^3), (v^1, v^2, v^3), (w^1, w^2, w^3), etc, at that position, with respect to all observers (or 'frames') in the underlying Euclidean 3-space $\mathbb{E}^3$. It then follows that laws of nature are naturally covariant (see page 117 of [7] and page 117 [5]), on the curved space $\mathbb{M}^3$ with respect to all observers (or 'frames') in the underlying Euclidean 3-space $\mathbb{E}^3$.

Now a space in which all local coordinate sets (or local 'frames') are identical to a singular coordinate set (or a singular local 'frame') at each point of it is an absolute space, an absolute space being a distinguished coordinate set (or a distinguished 'frame'), see page 2 of [7]. Thus the curved $\mathbb{M}^3$ in the second case of Riemannian metric spaces illustrated in Fig. 1c of Fig. 3, is an absolute space with respect to observers in the underlying Euclidean 3-space $\mathbb{E}^3$. It shall be re-denoted by $\hat{\mathbb{M}}^3$ with curved global absolute 'dimensions' $\hat{\eta}^1, \hat{\eta}^2$ and $\hat{\eta}^3$.

The different local coordinate sets in the absolute '3-space' $\hat{\mathbb{M}}^3$ shall likewise be denoted by, $(\hat{u}^1, \hat{u}^2, \hat{u}^3), (\hat{v}^1, \hat{v}^2, \hat{v}^3)$, $(\hat{w}^1, \hat{w}^2, \hat{w}^3)$, etc. A hat label shall be used to denote absolute coordinates/absolute intrinsic coordinates and absolute parameters/ absolute intrinsic parameters uniformly in this paper. The curved absolute space $\hat{\mathbb{M}}^3$ introduced here is different from the controversial Newtonian absolute space, see page 2 of [7], also [8]. This is so, because the Newtonian absolute space that supports Newton's mechanics is not curved.

Now, the curved absolute space $\hat{\mathbb{M}}^3$ will project a flat hyper-surface—a flat three-dimensional space—to be denoted by $\varnothing\mathbb{E}'^3_{ab}$, along the horizontal, such that the extended curved global 'dimensions', $\hat{\eta}^1, \hat{\eta}^2$ and $\hat{\eta}^3$, of $\hat{\mathbb{M}}^3$ become projected as extended straight line global dimensions, $\eta'^1_{ab}, \eta'^2_{ab}$ and η'^3_{ab}, respectively of $\varnothing\mathbb{E}'^3_{ab}$, and the singular (or distinguished) local coordinate sets, $(\hat{\xi}^1_A, \hat{\xi}^2_A, \hat{\xi}^3_A)$, $(\hat{\xi}^1_B, \hat{\xi}^2_B, \hat{\xi}^3_B)$, $(\hat{\xi}^1_C, \hat{\xi}^2_C, \hat{\xi}^3_C)$, etc, at different positions, A, B, C, etc, on the curved absolute space $\hat{\mathbb{M}}^3$, become projected as singular (or distinguished)

local coordinate sets, $(\xi'^1_{A'\,ab}, \xi'^2_{A'\,ab}, \xi'^3_{A'\,ab})$, $(\xi'^1_{B'\,ab}, \xi'^2_{B'\,ab}, \xi'^3_{B'\,ab})$, $(\xi'^1_{C'\,ab}, \xi'^2_{C'\,ab}, \xi'^3_{C'\,ab})$, etc, at the corresponding positions, A', B', C', etc, on the projective flat space $\varnothing\mathbb{E}'^3_{ab}$.

In other words, the different local coordinate sets, $(\hat{u}^1_A, \hat{u}^2_A, \hat{u}^3_A)$, $(\hat{v}^1_A, \hat{v}^2_A, \hat{v}^3_A)$, $(\hat{w}^1_A, \hat{w}^2_A, \hat{w}^3_A)$, etc, all of which are equivalent to a singular (or a distinguished) coordinate set $(\hat{\xi}^1_A, \hat{\xi}^2_A, \hat{\xi}^3_A)$ at a point A on $\hat{\mathbb{M}}^3$, are projected as local coordinate sets, $(u'^1_{A'\,ab}, u'^2_{A'\,ab}, u'^3_{A'\,ab})$, $(v'^1_{A'\,ab}, v'^2_{A'\,ab}, v'^3_{A'\,ab})$, $(w'^1_{A'\,ab}, w'^2_{A'\,ab}, w'^3_{A'\,ab})$, etc, all of which are equivalent to a singular (or distinguished) local coordinate set $(\xi'^1_{A'\,ab}, \xi'^2_{A'\,ab}, \xi'^3_{A'\,ab})$ at the corresponding point A' on $\varnothing\mathbb{E}'^3_{ab}$, and this is true at every other corresponding pair of positions on $\mathbb{M}^3$ and $\varnothing\mathbb{E}'^3_{ab}$.

The projective 3-space $\mathbb{E}'^3_{ab}$ of $\hat{\mathbb{M}}^3$ in which all local coordinate sets (or local 'frames') at any given point of it are identical to a singular (or distinguished) coordinate set at the given point, is itself an absolute space like $\hat{\mathbb{M}}^3$ that projects it. The subscript "ab" on $\mathbb{E}'^3_{ab}$ and on its global 'dimensions', $\eta'^1_{ab}, \eta'^2_{ab}$ and η'^3_{ab}, and its distinguished 'frames', $(\xi'^1_{A'\,ab}, \xi'^2_{A'\,ab}, \xi'^3_{A'\,ab})$, $(\xi'^1_{B'\,ab}, \xi'^2_{B'\,ab}, \xi'^3_{B'\,ab})$, $(\xi'^1_{C'\,ab}, \xi'^2_{C'\,ab}, \xi'^3_{C'\,ab})$, etc, at different points A', B', C', etc, on $\varnothing\mathbb{E}'^3_{ab}$, is used to denote "absolute".

