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Inverse Spectral Problem for Sturm-Liouville Operator with Boundary and Jump Conditions Dependent on the Spectral Parameter

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Abstract

In this paper, the inverse spectral problem of Sturm-Liouville operator with boundary conditions and jump conditions dependent on the spectral parameter is investigated. Firstly, the self-adjointness of the problem and the eigenvalue properties are given, then the asymptotic formulas of eigenvalues and eigenfunctions are presented. Finally, the uniqueness theorems of the corresponding inverse problems are given by Weyl function theory and inverse spectral data approach.

Keywords

Inverse Problem, Sturm-Liouville Operator, Weyl Function, Eigenparameter-Dependent Jump Conditions

1. Introduction

Inverse spectral problems are motivated to recovering operators from the priori known spectral characteristics. These problems often appear in mathematics, physics, mechanics, electronics, and some other sciences and engineering problems, and, hence, are very important to understanding the real world. Significant progress has been made in the inverse problem theory for regular self-adjoint or nonself-adjoint Sturm-Liouville operators [1] [2] [3] [4].

The inverse problem of Sturm-Liouville operator was initiated by Ambarzumian [5] and Borg [6], after that, there are various generalizations on the inverse problems of Sturm-Liouville operators. Besides the classical regular Sturm-Liouville operators [2] [3], in recent years there have been a lot of inverse problems for Sturm-Liouville operators with eigenparameter-dependent boundary conditions and Sturm-Liouville operators with transmission conditions [7]-[10].

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Fulton has studied the inverse spectral problem with boundary conditions linearly dependent on the spectral parameter [7]. Binding *et al.* discussed boundary conditions that depend nonlinearly on the spectral parameter [9]. Hald has studied the discontinuous Sturm-Liouville problem and shown the direct and inverse spectral theory on the Sturm-Liouville problem with internal discontinuous point conditions [10]. The corresponding direct problems of boundary value problems with transmission conditions and/or eigenparameter-dependent boundary conditions, we refer to [20]-[25] and the references therein.

Recent years, the boundary value problems with eigenparameter-dependent transmission conditions have drown scholars' much attention and have achieved significant progress, including direct and inverse spectral theory and half inverse spectral theory [26]-[35]. In 2005, Akdogan *et al.* investigated the discontinuous Sturm-Liouville problems, where the spectral parameter not only appears in differential equations, but also in boundary conditions and one of the jump conditions, they got the asymptotic approximation of fundamental solutions and the asymptotic formulae for eigenvalues of such problems [27]. In 2012, Ozkan *et al.* considered the spectral problems for Sturm-Liouville operator with both boundary and one of the jump conditions linearly dependent on the eigenparameter, and studied the inverse problem of this operator [28]. In 2014, Guo *et al.* investigated the inverse spectral problem of Sturm-Liouville operator with finite number of jump conditions dependent on the eigenparameter [29]. In 2016, Wei *et al.* investigated the inverse spectral problem for Dirac operator with boundary and jump conditions dependent on the spectral parameter. Through inducting the generalized normal constants they have proved the uniqueness theorem, then a construction method for solving this inverse problem was given [30]. In 2018 and 2021, Bartels *et al.* presented Sturm-Liouville problems with transfer condition Herglotz dependent on the eigenparameter, and showed the Hilbert space formulation of the problem and calculated out the eigenvalue and eigenfunction asymptotic formula on this problem [31] [34]. Zhang *et al.* studied the finite spectrum of Sturm-Liouville problems with both jump conditions dependent on the spectral parameter [35].

Since then the Sturm-Liouville problems with jump conditions containing the spectral parameter have been widely studied, however, for the problems with both jump conditions containing the spectral parameter attach less attention, which often appear in heat transfer, electronic signal amplifiers and other issues of sciences, hence have high research significance. It's also a good complement to the study of spectral and inverse spectral problems of boundary value problems of differential equations.

In this paper, we mainly investigate the inverse spectral problem of Sturm-Liouville operator in which the spectral parameter not only appears in the differential equation, but also appears in both of the jump conditions and the boundary conditions. While the spectral parameter appears in equation and boundary conditions and transmission conditions, hence it is much complicated. The studies on such problems play an important role in differential equations and spectral theory. To show the inverse spectral theory of this problem, the operator formulation of this problem is constructed and some spectral properties are given, next the asymptotic behavior of the solutions and eigenvalues is provided, then several uniqueness results for this inverse spectral problem are given. The uniqueness theorem is very important in inverse spectral theory of boundary value problems, and there are many approaches to solve the uniqueness theorem. In this paper, we will use three general methods to solve the uniqueness theorem, which are equivalent to each other, *i.e.* the Weyl function theory, two spectra and spectral data approach. To analyze this inverse spectral problem, the dependence of eigenvalues of such problems is the theoretical basis of it, and the corresponding results the reader may refer to [36].

2. Notation and Basic Properties

Consider the following boundary value problem (denoted by *L*) consisting of the following Sturm-Liouville equation

$$l(y) := -y'' + q(x)y = \lambda y,\ x \in J = [0,c) \cup (c,\pi], \tag{2.1}$$

together with boundary conditions (BCs)

$$l_1(y) := \lambda\big(\alpha_1 y(0) + \alpha_2 y'(0)\big) - \big(\alpha_3 y(0) + \alpha_4 y'(0)\big) = 0, \tag{2.2}$$

$$l_2(y) := \lambda\big(\beta_1 y(\pi) + \beta_2 y'(\pi)\big) - \big(\beta_3 y(\pi) + \beta_4 y'(\pi)\big) = 0, \tag{2.3}$$

and jump conditions with spectral parameter

$$y(c^-) + (\lambda\eta_1 - \xi_1)y'(c^-) + y'(c^+) = 0, \tag{2.4}$$

$$y'(c^-) - y(c^+) + (\lambda\eta_2 - \xi_2)y'(c^+) = 0, \tag{2.5}$$

where $q(x) \in L_2(J)$ is real valued, $0 < c < \pi$, $\alpha_i, \beta_i, \eta_k, \xi_k \in \mathbb{R}$, $\eta_k > 0$, $d_1 = \alpha_2\alpha_3 - \alpha_1\alpha_4 > 0$, $d_2 = \beta_1\beta_4 - \beta_2\beta_3 > 0$, $k = 1, 2$, $i = 1, 2, 3, 4$. Here λ is a spectral parameter.

