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Studying the (*n*-1)-th Order Riccati Type Characteristic Equations of Linear Differential Equations in the Form of Algebraic Equations

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Author's contribution

Author KAK designed the study, and wrote the first draft of the manuscript. Ideas, methods and proofs of solutions of LDE in the form of Algebraic Equations carried out by author KAK solely. Author read and approved the final manuscript.

Research Article

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ABSTRACT

From existence of analogies between problems of integrating Linear differential equations (LDE) in quadratures and solution to algebraic equations in radicals and based on studying properties of the latter in the article we have found one special differential equation. By studying characteristic equation of Riccati type and linking it to an algebraic equation we have obtained a recurrent LDE with variable coefficients and have established its properties.

Keywords: Characteristic equations of riccati type; recurrent differential equation.

1. INTRODUCTION

In a short classical recording of linear differential equations (LDE)

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$$Ly \equiv \sum_{i=1}^{n} a_i(x) y^{(i)} + a_0(x) y = 0, \tag{1}$$

It is almost impossible to detect any direct method or to foresee a target approach to its solution.

LDE with variable coefficients to be solved in quadratures, i.e., operation reduced to a finite execution sequence of known functions and to the integration of such functions is uncommon. Examples are the Euler equation, some classes of equations, coefficients of which satisfy certain conditions, reduced to the equations with constant coefficients, Yerugin formulation or the method of successive derivations [3]. In our view, the task to solve problems in LDE quadratures and find new classes of integrable equations is of interest for the following reasons:

- ✓ Integration in closed form provides solution in analytical form that can make a complete study of the problem to be solved.
- ✓ There is the possibility of finding approximate solutions to differential equations, similar in their forms to integrable equations.
- ✓ The problem of integrability of LDE in closed form has not been completely investigated [4,5,7,10,12,13].

The invariant subspace method is refined to present more unity and more diversity of exact solutions to evolution equations, and the key idea is to take subspaces of solutions to linear ordinary differential equations as invariant subspaces that evolution equations admit [14]. There are some other wonderful analytical properties for linear differential equations with variable coefficients presented and used to construct invariant solution subspaces. More generally, solution presentations to linear systems with variable coefficients by the matrix exponential were argued [15]. To give a definite answer to this question, which satisfy the chain rule .The presented matrices in [15] consist of finite-times continuously differentiable entries or smooth entries, and two classes of illustrative examples of coefficient matrices were presented [15]. And a related problem on chair rules for differentiation was analyzed quite systematically in [16].

2. The Characteristic Equation of Riccati Type

Equation (1) by replacing unknown function

$$y = e^{\int_{x_0}^{x} r(t)dt}, \quad x_0 \in (a,b)$$

adducts to nonlinear differential equation of (n-1)-th order

$$R(r) \equiv \sum_{i=1}^{n} a_i(x)[p + r(x)]^{i-1} \cdot r(x) + a_0(x) = 0, \tag{2}$$

which is called the characteristic equation of the Riccati type . Here

$[p+r(x)]^k \cdot r(x)$ means consistent application *k* times of the operator $[p+r(x)]$, $p=\frac{d}{dx}$ to the function *r*(*x*) [4,5,10,15].

As far back as L. Euler reduced a second order equation to the Riccati equation, using the method of integrating multiplier in the form of exp [1].

Papers [2,12,13] are devoted to linear differential equations or equations of Riccati type research and to solutions in quadratures. One of the criteria for conversion is more complete study of the new equations, their better solution compared with baseline. Of course, integrating of the characteristic equations of Riccati type may be no less difficult than the integration of linear equations themselves, but in some cases, finding a solution is much easier. It also should be noted that the apparent complexity of the problem is of fundamental importance, the following problem is formulated.

To develop methods for solving the LDE with variable coefficients, by unified approach to select and exclude the classes of equations, the order of which can be reduced or even to reduce the problem to quadratures.