An absolute space remains an absolute space and an absolute parameter remains an absolute parameter in the context of absolute metric phenomenon that causes the curvature of $\hat{\mathbb{M}}^3$. Thus the curved absolute space $\hat{\mathbb{M}}^3$ invariantly projects the flat primed (or proper) absolute space $\mathbb{E}'^3_{ab}$ in the context of the absolute phenomenon that causes the curvature of $\hat{\mathbb{M}}^3$. This will ultimately be expressed as the invariance, $\eta'^1_{ab} = \hat{\eta}^1$; $\eta'^2_{ab} = \hat{\eta}^2$; $\eta'^3_{ab} = \hat{\eta}^3$; $\xi'^1_{A\,ab} = \hat{\xi}^1_A$; $\xi'^2_{A\,ab} = \hat{\xi}^2_A$; $\xi'^3_{A\,ab} = \hat{\xi}^3_A$; etc.

The projective flat absolute proper metric space $\mathbb{E}'^3_{ab}$ of the curved absolute metric space $\hat{\mathbb{M}}^3$, in which all local coordinate sets at any given point of it are identical to a singular (or distinguished) coordinate set at the given point, is certainly different from the observed flat relative proper (or physical) 3-space $\mathbb{E}'^3$ of the Galileo space $(\mathbb{E}'^3, \hat{c}_s \hat{t})$ of Fig. 1a that we started with in this section. Certainly different coordinate sets,

(x'^1, x'^2, x'^3), (y'^1, y'^2, y'^3), (z'^1, z'^2, z'^3), etc, of $\mathbb{E}'^3$ are distinct and there is Galilean relativity on $(\mathbb{E}'^3, \hat{c}_s \hat{t})$. The $\mathbb{E}'^3$ (denoted by Σ' in [1]), is the flat relative proper metric 3-space of the flat four-dimensional relative proper metric spacetime $(\Sigma', c_s t')$ on which the special theory of relativity (SR), involving affine spacetime coordinates (or frames) operate in that article.

The projective flat absolute proper 3-space $\mathbb{E}'^3_{ab}$ is not the space in which 3-observers are located, but the flat relative proper (or physical) metric 3-space $\mathbb{E}'^3$, denoted by Σ' in [1, 2, 3, 4]. It is therefore mandatory for us to prescribe the flat (or Euclidean) relative proper metric 3-space $\mathbb{E}'^3$ in which 3-observers are located alongside the projective flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}$ of the curved absolute metric 3-space $\hat{\mathbb{M}}^3$, such that $\mathbb{E}'^3_{ab}$ lies underneath (or is embedded in) $\mathbb{E}'^3$ along the horizontal. The curved absolute space $\hat{\mathbb{M}}^3$ invariantly projects the flat absolute proper 3-space along the horizontal in the new geometry.

The flat absolute proper metric space $\mathbb{E}'^3_{ab}$ that is invariantly projected along the horizontal by the curved absolute metric space $\hat{\mathbb{M}}^3$ has been given a prime label like the relative (or physical) proper Euclidean 3-space $\mathbb{E}'^3$ in which observers are located, which lies over it along the horizontal. The prime label shall be used to indicate proper (or classical) spaces, coordinates and parameters uniformly in this paper. Consequently $\mathbb{E}'^3_{ab}$ is to be referred to as absolute proper space, as done already above.

Contrary to Fig. 1c or Fig. 3 that we started with, in which a curved physical (or relative) proper 3-dimensional metric space $\mathbb{M}^3$, which is curved toward the absolute time 'dimension' along the vertical, lies above its projective relative (or physical) Euclidean space $\mathbb{E}^3$ (without prime label), the flatness of the original relative proper Euclidean 3-space $\mathbb{E}'^3$ (in Fig. 1a) shall be left unaffected by the evolution of the curved absolute space $\hat{\mathbb{M}}^3$, which lies above its projective flat absolute proper space $\mathbb{E}'^3_{ab}$, where $\mathbb{E}'^3_{ab}$ underlies $\mathbb{E}'^3$. Consequently Fig. 3 shall be modified as Fig. 4 temporarily, where the curved absolute metric space lies above its projective flat absolute proper metric space $\mathbb{E}'^3_{ab}$ that lies

underneath the original flat relative (or physical) proper metric space $\mathbb{E}'^3$ in this situation.

Now, as noted earlier, two distinct observable metric spaces of different metric tensors in Fig. 1c or Fig. 3, have evolved from the singular Galileo space of Fig. 1a, whereas such duality of observable metric spaces is not observed in nature. This has been remarked as a paradox raised by Fig. 1c or Fig. 3 earlier. The duplication of metric spaces in Fig. 1c or Fig. 3 has now become a triplication of metric spaces in Fig. 4. These are the curved absolute metric space $\hat{\mathbb{M}}^3$ with absolute Riemannian metric tensor, the relative (or physical) proper Euclidean metric 3-space $\mathbb{E}'^3$ with Euclidean metric tensor, in which observers are located, and the projective flat absolute proper metric space $\mathbb{E}'^3_{ab}$, also with Euclidean metric tensor in Fig. 4.