In order to describe the self-adjointness of the operator corresponding to the problem *L*, firstly, let us consider the set associated with the functions considered in the present paper as

$$U = \left\{ y \in L^2(J) : y, y' \in AC_{loc}(J), l(y) \in L^2(J) \right\},$$

where $AC_{loc}(J)$ denotes all local absolutely continuous functions on *J*, then we can introduce an inner product in the Hilbert space $\mathcal{H} := L^2(J) \oplus \mathbb{C}^4$ as

$$(F,G) = \int_0^c f\overline{g}\,\mathrm{d}x + \int_c^\pi f\overline{g}\,\mathrm{d}x + \frac{1}{d_1} f_1\overline{g}_1 + \frac{1}{d_2} f_2\overline{g}_2 + \eta_1 f_3\overline{g}_3 + \eta_2 f_4\overline{g}_4, \tag{2.6}$$

for arbitrary

$$F = (f, f_1, f_2, f_3, f_4)^{\mathrm{T}}, G = (g, g_1, g_2, g_3, g_4)^{\mathrm{T}} \in \mathcal{H}.$$

To facilitate the description, the following notation need to be listed. For $y \in U$, let

$$\tilde{M}_1(y) = \alpha_3 y(0) + \alpha_4 y'(0),\ M_1(y) = \alpha_1 y(0) + \alpha_2 y'(0),$$

$$\tilde{M}_2(y)=\beta_3 y(\pi)+\beta_4 y'(\pi),\ M_2(y)=\beta_1 y(\pi)+\beta_2 y'(\pi),$$

$$\tilde{M}_3(y)=\frac{1}{\eta_1}\left[\xi_1 y'(c^-)-y(c^-)-y'(c^+)\right], M_3(y)=y'(c^-),$$

$$\tilde{M}_4(y)=\frac{1}{\eta_2}\left[\xi_2 y'(c^+)+y(c^+)-y'(c^-)\right], M_4(y)=y'(c^+),$$

then the boundary conditions (2.2), (2.3) and jump conditions (2.4), (2.5) can be written as

$$\tilde{M}_1(y)=\lambda M_1(y),\ \tilde{M}_2(y)=\lambda M_2(y),\ \tilde{M}_3(y)=\lambda M_3(y),\ \tilde{M}_4(y)=\lambda M_4(y).$$

In the Hilbert space $\mathcal{H}$ we define a linear operator $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ as

$$\mathcal{A}F=\begin{pmatrix} l(f) \\ \tilde{M}_1(f) \\ \tilde{M}_2(f) \\ \tilde{M}_3(f) \\ \tilde{M}_4(f) \end{pmatrix}=\begin{pmatrix} -f''+qf \\ \alpha_3 f(0)+\alpha_4 f'(0) \\ \beta_3 f(\pi)+\beta_4 f'(\pi) \\ \frac{1}{\eta_1}\left[\xi_1 f'(c^-)-f(c^-)-f'(c^+)\right] \\ \frac{1}{\eta_2}\left[\xi_2 f'(c^+)+f(c^+)-f'(c^-)\right] \end{pmatrix}, \tag{2.7}$$

and the domain of the operator $\mathcal{A}$ as

$$\begin{aligned} D(\mathcal{A}) := \{ & F=\left(f(x), f_1, f_2, f_3, f_4\right)^{\mathrm{T}}\in\mathcal{H}: f(x)\in U, \text{and have finite limits} \\ & f(c^\pm)=\lim_{x\to c\pm 0} f(x),\ f'(c^\pm)=\lim_{x\to c\pm 0} f'(x), \\ & f_1=M_1(f),\ f_2=M_2(f),\ f_3=M_3(f),\ f_4=M_4(f)\}. \end{aligned}$$

Thus, the problem L can be written as the following form

$$\mathcal{A}F=\lambda F,$$

where $F=(f, f_1, f_2, f_3, f_4)^{\mathrm{T}}\in D(\mathcal{A})$.

Then it can be proven that the following theorem about the self-adjointness of the operator $\mathcal{A}$ holds.

Theorem 1. [27] *The linear operator* $\mathcal{A}$ *is self-adjoint in the Hilbert space* $\mathcal{H}$.

Define two fundamental solutions $\varphi(x,\lambda), \chi(x,\lambda)$ of Equation (2.1) on whole $[0,c)\cup(c,\pi]$ satisfying the jump conditions (2.4), (2.5) and the following initial conditions, respectively

$$\begin{pmatrix} \varphi(0,\lambda) \\ \varphi'(0,\lambda) \end{pmatrix}=\begin{pmatrix} -\lambda\alpha_2+\alpha_4 \\ \lambda\alpha_1-\alpha_3 \end{pmatrix},\ \begin{pmatrix} \chi(\pi,\lambda) \\ \chi'(\pi,\lambda) \end{pmatrix}=\begin{pmatrix} -\lambda\beta_2+\beta_4 \\ \lambda\beta_1-\beta_3 \end{pmatrix}.$$