If it's known one particular solution $r_1(x)$ (or $y_1(x)$ for linear equation),) then by taking a certain replacement [2,6,8,9,11,18],

$$y(x)=y_1(x)\cdot\int_{x_0}^{x}u(t)dt$$

instead of functions $y(x), y_1(x), u(x)$ corresponding exponentials, i.e.

$$\exp\int_{x_0}^{x}r(t)dt=\exp\int_{x_0}^{x}r_1(t)dt\cdot\int_{x_0}^{x}\exp\int_{x_0}^{t}z(\tau)d\tau dt,$$

find a substitution

$$r(x)=r_1(x)+\frac{\exp\int_{x_0}^{x}z(t)dt}{\int_{x_0}^{x}\exp\int_{x_0}^{t}z(\tau)d\tau dt}. \qquad (2')$$

The obtained replacement can result in equation (*n*-1)-th order (2) to the new characteristic equation (*n*-2)-th order of the Riccati type

$$[p+z(x)]^{n-2}z(x)+\widetilde{b}_{n-2}(x)[p+z(x)]^{n-3}z(x)+...+\widetilde{b}_1(x)z(x)+\widetilde{b}_0(x)=0.$$

This can be verified by direct substitution in (2) in the form (2'). So, the complexity of the integration of linear differential equations in closed form is initially to find a particular solution of (2).

In practice, there are often linear differential equations with variable coefficients (1) who have one (or several) solution of the form $e^{\lambda x}$. In this case, a particular solution of the characteristic equation (*n*-1)- th order of the Riccati type is constant $r_1(x) = \lambda = const.$ Constanta λ can be complex[18].

Non- linearity and "roughness" of a characteristic equation of Riccati type (2) allow us to "catch" and extract new classes of LDE with variable coefficients, which can be solved and new integration methods can be found. We have proved a relation between the n-th order LDE and the *n*-th degree algebraic functional equation [9,18]. Study represents the (*n*-1)-th order characteristic equation of Riccati type as an algebraic equation of the n-th degree with respect to function *r(x)* with coefficients depending on its derivatives and LDE coefficients and have outlined an iterative method for an approximate calculation of its roots [6,7,8,17]. Thus, the problems set are actual and their solution will allow to develop the theory of differential equations, as well as to expand the range of their applications.

3. RESEARCH

Let's study linear differential equation

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_1(x)y' + b_0(x)y = 0,\ b_{i-1}(x) \in C(a,b), \tag{3}$$

Assumption that coefficients satisfy the following conditions

$$b_k(x) = b_{n-k}(x) \quad \forall x \epsilon (a,b), k = 0,1,2,\ldots,n,\ b_0(x) = b_n(x) \equiv 1, \tag{4}$$

yields the following linear equation

$$y^{(n)} + b_1(x)y^{(n-1)} + b_2(x)y^{(n-2)} + \ldots + b_2(x)y'' + b_1(x)y' + y = 0\,. \tag{5}$$

By analogy with algebraic equation let's name the equation yielded in (5) as a recurrent differential equation of the *n*-th order. Let's formulate some of its properties.

Theorem 1. If recurrent differential equation (5) has a particular solution $y_1(x) = e^{\alpha x}$, $\alpha \neq 0$, function $y_2(x) = e^{\frac{1}{\alpha}x}$ is also its solution.

Proof: In case if solution has $y = e^{\lambda x}$ form, the characteristic equation

$$\lambda^n + b_1(x)\lambda^{n-1} + b_2(x)\lambda^{n-2} + \ldots + b_2(x)\lambda^2 + b_1(x)\lambda + 1 = 0. \tag{6}$$

corresponds to the linear recurrent equation (5).

Since, by hypothesis, there is a solution to the recurrent differential equation $y_1(x) = e^{\alpha x}$, then $\alpha = \lambda$ satisfies the equation (6), i.e.

$\alpha^n + b_1(x)\alpha^{n-1} + b_2(x)\alpha^{n-2} + ... + b_2(x)\alpha^2 + b_1(x)\alpha + 1 = 0$.

Substituting value of $\lambda = \frac{1}{\alpha}$ in equation (6) and reducing to a common denominator α^n yields

$$1 + b_1(x)\alpha + b_2(x)\alpha^2 + ... + b_2(x)\alpha^{n-2} + b_1(x)\alpha^{n-1} + \alpha^n = 0,$$

$b_1(x) \not\equiv 0$ which coincides with the previous condition, and this proves the theorem.

Consequence 1: Recurrent second-order differential equation with variable coefficients

$b_1(x) \not\equiv 0$

$$y'' + b_1(x)y' + y = 0 \qquad (*)$$

has no solution of exponential form $y(x) = e^{\lambda x}$.

If function $y = e^{\lambda x}$ were a solution, then

$$\lambda^2 + b_1(x)\lambda + 1 = 0,$$

would be satisfied and this implies that coefficient

$$b_1(x) = -\frac{\lambda^2 + 1}{\lambda} = const,$$

however, this contradicts the hypothesis.