In order for the 3-observers to observe only the flat relative proper metric 3-space $\mathbb{E}'^3$ in which they are located in Fig. 4, so that the paradox noted above is resolved, the projective underlying absolute proper metric space $\mathbb{E}'^3_{ab}$ must be an intrinsic (i.e, a non-observable and non-detectable) space to observers in $\mathbb{E}'^3$. Thus $\mathbb{E}'^3_{ab}$ shall be referred to as absolute proper intrinsic metric space. The curved absolute metric space $\hat{\mathbb{M}}^3$ that projects the flat absolute proper intrinsic metric space $\mathbb{E}'^3_{ab}$ along the horizontal, must itself be an absolute intrinsic space. It shall therefore be renamed absolute intrinsic metric space consequently. Thus the non-observable absolute intrinsic metric space

$\hat{\mathbb{M}}^3$ projects the non-observable flat absolute proper intrinsic metric space $\mathbb{E}'^3_{ab}$ along the horizontal, leaving the flat relative (or physical) proper metric 3-space as the only observable space to 3-observers in it in Fig. 4.

It is natural to associate an absolute intrinsic time metric 'dimension' temporarily being denoted by $\hat{\eta}^0$ in Fig. 4, with the curved absolute intrinsic metric space $\hat{\mathbb{M}}^3$. The absolute intrinsic metric time dimension is not curved simultaneously from its vertical position with $\hat{\mathbb{M}}^3$, by the absolute intrinsic metric phenomenon that causes the curvature of $\hat{\mathbb{M}}^3$. Consequently $\hat{\eta}^0$ lies parallel to the absolute metric time 'dimension', $\hat{x}^0 (= \hat{c}_s \hat{t})$, along the vertical, as illustrated in Fig 4.

Thus one consequence of the fact deduced earlier that the metric tensor and the line element are both invariant with re-parametrization in Riemann geometry in which a curved metric 3-space that is curved onto the absolute time 'dimension' along the vertical (as a curved hyper-surface), lies above its projective Euclidean 3-space (as a flat hyper-surface) along the horizontal, in which the observers are located, illustrated in Fig. 1c or Fig. 3, is that such Riemann geometry is realizable on a curved non-observable and non-detectable absolute intrinsic metric space $\hat{\mathbb{M}}^3$, in which all local coordinate sets are equivalent to a singular (or distinguished) local absolute intrinsic coordinate sets, $(\hat{\eta}^1_A, \hat{\eta}^2_A, \hat{\eta}^3_A)$, $(\hat{\eta}^1_B, \hat{\eta}^2_B, \hat{\eta}^3_B)$, $(\hat{\eta}^1_C, \hat{\eta}^2_C, \hat{\eta}^3_C)$, etc, at different positions A, B, C, etc, on it.
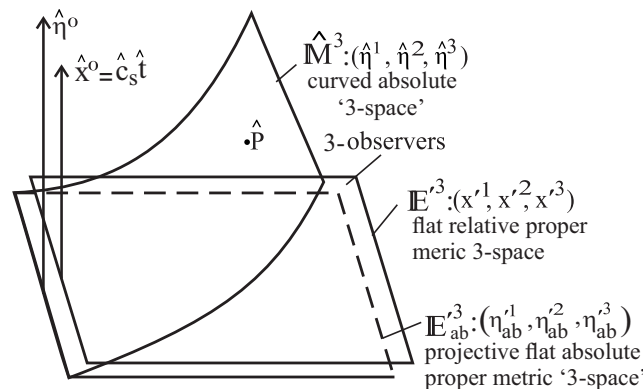

Fig. 4.

The curved absolute intrinsic metric space lies above its projective flat absolute proper intrinsic metric space $\mathbb{E}'^3_{ab}$, in which all local coordinate sets (or local 'frames') are equivalent to singular (or distinguished) local coordinate sets (or 'frames'), $(\eta'^1_{A'ab}, \eta'^2_{A'ab}, \eta'^3_{A'ab})$, $(\eta'^1_{B'ab}, \eta'^2_{B'ab}, \eta'^3_{B'ab})$, $(\eta'^1_{C'ab}, \eta'^2_{C'ab}, \eta'^3_{C'ab})$, etc, at the corresponding different positions A', B', C' etc, on $\mathbb{E}'^3_{ab}$, with respect to observers in the flat relative (or physical) proper metric 3-space $\mathbb{E}'^3$ that lies above $\mathbb{E}'^3_{ab}$ along the horizontal. The Riemann geometry on the curved absolute intrinsic metric space $\hat{\mathbb{M}}^3$, with respect to 3-observers in the underlying relative proper Euclidean metric 3-space $\mathbb{E}'^3$, shall be entitled absolute intrinsic Riemann geometry.