Since these solutions $\varphi(x,\lambda)$ and $\chi(x,\lambda)$ satisfy the jump conditions (2.4) and (2.5), the following relations

$$\begin{pmatrix} \varphi(c^+,\lambda) \\ \varphi'(c^+,\lambda) \end{pmatrix}=\begin{pmatrix} (1-a_\lambda b_\lambda)\varphi'(c^-,\lambda)-b_\lambda\varphi(c^-,\lambda) \\ -a_\lambda\varphi'(c^-,\lambda)-\varphi(c^-,\lambda) \end{pmatrix},$$

$$\begin{pmatrix} \chi(c^-,\lambda) \\ \chi'(c^-,\lambda) \end{pmatrix}=\begin{pmatrix} -a_\lambda\chi(c^+,\lambda)+(a_\lambda b_\lambda-1)\chi'(c^+,\lambda) \\ \chi(c^+,\lambda)-b_\lambda\chi'(c^+,\lambda) \end{pmatrix}$$

hold, where $a_\lambda = \lambda\eta_1 - \xi_1, b_\lambda = \lambda\eta_2 - \xi_2$.

For each $x \in J$, these solutions satisfy the relation $l_1(\varphi) = l_2(\chi) = 0$. Then the characteristic function can be introduced as

$$\Delta(\lambda) = \langle \varphi(x,\lambda), \chi(x,\lambda) \rangle = \varphi(x,\lambda)\chi'(x,\lambda) - \varphi'(x,\lambda)\chi(x,\lambda), \tag{2.8}$$

according to the Liouville's theorem, the Wronskian $\langle \varphi(x,\lambda), \chi(x,\lambda) \rangle$ is an entire function in λ and the zeros namely λ_n of $\Delta(\lambda)$ coincide with the eigenvalues of the problem *L*. Substituting $x = \pi$ into (2.8) we get

$$\Delta(\lambda) = \lambda\left(\beta_1\varphi(\pi,\lambda) + \beta_2\varphi'(\pi,\lambda)\right) - \left(\beta_3\varphi(\pi,\lambda) + \beta_4\varphi'(\pi,\lambda)\right). \tag{2.9}$$

The normal constants ρ_n of the problem *L* can be defined as follows

$$\begin{aligned} \rho_n = &\int_0^c \varphi^2(x,\lambda_n)\mathrm{d}x + \int_c^\pi \varphi^2(x,\lambda_n)\mathrm{d}x + \frac{1}{d_1}\left(\alpha_1\varphi(0,\lambda_n) + \alpha_2\varphi'(0,\lambda_n)\right)^2 \\ &+ \frac{1}{d_2}\left(\beta_1\varphi(\pi,\lambda_n) + \beta_2\varphi'(\pi,\lambda_n)\right)^2 + \eta_1\varphi'^2(c^-,\lambda_n) + \eta_2\varphi'^2(c^+,\lambda_n). \end{aligned} \tag{2.10}$$

If the functions $\varphi(x,\lambda_n)$ and $\chi(x,\lambda_n)$ are the eigenfunctions of the problem *L*, then there exists a sequence $\{\varpi_n\}$ such that

$$\chi(x,\lambda_n) = \varpi_n\varphi(x,\lambda_n), \text{ where } \varpi_n = -\frac{\chi(0,\lambda_n)\alpha_1 + \chi'(0,\lambda_n)\alpha_2}{d_1}. \tag{2.11}$$

Theorem 2. *Let* λ_n *be the zeros of the function* $\Delta(\lambda)$, *then*

$$\dot{\Delta}(\lambda_n) = \varpi_n\rho_n. \tag{2.12}$$

where $\dot{\Delta}(\lambda) = \frac{\mathrm{d}\Delta}{\mathrm{d}\lambda}$, *and* ρ_n, ϖ_n *are defined by* (*2.10*) *and* (*2.11*), *respectively.*

Proof. Let us write the following equations

$$-\chi''(x,\lambda) + q(x)\chi(x,\lambda) = \lambda\chi(x,\lambda), \tag{2.13}$$

$$-\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) = \lambda_n\varphi(x,\lambda_n). \tag{2.14}$$

Let (2.13), (2.14) multiplied by $\varphi(x,\lambda_n)$ and $\chi(x,\lambda)$, respectively, and subtracting them, then the equality

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle \varphi(x,\lambda_n), \chi(x,\lambda) \rangle = (\lambda_n - \lambda)\varphi(x,\lambda_n)\chi(x,\lambda) \tag{2.15}$$

is obtained. Integrating over the interval *J*

$$\begin{aligned} &(\lambda_n - \lambda)\left(\int_0^c \chi(x,\lambda)\varphi(x,\lambda_n)\mathrm{d}x + \int_c^\pi \chi(x,\lambda)\varphi(x,\lambda_n)\mathrm{d}x\right) \\ &= -\Delta(\lambda) - (\lambda_n - \lambda)\left(\beta_1\varphi(\pi,\lambda_n) + \beta_2\varphi'(\pi,\lambda_n)\right) + (\lambda_n - \lambda)\left(\alpha_1\chi(0,\lambda) + \alpha_2\chi'(0,\lambda)\right) \\ &\quad - (\lambda_n - \lambda)\eta_1\varphi(c^+,\lambda_n)\chi(c^+,\lambda) + (\lambda_n - \lambda)\eta_1(\lambda\eta_2 - \xi_2)\varphi(c^+,\lambda_n)\chi'(c^+,\lambda) \\ &\quad + (\lambda_n - \lambda)\eta_1(\lambda_n\eta_2 - \xi_2)\varphi'(c^+,\lambda_n)\chi(c^+,\lambda) \\ &\quad - (\lambda_n - \lambda)\left(\eta_1(\lambda_n\eta_2 - \xi_2)(\lambda\eta_2 - \xi_2) + \eta_2\right)\varphi'(c^+,\lambda_n)\chi'(c^+,\lambda). \end{aligned}$$