Consequence 2: If $y_1(x) = \exp \int_{x_0}^{x} r_1(t)dt$, where $r_1(x) \neq 0$ $\forall x \in (a,b)$ is a particular solution to recurrent equation (*), then function $y_2(x) = \exp\left[-\int_{x_0}^{x} \frac{1}{r_1(t)} dt\right]$ is a solution to another second-order recurrent equation

$$y''(x) - b_1(x)y'(x) + y(x) = 0.$$

Indeed, since $y_1(x)$ is a solution to equation (*), then $r_1(x)$ satisfies Riccati characteristic equation

$$r_1'(x) + r_1^2(x) + b_1(x)r_1(x) + 1 = 0.$$

Let's divide it by $r_1^2(x)$ and reduce to

$$\left(-\frac{1}{r_1(x)}\right)' + \left(-\frac{1}{r_1(x)}\right)^2 - b_1(x)\left(-\frac{1}{r_1(x)}\right) + 1 = 0.$$

This means that function

$$y_2(x) = \exp\left[-\int_{x_0}^{x} \frac{1}{r_1(t)}\,dt\right]$$

Is a solution to this recurrent differential equation?

Theorem 2. Recurrent differential equation of an odd order has solution $y_1(x) = e^{-x}$ and replacing $y' + y = u(x)$ yields a recurrent equation of one order lower.

Proof: Let $n = 2k + 1$, $k = 0,1,2,\ldots$. We substitute function $y_1(x) = e^{-x}$ in a left side of equation (5). Then, factoring out the exponent yields

$$e^{-x}[(-1)^{2k+1} + b_1(x)(-1)^{2k} + b_2(x)(-1)^{2k-1} + \ldots + b_k(x)(-1)^{k+1} + \\ + b_k(x)(-1)^{k} + \ldots + b_2(x)(-1)^2 + b_1(x)(-1) + 1] = 0,$$

since members in square brackets annihilate each other. Thus, the first conclusion of the theorem is proved.

To prove the second part of the conclusion, we use a theorem whereby equation (5) which has solution of $e^{\lambda x}$ form reduces to

$$\begin{gathered}(y' - \lambda y)^{(n-1)} + [\lambda + b_{n-1}(x)](y' - \lambda y)^{(n-2)} + \\ + [\lambda^2 + b_{n-1}(x)\lambda + b_{n-2}(x)](y' - \lambda y)^{(n-3)} + \ldots + [\lambda^{n-2} + b_{n-1}(x)\lambda^{n-3} + \\ + \ldots + b_3(x)\lambda + b_2(x)](y' - \lambda y)' + \\ + [\lambda^{n-1} + b_{n-1}(x)\lambda^{n-2} + \ldots + b_2(x)\lambda + b_1(x)](y' - \lambda y) = 0.\end{gathered} \tag{7}$$

Since $y_1(x) = e^{-x}$ is a particular solution to a recurrent differential equation of an odd order, then, substituting value of $\lambda = -1$ in (7) taking into account (4) yields

$$\begin{aligned}&(y'+y)^{(2k)} + [-1 + b_1(x)](y'+y)^{(2k-1)} + [(-1)^2 + b_1(x)(-1) + b_2(x)](y'+y)^{(2k-2)} + \\ &+ \ldots + [(-1)^k + b_1(x)(-1)^{k-1} + \ldots + b_k(x)](y'+y)^k + \\ &+ [(-1)^{2k-2} + b_1(x)(-1)^{2k-3} + \ldots + b_4(x)(-1) + b_3(x)](y'+y)'' + \\ &+ [(-1)^{2k-1} + b_1(x)(-1)^{2k-2} + \ldots + b_3(x)(-1) + b_2(x)](y'+y)' + \\ &+ [(-1)^{2k} + b_1(x)(-1)^{2k-1} + \ldots + b_2(x)(-1) + b_1(x)](y'+y) = 0.\end{aligned}$$

It is easy to see in square brackets that with $(y' + y)'$ members annihilate each other and $[-1 + b_1(x)]$ is left, meanwhile with $(y' + y)$ the coefficient is equal to one.
Thus, we obtain

$$u^{(2k)} + [-1 + b_1(x)]u^{(2k-1)} + [1 - b_1(x) + b_2(x)]u^{(2k-2)} + \\ + \ldots + [(-1)^k + b_1(x)(-1)^{k-1} + \ldots + b_k(x)]u^{(k)} + \ldots + \tag{8}$$

$$+[1-b_1(x)+b_2(x)]u''+[-1+b_1(x)]u'+u=0,$$

where $u = y'(x)+y(x)$.