As the next step, let us adopt more appropriate notations and representations for the intrinsic spaces and the associated intrinsic time coordinates than used above. The notation $\hat{\mathbb{M}}^3(\hat{\eta}^1, \hat{\eta}^2, \hat{\eta}^3)$ for the curved absolute intrinsic metric space shall be replaced with $\varnothing\hat{\mathbb{M}}^3(\varnothing\hat{x}^1, \varnothing\hat{x}^2, \varnothing\hat{x}^3)$. The projective flat absolute proper intrinsic metric space $\mathbb{E}'^3_{ab}(\eta'^1_{ab}, \eta'^2_{ab}, \eta'^3_{ab})$ shall be re-denoted by $\varnothing\mathbb{E}'^3_{ab}(\varnothing x'^1_{ab}, \varnothing x'^2_{ab}, \varnothing x'^3_{ab})$. The extra subscript "ab" label is used to indicate "absolute" as noted earlier. Hence $\varnothing x'^1_{ab}$ is an absolute proper intrinsic space coordinate, while the prime label is used to denote "proper". The absolute intrinsic metric time 'dimension' $\hat{\eta}^0$ shall likewise be replaced by, $\varnothing\hat{x}^0 = \varnothing\hat{c}_s\varnothing\hat{t}$. By effecting these new notations in Fig. 4, we have Fig. 5. The non-observable and non-detectable (or hidden) intrinsic spaces have been shown with dotted boundaries in Fig. 5, as shall be done henceforth.

Since different local absolute proper intrinsic coordinate sets (or local absolute proper intrinsic 'frames'), $(\varnothing u'^1_{A'ab}, \varnothing u'^2_{A'ab}, \varnothing u'^3_{A'ab})$, $(\varnothing v'^1_{A'ab}, \varnothing v'^2_{A'ab}, \varnothing v'^3_{A'ab})$, $(\varnothing w'^1_{A'ab}, \varnothing w'^2_{A'ab}, \varnothing w'^3_{A'ab})$, etc, at a position A', say, in the projective absolute proper intrinsic metric space $\varnothing\mathbb{E}'^3_{ab}$ are equivalent to a singular (or distinguished) absolute proper intrinsic local coordinate set $(\varnothing\xi'^1_{A'ab}, \varnothing\xi'^2_{A'ab}, \varnothing\xi'^3_{A'ab})$, with respect to observers in the relative proper Euclidean metric 3-space $\mathbb{E}'^3$, natural laws in $\varnothing\mathbb{E}'^3_{ab}$ are naturally covariant with respect to observers in $\mathbb{E}'^3$. The fact that natural laws

on the curved absolute intrinsic metric space $\varnothing\hat{\mathbb{M}}^3$ that projects $\varnothing\mathbb{E}'^3_{ab}$ along the horizontal are naturally covariant with respect to observers in $\mathbb{E}'^3$ has been deduced in a similar manner earlier.

It is appropriate to mention the existence of the concepts of "intrinsic metric" and associated "intrinsic dimensions" of metric spaces [9], in the mathematical study of metric spaces. The distance between two points of a metric space relative to the intrinsic metric is defined as the infimum of the lengths of all paths from the first point to the second. If the space is such that there always exists a path that achieves the infimum of length (a geodesic) then it is a geodesic metric space. Intrinsic dimension is applied to data space in signal processing in information theory [10, 11, 12]. These mathematical concepts differ from the intrinsic Riemannian metric tensor $\varnothing\hat{g}_{\mu\nu}$ on absolute intrinsic Riemannian metric space $\varnothing\hat{\mathbb{M}}^3$ and absolute intrinsic metric dimensions, $\varnothing\hat{x}^1$, $\varnothing\hat{x}^2$ and $\varnothing\hat{x}^3$ of $\varnothing\hat{\mathbb{M}}^3$, being isolated in this paper.

Now the absolute metric time 'dimension', $\hat{x}^0 = \hat{c}_s\hat{t}$, is the outward manifestation of the absolute intrinsic metric time 'dimension', $\varnothing\hat{x}^0 = \varnothing\hat{c}_s\varnothing\hat{t}$, which lies parallel to, $\hat{x}^0 = \hat{c}_s\hat{t}$, along the vertical in Fig. 5. There is likewise the outward manifestation of the projective flat absolute proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3_{ab}(\varnothing x'^1_{ab}, \varnothing x'^2_{ab}, \varnothing x'^3_{ab})$ along the horizontal namely, the flat absolute proper metric space, which must be obtained by simply dropping the symbol $\varnothing$ from $\varnothing\mathbb{E}'^3_{ab}(\varnothing x'^1_{ab}, \varnothing x'^2_{ab}, \varnothing x'^3_{ab})$, giving $\mathbb{E}'^3_{ab}(x'^1_{ab}, x'^2_{ab}, x'^3_{ab})$.

As also first introduced as *Ansatz* in sub-section 4.4 of [1] and validated in sub-section 1.2 of [4], there is a flat relative proper intrinsic metric space that underlies the relative proper Euclidean 3-space $\mathbb{E}'^3(x'^1, x'^2, x'^3)$ namely, the flat relative proper intrinsic metric space, which must be obtained by incorporating the symbol $\varnothing$ into $\mathbb{E}'^3(x'^1, x'^2, x'^3)$ giving $\varnothing\mathbb{E}'^3(\varnothing x'^1, \varnothing x'^2, \varnothing x'^3)$.

The flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}(x'^1_{ab}, x'^2_{ab}, x'^3_{ab})$ and the flat relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3(\varnothing x'^1, \varnothing x'^2, \varnothing x'^3)$, must be incorporated into Fig. 5 in order to make

that diagram more complete. This must be done by letting the flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}$ to underlie (i.e. to be embedded in) the flat relative proper metric 3-space $\mathbb{E}'^3$ and the projective absolute proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3_{ab}$ to underlie (i.e. to be embedded in) the flat relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3$. Thus Fig. 5 must be replaced by the more complete Figs. 6a and 6b.