Dividing both sides of the above equality by $\lambda_n - \lambda$, and let $\lambda \to \lambda_n$, then we have

$$
\begin{aligned}
-\dot{\Delta}(\lambda_n) = & -\int_0^c \chi(x,\lambda_n)\varphi(x,\lambda_n)\mathrm{d}x - \int_c^{\pi} \chi(x,\lambda_n)\varphi(x,\lambda_n)\mathrm{d}x \\
& -\left(\beta_1\varphi(\pi,\lambda_n)+\beta_2\varphi'(\pi,\lambda_n)\right)+\left(\alpha_1\chi(0,\lambda)+\alpha_2\chi'(0,\lambda)\right) \\
& -\eta_1\varphi\left(c^+,\lambda_n\right)\chi\left(c^+,\lambda_n\right)+\left[\eta_1\left(\lambda_n\eta_2-\xi_2\right)\right]\varphi\left(c^+,\lambda_n\right)\chi'\left(c^+,\lambda_n\right) \\
& +\left[\eta_1\left(\lambda_n\eta_2-\xi_2\right)\right]\varphi'\left(c^+,\lambda_n\right)\chi\left(c^+,\lambda_n\right) \\
& -\left[\eta_1\left(\lambda_n\eta_2-\xi_2\right)^2+\eta_2\right]\varphi'\left(c^+,\lambda_n\right)\chi'\left(c^+,\lambda_n\right).
\end{aligned}
$$

Using (2.11)

$$
\begin{aligned}
\dot{\Delta}(\lambda_n) = & \int_0^c \varphi^2(x,\lambda_n)\mathrm{d}x + \int_c^{\pi}\varphi^2(x,\lambda_n)\mathrm{d}x + \eta_1\varphi^2\left(c^+,\lambda_n\right) \\
& + \frac{\left(\beta_1\varphi(\pi,\lambda_n)+\beta_2\varphi'(\pi,\lambda_n)\right)\left(\beta_1\chi(\pi,\lambda_n)-\beta_2\chi'(\pi,\lambda_n)\right)}{d_2} \\
& + \frac{\left(\alpha_1\varphi(0,\lambda_n)-\alpha_2\varphi'(0,\lambda_n)\right)\left(\alpha_1\chi(0,\lambda_n)-\alpha_2\chi'(0,\lambda_n)\right)}{d_1} \\
& -2\eta_1\left(\lambda_n\eta_2-\xi_2\right)\varphi\left(c^+,\lambda_n\right)\varphi'\left(c^+,\lambda_n\right)+\left(\left(\lambda_n\eta_2-\xi_2\right)^2\eta_1\right)\varphi'^2\left(c^+,\lambda_n\right) \\
& +\eta_2\varphi'^2\left(c^+,\lambda_n\right) \\
& =\varpi_n\Bigg[\int_0^c \varphi^2(x,\lambda_n)\mathrm{d}x + \int_c^{\pi}\varphi^2(x,\lambda_n)\mathrm{d}x + \frac{1}{d_1}\left(\alpha_1\varphi(0,\lambda_n)-\alpha_2\varphi'(0,\lambda_n)\right)^2 \\
& +\frac{1}{d_2}\left(\beta_1\varphi(\pi,\lambda_n)-\beta_2\varphi'(\pi,\lambda_n)\right)^2+\eta_1\varphi'^2\left(c^-,\lambda_n\right)+\eta_2\varphi'^2\left(c^+,\lambda_n\right)\Bigg] \\
& =\varpi_n\rho_n.
\end{aligned}
$$

Thus the equality (2.12) holds.

□

3. Construction and Asymptotic Approximation of Fundamental Solutions and Eigenvalues

In this section, we will obtain the asymptotic approximation of fundamental solutions and eigenvalues of the problem L.

Lemma 1. *Let* $\rho=\sqrt{\lambda}=\sigma+i\tau$ *. Then the following asymptotics hold.*

When $\beta_2\neq 0$, *one has*

$$
\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi(x,\lambda) = & \rho^7\beta_2\eta_1\eta_2\cos\rho(\pi-c)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin\rho(x-c) \\
& +O\left(|\rho|^{k+6}\exp\left(|\tau|(\pi-x)\right)\right),\ x\in[0,c),
\end{aligned}
\tag{3.1}
$$

$$
\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi(x,\lambda) = -\rho^2\beta_2\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos\rho(\pi-x)+O\left(|\rho|^{k+1}\exp\left(|\tau|(\pi-x)\right)\right),\ x\in(c,\pi]. \tag{3.2}
$$

When $\beta_2=0$, *one has*

$$
\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi(x,\lambda) = & \rho^6\beta_1\eta_1\eta_2\cos\rho(\pi-c)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos\rho(x-c) \\
& +O\left(|\rho|^{k+5}\exp\left(|\tau|(\pi-x)\right)\right), x\in[0,c),
\end{aligned}
\tag{3.3}
$$

$$
\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi(x,\lambda) = \rho\beta_1\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin\rho(\pi-x)+O\left(|\rho|^{k}\exp\left(|\tau|(\pi-x)\right)\right),\ x\in(c,\pi]. \tag{3.4}
$$

Where $k=0,1$.

Proof. When $\beta_2 \neq 0$. Let $f_0(x,\lambda), g_0(x,\lambda)$ be the solutions of (2.1) under the conditions

$$f_0(\pi,\lambda)=1,\ f_0'(\pi,\lambda)=0,\ g_0(\pi,\lambda)=0,\ g_0'(\pi,\lambda)=1.$$

According to [37], one has

$$\begin{aligned} f_0(x,\lambda) &= \cos\rho(\pi-x)+\int_x^\pi \frac{\sin\rho(x-t)}{\rho} q(t) f_0(t,\lambda)\mathrm{d}t \\ &= \cos\rho(\pi-x)+\frac{\sin\rho x}{2\rho}\int_x^\pi q(t)\mathrm{d}t \\ &\quad +\frac{1}{2\rho}\int_x^\pi q(t)\sin\rho(x-2t)\mathrm{d}t + O\left(\frac{1}{\rho^2}\mathrm{e}^{|\tau|(\pi-x)}\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} g_0(x,\lambda) &= \frac{\sin\rho(\pi-x)}{\rho}+\int_x^\pi \frac{\sin\rho(x-t)}{\rho} q(t) g_0(t,\lambda)\mathrm{d}t \\ &= \frac{\sin\rho(\pi-x)}{\rho}-\frac{\cos\rho(\pi-x)}{2\rho^2}\int_x^\pi q(t)\mathrm{d}t \\ &\quad +\frac{1}{2\rho^2}\int_x^\pi q(t)\cos\rho(x-2t)\mathrm{d}t + O\left(\frac{1}{\rho^3}\mathrm{e}^{|\tau|(\pi-x)}\right), \end{aligned} \tag{3.6}$$

where $\rho=\sqrt{\lambda}, \tau = Im\rho$.

Suppose $f(x,\lambda), g(x,\lambda)$ are the solutions of Equation (2.1), and satisfy the jump conditions (2.4), (2.5), and the following initial conditions

$$f(\pi,\lambda)=1,\ f'(\pi,\lambda)=0,\ g(\pi,\lambda)=0,\ g'(\pi,\lambda)=1.$$

Hence as $x>c$,

$$f(x,\lambda)=f_0(x,\lambda),\ g(x,\lambda)=g_0(x,\lambda),$$

and as $x<c$, let

$$\begin{cases} f(x,\lambda)=A_1 f_0(x,\lambda)+B_1 g_0(x,\lambda), \\ g(x,\lambda)=A_2 f_0(x,\lambda)+B_2 g_0(x,\lambda). \end{cases} \tag{3.7}$$

Due to the fact that $f(x,\lambda), g(x,\lambda)$ meet the jump conditions (2.4), (2.5), and $f(c^+)=f_0(c^+)=f_0(c)$, $g(c^+)=g_0(c^+)=g_0(c)$, we can get

$$\begin{cases} A_1 f_0(c,\lambda)+B_1 g_0(c,\lambda)=-a_\lambda f_0(c,\lambda)+(a_\lambda b_\lambda-1) f_0'(c,\lambda), \\ A_1 f_0'(c,\lambda)+B_1 g_0'(c,\lambda)=f_0(c,\lambda)-b_\lambda f_0'(c,\lambda), \\ A_2 f_0(c,\lambda)+B_2 g_0(c,\lambda)=-a_\lambda g_0(c,\lambda)+(a_\lambda b_\lambda-1) g_0'(c,\lambda), \\ A_2 f_0'(c,\lambda)+B_2 g_0'(c,\lambda)=g_0(c,\lambda)-b_\lambda g_0'(c,\lambda). \end{cases}$$

Thus by calculation, it can be obtained that

$$\begin{aligned} A_1 &= -\frac{1}{2}\rho^5\eta_1\eta_2\sin 2\rho(\pi-c)+O(\rho^4), \\ B_1 &= -\frac{1}{2}\rho^6\eta_1\eta_2+\frac{1}{2}\rho^6\eta_1\eta_2\cos 2\rho(\pi-c)+O(\rho^5), \\ A_2 &= \frac{1}{2}\rho^4\eta_1\eta_2+\frac{1}{2}\rho^4\eta_1\eta_2\cos 2\rho(\pi-c)+O(\rho^3), \\ B_2 &= \frac{1}{2}\rho^5\eta_1\eta_2\sin 2\rho(\pi-c)+O(\rho^4). \end{aligned}$$

Substituted into Equations (3.7), we have

$$f(x,\lambda) = -\frac{1}{2}\rho^5\eta_1\eta_2\left[\sin\rho(x+\pi-2c) - \sin\rho(\pi-x)\right] + O(\rho^4),$$

$$g(x,\lambda) = \frac{1}{2}\rho^4\eta_1\eta_2\left[\cos\rho(\pi-x) + \cos\rho(\pi+x-2c)\right] + O(\rho^3).$$

According to the initial conditions satisfied by $\chi(x,\lambda)$.

For $k=0,1$, as $x\in[0,c)$,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi(x,\lambda) = \rho^7\beta_2\eta_1\eta_2\cos\rho(\pi-c)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin\rho(x-c) + O\left(|\rho|^{k+6}\exp\left(|\tau|(\pi-x)\right)\right),$$

and as $x\in(c,\pi]$,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\chi(x,\lambda) = -\rho^2\beta_2\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos\rho(\pi-x) + O\left(|\rho|^{k+1}\exp\left(|\tau|(\pi-x)\right)\right).$$

Similarly, when $\beta_2 = 0$, (3.2) can be obtained.