Figs. 6a and 6b are actually one diagram. They are separated for clarity only. The flat relative proper metric 3-space $\mathbb{E}'^3$ in Fig. 6b is the outward manifestation of the flat relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3$ in Fig. 6a and the flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}$ in Fig. 6b is the outward manifestation of the flat absolute proper intrinsic metric space $\varnothing\mathbb{E}'^3_{ab}$ in Fig. 6a. The observers are the 3-observers located on the relative proper metric space $\mathbb{E}'^3$ in Fig. 6b.

The flat absolute and relative proper intrinsic metric spaces in Fig. 6a have outward manifestations, while the curved absolute intrinsic metric 3-space $\varnothing\hat{\mathbb{M}}^3$ in Fig. 6a does not have an outward manifestation. A fundamental explanation of this must be sought, but for now, the outward manifestation of the curved $\varnothing\hat{\mathbb{M}}^3$ namely, $\hat{\mathbb{M}}^3$ in Fig. 4, is what has been converted

to $\varnothing\hat{\mathbb{M}}^3$ to give Fig. 5. The fact that outward manifestation of the curved $\varnothing\hat{\mathbb{M}}^3$ is not required in the absolute intrinsic Riemann geometry shall be shown shortly in this section.

The flat absolute proper metric space $\mathbb{E}'^3_{ab}$ is actually imperceptibly embedded in the flat relative proper metric space $\mathbb{E}'^3$ in Fig. 6b, the two thereby appearing as $\mathbb{E}'^3$ to observers in $\mathbb{E}'^3$ and $\varnothing\mathbb{E}'^3_{ab}$ is imperceptibly embedded in $\varnothing\mathbb{E}'^3$ in Fig. 6a, the two thereby appearing as $\varnothing\mathbb{E}'^3$ with respect to observers in $\mathbb{E}'^3$. The spaces in Figs. 6a and 6b are all that is required to develop absolute intrinsic Riemann geometry on the curved absolute intrinsic metric space $\varnothing\hat{\mathbb{M}}^3(\varnothing\hat{x}^1, \varnothing\hat{x}^2, \varnothing\hat{x}^3)$ with respect to observers in the relative proper Euclidean 3-space $\mathbb{E}'^3$ in this article and the subsequent two articles.

The fact that the absolute metric time coordinate, $\hat{x}^0 = \hat{c}_s\hat{t}$, and the absolute intrinsic metric time coordinate, $\varnothing\hat{x}^0 = \varnothing\hat{c}_s\varnothing\hat{t}$, do not evolve with respect to 3-observers in $\mathbb{E}'^3$ in the context of the absolute intrinsic metric phenomenon that gives rise to the curved of $\varnothing\hat{\mathbb{M}}^3$ in Figs. 5 and 6a, are shown by allowing, $\hat{x}^0 = \hat{c}_s\hat{t}$ and $\varnothing\hat{x}^0 = \varnothing\hat{c}_s\varnothing\hat{t}$, to remain unchanged and not curved from their vertical positions in those diagrams.
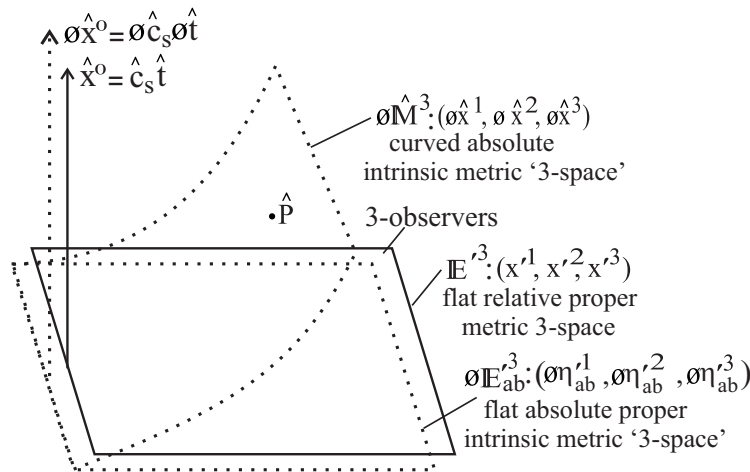

Fig. 5. The '3-dimensional' absolute intrinsic metric space curving toward the absolute intrinsic time 'dimension' along the vertical, projects flat 3-dimensional absolute proper intrinsic metric space, which lies underneath (or is embedded in) the flat relative proper metric 3-space along the horizontal.
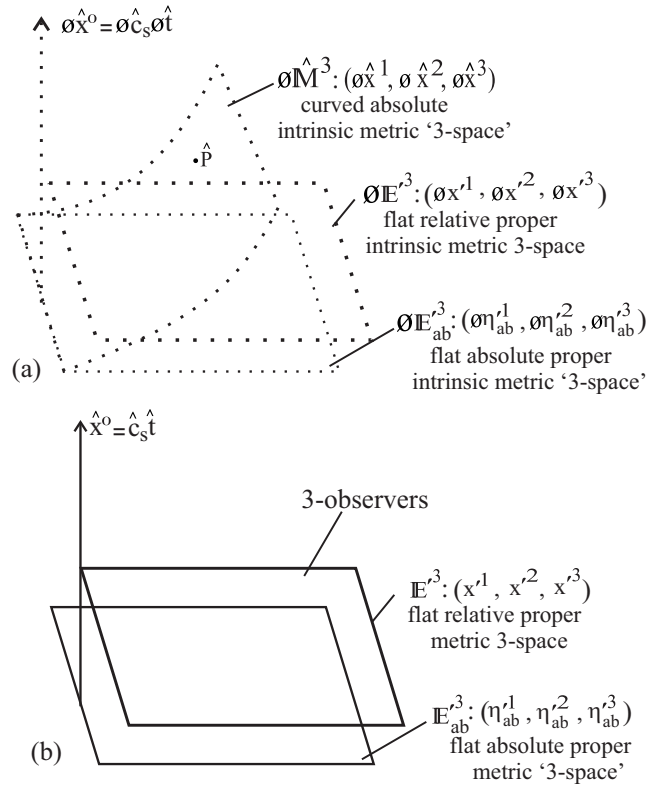
Fig. 6. Figure 5 made more complete by including absolute proper metric '3-space' $\mathbb{E}'^3_{ab}$ as outward manifestation of $\varnothing\mathbb{E}'^3_{ab}$ in Fig. 5 and the relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3$ that underlies $\mathbb{E}'^3$ in Fig. 5.

The fact that the flat 3-dimensional relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3$ with respect to 'intrinsic 3-observers' in it is naturally contracted to a one-dimensional (straight line) relative proper intrinsic metric space, denoted by $\varnothing\rho'$, with respect to 3-observers in the relative proper metric 3-space $\mathbb{E}'^3$ overlying $\varnothing\mathbb{E}'^3$, has been mentioned in sub-section 4.4 of [1], as illustrated in Figs. 6a and 6b of that article. That fact is applied in Fig. 7 of that article and all diagrams in the [2, 3, 4] that follow [1]. Figure 2a of [4] also shows that the one-dimensional scalar relative proper metric space $\rho^{0\prime}$ of the positive time-universe along the vertical, projects one-dimensional relative proper intrinsic metric space $\varnothing\rho'$ into the relative proper metric Euclidean 3-space $\mathbb{E}'^3$ of our universe (as a hyper-surface) along the horizontal.

However the three-dimensionality of $\varnothing\mathbb{E}'^3$ shall be preserved in this paper and shown to be one-dimensional intrinsic metric space $\varnothing\rho'$ with respect to 3-observers in $\mathbb{E}'^3$, by another formal procedure elsewhere. The $\varnothing\mathbb{E}'^3$ shall then be replace by $\varnothing\rho'$.

The following features of the new notations in Fig. 5 make them more appropriate than those in Fig. 4:

1. Apart from the attachment of the symbol $\varnothing$ to the usual coordinates, no new symbol has been introduced to represent the intrinsic coordinates. This minimizes the number of symbols that enters into the theory, which is aesthetically desirable.

2. The fact that the flat relative (or physical) proper metric 3-space $\mathbb{E}'^3$ is the outward manifestation of the flat relative proper

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intrinsic metric 3-space $\varnothing\mathbb{E}'^3$ and the flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}$ is the outward manifestation of the flat absolute proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3_{ab}$, can be seen from the new notations. For if we remove the symbol $\varnothing$ from $\varnothing\mathbb{E}'^3_{ab}(\varnothing x'^1_{ab}, \varnothing x'^2_{ab}, \varnothing x'^3_{ab})$ we obtain $\mathbb{E}'^3_{ab}(x'^1_{ab}, x'^2_{ab}, x'^3_{ab})$, which must be interpreted as: $\mathbb{E}'^3_{ab}$ is the outward manifestation of $\varnothing\mathbb{E}'^3_{ab}$. Likewise dropping the symbol $\varnothing$ from $\varnothing\mathbb{E}'^3$ gives $\mathbb{E}'^3$, which must be interpreted as: $\mathbb{E}'^3$ is the outward manifestation of $\varnothing\mathbb{E}'^3$ and x'^1 is the outward manifestation of $\varnothing x'^1$, etc. Also if we remove the symbol $\varnothing$ from, $\varnothing\hat{x}^0 \equiv \varnothing\hat{c}_s\varnothing\hat{t}$, we have $\hat{x}^0 \equiv \hat{c}_s\hat{t}$.

The fact that the absolute proper metric space $\mathbb{E}'^3_{ab}$ is the outward manifestation of the absolute proper intrinsic metric space $\varnothing\mathbb{E}'^3_{ab}$ and, hence that, the relative (or physical) proper metric 3-space $\mathbb{E}'^3$ is the outward manifestation of the relative proper intrinsic metric space $\varnothing\mathbb{E}'^3$ in Figs. 6a and 6b, cannot be easily seen or demonstrated with other notations, such as the one adopted initially and illustrated in Fig. 4.

3. Following the derivation of the two-dimensional relative proper intrinsic metric spacetime (also referred to as relative proper metric nospace-notime), which underlies the flat four-dimensional relative proper metric spacetime $(\Sigma', c_s t')$ in sub-section 1.2 of [4], the symbol $\varnothing$ attached to the intrinsic coordinates in the new notations has the meaning of 'void', 'null' or 'nothing'. Thus $\varnothing$space can be referred to as 'void-space' or 'null-space', but 'nospace' has been preferred, as discussed in sub-section 1.2 of [4]. Any distance $\varnothing d'$ of proper intrinsic space (or proper nospace) $\varnothing\mathbb{E}'^3$ is equivalent to zero distance of the proper physical Euclidean metric 3-space $\mathbb{E}'^3$. This can be seen directly from the symbol $\varnothing$ attached to $\varnothing d'$, with the meaning of 'void', 'null' or 'nothing', whereas the fact that an interval of intrinsic space $\Delta\eta'$ in the notation of Fig. 4 is equivalent to zero distance of the physical 3-space cannot be seen directly. The fact that any interval

of intrinsic space is equivalent to zero interval of physical space makes it non-detectable to observers in the physical space. The symbol $\varnothing$ attached to a space or coordinate or a physical parameter is used to indicate that the space or coordinate or parameter is intrinsic, that is, non-detectable (or hidden) to observers in the relative proper Euclidean 3-space $\mathbb{E}'^3$.