Lemma 2. *The function* $\varphi(x,\lambda)$ *has the following asymptotics, for* $k=0,1$. *When* $\alpha_2\neq 0$, *one has*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\varphi(x,\lambda) = \begin{cases} -\rho^2\alpha_2\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos\rho x + O\left(|\rho|^{k+1}\exp(|\tau|x)\right), & x\in[0,c), \\ -\rho^7\alpha_2\eta_1\eta_2\sin\rho c\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos\rho(x-c) + O\left(|\rho|^{k+6}\exp(|\tau|x)\right), & x\in(c,\pi]. \end{cases} \tag{3.8}$$

When $\alpha_2 = 0$, *one has*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\varphi(x,\lambda) = \begin{cases} \rho\alpha_1\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin\rho x + O\left(|\rho|^{k}\exp(|\tau|x)\right), & x\in[0,c), \\ -\rho^6\alpha_1\eta_1\eta_2\cos\rho c\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos\rho(x-c) + O\left(|\rho|^{k+5}\exp(|\tau|x)\right), & x\in(c,\pi]. \end{cases} \tag{3.9}$$

Proof. The proof is the same as lemma 1, hence we omit the details. □

Hence, when $\alpha_2\neq 0, \beta_2\neq 0$, according to (2.9) and (3.8) the characteristic function $\Delta(\lambda)$ as $\rho\to\infty$ is

$$\Delta(\lambda) = \rho^{10}\beta_2\alpha_2\eta_1\eta_2\sin\rho c\sin\rho(\pi-c) + O\left(\rho^9\exp(|\tau|\pi)\right). \tag{3.10}$$

Let $\Delta(\lambda) = \Delta_1(\lambda) + \Delta_2(\lambda)$, where

$$\Delta_1(\lambda) = \rho^{10}\beta_2\alpha_2\eta_1\eta_2\sin\rho c\sin\rho(\pi-c),$$

$$\Delta_2(\lambda) = O\left(\rho^9\exp(|\tau|\pi)\right).$$

Next, we ready to find the asymptotic formulas for the eigenvalues of the considered problem *L*.

Theorem 3. *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be the eigenvalues of the problem L,* $\lambda_n = \rho_n^2$, *then it has following asymptotics as* $n\to\infty$

$$\rho_n' = \frac{n\pi}{c} + O\left(\frac{1}{n}\right),\quad \rho_n'' = \frac{n\pi}{\pi-c} + O\left(\frac{1}{n}\right). \tag{3.11}$$

Proof. Let

$$G_n'=\left\{\rho\in\mathbb{C}:\rho=\left|\frac{n\pi}{c}\right|+\frac{1}{2}\left|\frac{n\pi}{\pi-c}-\frac{n\pi}{c}\right|\right\}, \tag{3.12}$$

$$G_n''=\left\{\rho\in\mathbb{C}:\rho=\left|\frac{n\pi}{\pi-c}\right|+\frac{1}{2}\left|\frac{n\pi}{\pi-c}-\frac{n\pi}{c}\right|\right\}. \tag{3.13}$$

Next we only prove the case of ρ_n', and ρ_n'' can be proved in the same way.

Denote $G_\delta=\left\{\rho:\left|\rho-\rho_n^0\right|\geq\delta\right\}$, where $\delta>0$, and ρ_n^0 are square roots of $\Delta_1(\lambda)$, then from [3] we know that for any $\rho\in G_\delta$, there exists a constant $C_\delta>0$, such that

$$\left|\Delta_1(\lambda)\right|>C_\delta\left|\rho^{10}\right|\exp\left(|\tau|\pi\right),\ \rho\in G_\delta,$$

thus for sufficiently large $\rho^*>0$, when $\rho\in G_\delta$ and $|\rho|>\rho^*$, it has

$$\left|\Delta(\lambda)\right|>C_\delta\left|\rho^{10}\right|\exp\left(|\tau|\pi\right). \tag{3.14}$$

It's easy to know $G_n',G_n''\subset G_\delta$. Clearly, $\left|\Delta_1(\lambda)\right|>\left|\Delta_2(\lambda)\right|$ for $\rho\in G_n'$, according to Rouchè's theorem, it is clear that the number of zeros of $\Delta(\lambda)$ inside G_n' coincides with the number of zeros of $\Delta_1(\lambda)$. Applying Rouchè's theorem again to the circle $\gamma_n(\delta)=\left\{\rho:\left|\rho-\frac{n\pi}{c}\right|<\delta\right\}$, for sufficiently large *n*, in each $\gamma_n(\delta)$, there exits a unique zero of $\Delta(\lambda)$, namely $\rho_n'=\sqrt{\lambda_n}$. Because of $\delta>0$ is sufficiently small, when $n\to\infty$, we have

$$\rho_n'=\frac{n\pi}{c}+\varepsilon_n,\ \varepsilon_n=o(1). \tag{3.15}$$

Let $S(z)=\cos z(\pi-2c)-\cos z\pi$. Substituting (3.15) into (3.10), we can obtain that

$$S\left(\frac{n\pi}{c}+\varepsilon_n\right)=O\left(\frac{1}{\frac{n\pi}{c}+\varepsilon_n}\right).$$

By the well-known formula $S(z+h)=S(z)+\left(S'(z)+o(1)\right)h$, the above equation can be changed to the following formula

$$S\left(\frac{n\pi}{c}\right)+\left(S'\left(\frac{n\pi}{c}\right)+o(1)\right)\varepsilon_n=O\left(\frac{1}{\frac{n\pi}{c}+\varepsilon_n}\right).$$

When $n\to\infty$, then $\varepsilon_n=O\left(\frac{1}{n}\right)$ is true.

Therefore, (3.11) can be rolled out.

4. Inverse Problems

In this section, we mainly consider the reconstruction of the problem *L*, from the Weyl function, from the spectral data $\{\lambda_n,\rho_n\}_{n=0}^{\infty}$, and from two spectra $\{\lambda_n\}_{n=0}^{\infty}\cup\{\mu_n\}_{n=0}^{\infty}$.

Denote

$$M(\lambda)=\frac{\chi(0,\lambda)\alpha_1+\chi'(0,\lambda)\alpha_2}{d_1\Delta(\lambda)}, \tag{4.1}$$

where α_1,α_2 are not 0 at the same time. Let $\phi(x,\lambda)$ be the solution of (2.1), satisfying the following initial conditions and jump conditions (2.4) and (2.5)

$$\phi(0,\lambda)=d_1^{-1}\alpha_2,\ \phi'(0,\lambda)=-d_1^{-1}\alpha_1.$$

Because of $\langle\varphi(x,\lambda),\phi(x,\lambda)\rangle=1$, we have

$$\chi(x,\lambda)=\Delta(\lambda)\phi(x,\lambda)-\frac{\chi(0,\lambda)\alpha_1+\chi'(0,\lambda)\alpha_2}{d_1}\varphi(x,\lambda),$$

or

$$\frac{\chi(x,\lambda)}{\Delta(\lambda)}=\phi(x,\lambda)-M(\lambda)\varphi(x,\lambda). \tag{4.2}$$

Denote

$$\Phi(x,\lambda)=\frac{\chi(x,\lambda)}{\Delta(\lambda)}. \tag{4.3}$$

Thus $\Phi(x,\lambda)$ is the solution of (2.1) that satisfies the conditions $l_1(\Phi)=-1$, $l_2(\Phi)=0$ and the jump conditions (2.4) and (2.5), where $\Delta(\lambda)$ is defined in (2.8).

The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the boundary value problem *L*.

Next, the uniqueness theorem for problem *L* will be given by the Weyl function. For studying the inverse problem we agree that together with *L* consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{q}(x),\tilde{c},\tilde{\alpha}_i,\tilde{\beta}_i,\tilde{\eta}_k,\tilde{\xi}_k, i=1,2,3,4; k=1,2$.

Theorem 4. *If* $M(\lambda)=\tilde{M}(\lambda)$, *then* $L=\tilde{L}$, *i.e.* $q(x)=\tilde{q}(x)$ *a.e. J, and* $c=\tilde{c}$, $\alpha_i=\tilde{\alpha}_i$, $\beta_i=\tilde{\beta}_i$, $i=1,2,3,4$, $\eta_k=\tilde{\eta}_k$, $\xi_k=\tilde{\xi}_k$, $k=1,2$.

Proof. Let us define the matrix $P(x,\lambda)=\left[p_{j,k}(x,\lambda)\right]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix}\tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda)\end{pmatrix}=\begin{pmatrix}\varphi(x,\lambda) & \Phi(x,\lambda)\\ \varphi'(x,\lambda) & \Phi'(x,\lambda)\end{pmatrix},$$

then we can calculate that

$$\begin{cases}p_{j,1}(x,\lambda)=\Phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda),\\ p_{j,2}(x,\lambda)=\varphi^{(j-1)}(x,\lambda)\tilde{\Phi}(x,\lambda)-\Phi^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda),\end{cases} \tag{4.4}$$

and

$$\begin{cases}\varphi(x,\lambda)=p_{11}(x,\lambda)\tilde{\varphi}(x,\lambda)+p_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda),\\ \Phi(x,\lambda)=p_{11}(x,\lambda)\tilde{\Phi}(x,\lambda)+p_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda).\end{cases} \tag{4.5}$$

According to (4.2) and (4.4), the following equations can be obtained

$$\begin{cases}p_{11}(x,\lambda)=-\phi(x,\lambda)\tilde{\varphi}'(x,\lambda)+\varphi(x,\lambda)\tilde{\phi}'(x,\lambda)+\left(M(\lambda)-\tilde{M}(\lambda)\right)\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda),\\ p_{12}(x,\lambda)=-\varphi(x,\lambda)\tilde{\phi}(x,\lambda)+\phi(x,\lambda)\tilde{\varphi}(x,\lambda)+\left(\tilde{M}(\lambda)-M(\lambda)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).\end{cases} \tag{4.6}$$

Denote $G'_\delta=\{\rho:|\rho-\rho'_n|\ge\delta\}$, $\tilde{G}'_\delta=\{\rho:|\rho-\tilde{\rho}'_n|\ge\delta\}$, where δ is sufficiently small number, ρ'_n and $\tilde{\rho}'_n$ are square roots of the eigenvalues of the problems L and $\tilde{L}$, respectively. By virtue of (3.1), (3.8) and (3.10), for sufficiently large ρ^*, there exists a constant $C_\delta>0$ such that

$$|p_{11}(x,\lambda)|\le C_\delta,\ |p_{12}(x,\lambda)|\le\frac{C_\delta}{|\rho|},\ \rho\in G'_\delta\cap\tilde{G}'_\delta. \tag{4.7}$$

Thus, if $M(\lambda)=\tilde{M}(\lambda)$, then for each fixed x, the functions $p_{11}(x,\lambda)$ and $p_{12}(x,\lambda)$ are entire in λ. Combined with (4.7), and according to Liouville's theorem, we can get

$$p_{11}(x,\lambda)=A(x),\ p_{12}(x,\lambda)=0. \tag{4.8}$$

Substituting (4.8) into (4.5), then for each $x\in J$ and $\lambda\in\mathbb{C}$ we have

$$\varphi(x,\lambda)=A(x)\tilde{\varphi}(x,\lambda),\ \Phi(x,\lambda)=A(x)\tilde{\Phi}(x,\lambda). \tag{4.9}$$

Due to $\langle\varphi(x,\lambda),\Phi(x,\lambda)\rangle=1$ and $\langle\tilde{\varphi}(x,\lambda),\tilde{\Phi}(x,\lambda)\rangle=1$, ones have $A^2(x)=1$.