4. The proper intrinsic coordinates, $\varnothing x'^1, \varnothing x'^2, \varnothing x'^3$ and $\varnothing c_s\varnothing t'$, of the relative proper intrinsic metric spacetime $(\varnothing\mathbb{E}'^3, \varnothing c_s\varnothing t')$, must be deemed to have been formally derived following the formal derivation of the two-dimensional relative proper intrinsic metric spacetime that underlies the flat four-dimensional relative proper metric spacetime $(\mathbb{E}'^3, c_s t')$ in sub-section 1.2 of [4], from which it is clear that these new intrinsic coordinates and their notations are not arbitrary creations. The new notations for the new intrinsic spacetime coordinates in Fig. 5 and its more complete form of Figs. 6a and 6b are the natural notations.

True to the title of this section, we can only talk of a new absolute intrinsic Riemannian metric space and the associated new absolute intrinsic Riemann geometry, which are being unearthed in this article. The observers with respect to whom the new geometry is valid are all 3-observers in the underlying flat (or Euclidean) relative proper metric 3-space $\mathbb{E}'^3$, as mentioned above.

Now in the absence of the curved absolute intrinsic metric space $\varnothing\hat{\mathbb{M}}^3$ (or in the absence of absolute intrinsic Riemann geometry), due to the absence of a long-range absolute intrinsic metric force field, the curved absolute intrinsic metric 3-space $\varnothing\hat{\mathbb{M}}^3$ in Fig. 5, or its more complete form Fig. 6a, becomes the flat absolute intrinsic metric 3-space $\varnothing\hat{\mathbb{E}}^3$, which is made manifested outwardly in the flat absolute metric 3-space $\hat{\mathbb{E}}^3$, obtained by simply dropping the symbol $\varnothing$ in $\varnothing\hat{\mathbb{E}}^3$. Thus in the absence of absolute intrinsic Riemann geometry (or in the absence of absolute intrinsic metric force field), the more complete Figs. 6a and 6b must be replaced by Fig. 7.

Fig. 7 is the reference geometry in the absence of absolute intrinsic metric force field (or absence

of absolute intrinsic Riemann geometry). The flat relative proper metric 3-space $\mathbb{E}'^3$ and its underlying flat relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3$ in Figs. 6a and 6b are absent in the reference flat absolute metric spacetime and its underlying flat absolute intrinsic metric spacetime geometry of Fig. 7. The only possible 3-observers in Fig. 7 are hypothetical 'absolute 3-observers' on the flat absolute 3-space $\hat{\mathbb{E}}^3$.

The location of the source of a long-range absolute metric force field at a point on the extended flat absolute metric space $\hat{\mathbb{E}}^3$ and the consequent automatic location of the source of the long-range absolute intrinsic metric force field on $\varnothing\hat{\mathbb{E}}^3$ underneath the source in $\hat{\mathbb{E}}^3$, will cause the absolute intrinsic metric space $\varnothing\hat{\mathbb{E}}^3$ to be curved toward the absolute intrinsic metric time 'dimension' $\varnothing\hat{c}_s\varnothing\hat{t}$ along the vertical, thereby becoming the curved absolute intrinsic metric 3-space $\varnothing\hat{\mathbb{M}}^3$ (as a curved hyper-surface), as illustrated in Fig. 6a, in all finite neighborhood of the sources. The curved $\varnothing\hat{\mathbb{M}}^3$ will project a flat absolute proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3_{ab}$ (as a flat hyper-surface) along the horizontal, which will be made manifested outwardly in flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}$ overlying $\varnothing\mathbb{E}'^3_{ab}$ along the horizontal.

It is important to note that the flat absolute metric space $\hat{\mathbb{E}}^3$ in Fig. 7 is not required to be curved

toward the absolute metric time coordinate $\hat{c}_s\hat{t}$ along the vertical to thereby project the flat absolute proper metric 3-space $\mathbb{E}'^3_{ab}$ along the horizontal, in what would have been absolute Riemannian (and not absolute intrinsic Riemann) geometry on curved absolute metric '3-space'. Rather the flat $\mathbb{E}'^3_{ab}$ evolves as the outward manifestation of the flat $\varnothing\mathbb{E}'^3_{ab}$ projected by the curved $\varnothing\hat{\mathbb{M}}^3$ in the context of absolute intrinsic Riemann geometry. The flat $\mathbb{E}'^3_{ab}$ that thus evolves replaces the original flat absolute space $\hat{\mathbb{E}}^3$ in Fig. 7 without any need for $\hat{\mathbb{E}}^3$ to be curved. Absolute Riemann geometry involving curved absolute metric '3-space' $\hat{\mathbb{M}}^3$ (as outward manifestation of the curved $\varnothing\hat{\mathbb{M}}^3$) with absolute metric tensor $\hat{g}_{\mu\nu}$ does not exist consequently.

The evolutions of the curved absolute intrinsic metric $\hat{\mathbb{M}}^3$ with absolute intrinsic metric tensor $\varnothing\hat{g}_{\mu\nu}$, its projective flat absolute proper intrinsic metric space $\varnothing\mathbb{E}'^3_{ab}$ and the flat absolute proper metric space $\mathbb{E}'^3_{ab}$ as the outward manifestation of $\varnothing\mathbb{E}'^3_{ab}$, in the context of absolute intrinsic Riemann geometry, as described above, will occur along with the automatic appearance of the flat relative (or physical) proper metric space $\mathbb{E}'^3$ (as the relative counterpart of $\mathbb{E}'^3_{ab}$) and the automatic appearance of the flat relative proper intrinsic metric space $\varnothing\mathbb{E}'^3$ (as the relative counterpart of $\varnothing\mathbb{E}'^3_{ab}$).
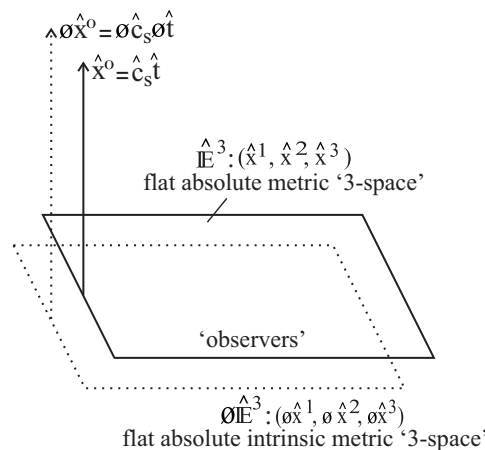

Fig. 7. Flat '3-dimensional' absolute metric space - absolute metric time is underlay by flat '3-dimensional' absolute intrinsic metric space - absolute intrinsic metric time.

It is mandatory to allow the flat relative proper metric 3-space $\mathbb{E}'^3$ and its underlying flat relative proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3$ to appear automatically along with the projective flat absolute proper intrinsic metric 3-space $\varnothing\mathbb{E}'^3_{ab}$ and its outward manifestation $\mathbb{E}'^3_{ab}$, as illustrated in Figs. 6a and 6b, lest there will be no space for the observers. The origin of $\mathbb{E}'^3$ and its underlying $\varnothing\mathbb{E}'^3$ shall be explored with further development of the absolute intrinsic Riemann geometry elsewhere.

As follows from the preceding two paragraphs and the discussions in paragraphs leading to them, the geometry of Fig. 7 will evolve into the geometry of Figs. 6a and 6b, with respect to 3-observers in the relative proper metric space $\mathbb{E}'^3$, in the context of absolute intrinsic Riemann geometry, in all finite neighborhood of a long-range absolute metric force field/absolute intrinsic metric force field. This will happen with the location of the source of the long-range absolute metric force-field at a point on the flat absolute metric space $\hat{\mathbb{E}}^3$ in Fig. 7. Figure 7 shall be referred to as the reference geometry to absolute intrinsic Riemann geometry of Figs. 6a and 6b.

However the absolute intrinsic Riemann geometry of Figs. 6a and 6b, with respect to 3-observers in the relative proper metric space $\mathbb{E}'^3$ (with respect to whom the absolute time 'dimension' $\hat{c}_s\hat{t}$ and absolute intrinsic time 'dimension' $\varnothing\hat{c}_s\varnothing\hat{t}$ do not evolve), is half of the geometry that evolves from the reference geometry of Fig. 7. There is a second half of the geometry with respect to 1-observers in the relative proper metric time dimension c_st' (with respect to whom the absolute metric space $\hat{\mathbb{E}}^3$ and absolute intrinsic metric space $\varnothing\hat{\mathbb{E}}^3$ in Fig. 7 do not evolve).

The full absolute intrinsic Riemannian metric spacetime geometry that evolves upon the reference geometry of Fig. 7, to be obtained by combining the half geometry of Figs. 6a and 6b with respect to 3-observers on the flat relative proper metric space $\mathbb{E}'^3$ and the half-geometry with respect to 1-observers in the relative proper metric time dimension c_st', shall be presented elsewhere.

5 CONCLUSION

This article exposes the existence of 'three-dimensional' absolute intrinsic Riemannian metric space in addition to the conventional three-dimensional Riemannian metric space and the development of absolute intrinsic Riemannian geometry on absolute intrinsic Riemannian metric space is commenced. The absolute intrinsic Riemannian metric space geometry contains the curved absolute intrinsic metric space $\varnothing\hat{\mathbb{M}}^3$, its projective flat absolute proper intrinsic metric space $\varnothing\mathbb{E}'^3_{ab}$ with its outward manifestation $\mathbb{E}'^3_{ab}$, in addition to the flat relative (or physical) proper metric space $\mathbb{E}'^3$ and its underlying flat relative proper intrinsic metric space $\varnothing\mathbb{E}'^3$. The observers are located in $\mathbb{E}'^3$. The resulting absolute intrinsic Riemannian metric space geometry that includes flat relative proper metric space is a more all-encompassing geometry than the conventional Riemannian metric space geometry with only a curved relative proper metric space $\mathbb{M}'^3$. There is also prospect for extension to a more all-encompassing absolute intrinsic Riemannian metric spacetime geometry that includes flat relative proper metric spacetime in the new scheme than the conventional Riemannian metric spacetime geometry with only a curved relative metric spacetime.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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