On the other hand, the asymptotic expressions

$$\begin{aligned}\varphi(x,\lambda)&=C(\rho)\exp(-i\rho x)\left(1+O\left(\frac{1}{\rho}\right)\right),\\ \tilde{\varphi}(x,\lambda)&=\tilde{C}(\rho)\exp(-i\rho x)\left(1+O\left(\frac{1}{\rho}\right)\right),\end{aligned} \tag{4.10}$$

can be easily verified. Here

$$C(\rho)=\begin{cases}-\frac{1}{2}\rho^2\alpha_2, & x\in[0,c),\\ -\frac{1}{4}\rho^7\alpha_2\eta_1\eta_2, & x\in(c,\pi];\end{cases}\quad \tilde{C}(\rho)=\begin{cases}-\frac{1}{2}\rho^2\alpha_2, & x\in[0,\tilde{c}),\\ -\frac{1}{4}\rho^7\alpha_2\eta_1\eta_2, & x\in(\tilde{c},\pi].\end{cases}$$

Without loss of generality, assume $c<\tilde{c}$. From (4.9), (4.10) we get $A(x)=1$ for $x\in[0,c)\cup(\tilde{c},\pi]$. When $x\in(c,\tilde{c})$, one has

$$\frac{1}{2}\left(1+O\left(\frac{1}{\rho}\right)\right)=A(x)\frac{1}{\rho^7}\left(1+O\left(\frac{1}{\rho}\right)\right). \tag{4.11}$$

By letting $|\rho|\to\infty$, in (4.11) we contradict $A^2(x)=1$. Thus $c=\tilde{c}$ and $A(x)=1$. Hence $\varphi(x,\lambda)=\tilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda)=\tilde{\Phi}(x,\lambda)$.

Finally, if $M(\lambda)=\tilde{M}(\lambda)$ holds, then we can conclude $q(x)=\tilde{q}(x)$, a.e. J and $c=\tilde{c}$, $\alpha_i=\tilde{\alpha}_i$, $\beta_i=\tilde{\beta}_i$, $i=1,2,3,4$, $\eta_k=\tilde{\eta}_k$, $\xi_k=\tilde{\xi}_k$, $k=1,2$. So consequently, $L=\tilde{L}$. □

Lemma 3. [29] *For the function* $M(\lambda)$ *defined in* (*4.1*), *the following expression can be established*

$$M(\lambda)=\sum_{n=0}^{\infty}\frac{1}{\rho_n(\lambda-\lambda_n)}. \tag{4.12}$$

Theorem 5. *If* $\lambda_n=\tilde{\lambda}_n$ *and* $\rho_n=\tilde{\rho}_n, n\in\mathbb{N}_0$, *then* $q(x)=\tilde{q}(x)$ *a.e. J, and* $c=\tilde{c}$, $\alpha_i=\tilde{\alpha}_i$, $\beta_i=\tilde{\beta}_i$, $i=1,2,3,4$, $\eta_k=\tilde{\eta}_k$, $\xi_k=\tilde{\xi}_k$, $k=1,2$, *i.e.* $L=\tilde{L}$.

Proof. From lemma 3, if $\lambda_n = \tilde{\lambda}_n$ and $\rho_n = \tilde{\rho}_n$, then $M(\lambda) = \tilde{M}(\lambda)$. According to Theorem 4, this theorem can be proved. □

Lastly, through the two spectra $\{\lambda_n\}_{n=0}^{\infty} \cup \{\mu_n\}_{n=0}^{\infty}$, let us prove the uniqueness theorem. Let $\{\mu_n\}_{n=0}^{\infty}$ be the spectra of the problem L_1 consisting of the Equation (2.1) with condition $\alpha_2 y'(0,\lambda) + \alpha_1 y(0,\lambda) = 0$ (where α_1, α_2 are not 0 at the same time) and conditions (2.3), (2.4) and (2.5). It is obvious that μ_n are the zeros of $\Delta_0(\mu) = \chi'(0,\mu)\alpha_2 + \chi(0,\mu)\alpha_1$, where $\Delta_0(\mu)$ is the characteristic function of the problem L_1.

Theorem 6. *If* $\lambda_n = \tilde{\lambda}_n, \mu_n = \tilde{\mu}_n, n \geq 0$, *then* $q(x) = \tilde{q}(x)$ *a.e. J, and* $c = \tilde{c}$, $\alpha_1 = \tilde{\alpha}_1$, $\alpha_2 = \tilde{\alpha}_2$, $\beta_i = \tilde{\beta}_i$, $i = 1,2,3,4$, $\eta_k = \tilde{\eta}_k$, $\xi_k = \tilde{\xi}_k$, $k = 1,2$.

Proof. Since the functions $\Delta(\lambda)$ and $\Delta_0(\mu)$ are entire of order $\frac{1}{2}$, we can write by Hadamards factorization theorem (methods popularized by the literature [3])

$$\Delta(\lambda) = C\prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right),$$

$$\Delta_0(\mu) = C_0\prod_{n=0}^{\infty}\left(1 - \frac{\mu}{\mu_n}\right).$$

Thus $\Delta(\lambda)$ and $\Delta_0(\mu)$ are uniquely determined up to a multiplicative constant by their zeros (the case when $\Delta(0) = 0$ requires minor modifications). Therefore, one has $\Delta(\lambda) = \tilde{\Delta}(\lambda)$, $\Delta_0(\mu) = \tilde{\Delta}_0(\mu)$, *i.e.* $\chi'(0,\mu)\alpha_2 + \chi(0,\mu)\alpha_1 = \tilde{\chi}'(0,\mu)\tilde{\alpha}_2 + \tilde{\chi}(0,\mu)\tilde{\alpha}_1$ when $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$. Consequently $M(\lambda) = \tilde{M}(\lambda)$, according to Theorem 4, the proof is completed.

5. Conclusion

In the present work, the inverse spectral problem of Sturm-Liouville operator with boundary conditions and jump conditions dependent on the spectral parameter is investigated. Such problems are connected with fields such as mechanical engineering, and acoustic wave propagation problems, etc. Here the uniqueness theorems of this problem are given by using Weyl function theory, two spectra and spectral data approaches. However, we only discuss the uniqueness theorem of the problem, the reconstruction formulae and stability of this problem have not been considered, we plan to consider these problems in future studies.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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