The resultant differential equation is one order lower and it is recurrent.

Theorem 3. Substituting $t=\int_{x_0}^{x}\sqrt{b_0(s)}ds$ in linear second-order equation with coefficient $b_0(x) > 0\ \forall\ x \in (a,b)$ yields recurrent differential equation.

Proof: Let's assume there is equation

$$y''(x)+b_1(x)y'(x)+b_0(x)y(x)=0,\quad b_0(x)>0.$$

Replacing independent variable $t = t(x)$ in it yields

$$y''_{t^2}+\frac{t''_{x^2}+b_1(x)t'_x}{(t'_x)^2}y'_t+\frac{b_0(x)}{(t'_x)^2}y=0\cdot \tag{9}$$

Equation (9) is recurrent if and only if

$\dfrac{b_0(x)}{(t'_x)^2}=1$, from which we get $t=\int_{x_0}^{x}\sqrt{b_0(s)}ds$, $b_0(s) > 0$.

Substituting function t(x) in equation (9) yields desired recurrent second-order differential equation

$$y''_{t^2}+\frac{1}{b_0(x)}\left[\left(\sqrt{b_0(x)}\right)'+b_1(x)\sqrt{b_0(x)}\right]y'_t+y=0.$$

Example: Let us show property of recurrent linear differential equation.

$$y^{IV}+(x^2+1)y^{III}-\frac{10(x^2+1)+17}{4}y^{II}+(x^2+1)y^{I}+y=0$$

Out of the corresponding function of the characteristic equation (4)

$$\lambda^4+(x^2+1)\lambda^3-\frac{10(x^2+1)+17}{4}\lambda^2+(x^2+1)\lambda+1=0$$

by trial we find that $\lambda_1 = 2$ and from Theorem 1 the second root $\lambda_2=\frac{1}{2}$.

Thus we found two partial solutions LDE $y_1(x)=e^{2x}$ and $y_2(x)=e^{\frac{1}{2}x}$.

4. CONCLUSION

A new notion of recurrent LDE with variable coefficients is introduced by analogy with an algebraic recurrent equation. Theorems of its properties have been proven. Once more this confirms interrelation between LDEs and algebraic equations.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

1. Eyler L. Integral calculus in 3 volumes, Moscow, State Technical Publishing House. 1957;2:368.
2. Sansone J. Ordinary Differential Equations, in 2 volumes, Moscow: IL. 1953;1:346; IL. 1954;2:414.
3. Yerugin NP. Reducible systems / / proceedings of the mathematical institute named after V.A. Steklov, v.13, publishing house of the USSR Academy of Sciences, Moscow-Leningrad. 1946;96.
4. Reid WT. Riccati differential equations. Academic Press, New York. 1972:216 p.
5. Van der Kloet P, Neerthoff FL. The Riccati equation as characteristic equation for general linear dynamic systems. Venue: Nonlinear Theory and its Application (NOLTA); 2001.
6. Pitaevskii LP, Stringari S. Theory of Bose-Einstein Condensation, Oxford University Press, Oxford; 2003.
7. Lloyd NG. On a class of differential equations of Riccati Type. J. London Math. Soc. 1975;10(2):1-10.
8. Polyanin AD. General Riccati equation.
   Available: http://eqworld.ipmnet.ru \en\ solations\ ode\ode0123. pdf (2004)/.
9. Blomberg H, Ylinen R. Algebraic theory for multivariable linear systems, academic press, London; 1983.
10. Mikhaylov FA et al. Dynamics of continuous linear systems with deterministic and random parameters. Moscow: Nauka. 1979;561.
11. Zhuosheng Lu, Fuding Xie. Explicit bi-soliton-like solutions for a generalized KP equation with variable coefficients. Mathematical and Computer Modelling; 2010.
12. Muryadyan MG, Yengibaryan NB. Linear differential equations of Riccati type / / USSR DAN. 1977;235(2):263-265.
13. Prikarpatsky AK. About equations of Riccati type, integrable in quadratures / / USSR DAN. 1980;251(5):1072-1077.
14. Ma Wen-Xiu. A refined invariant subspace method and application equations .Science China Mathematics. 2012;55:1769-1778.
15. Wen-Xiu Ma, Xiang Gu Liang Gao. A Note on exact solutions to linear differential equations by the matrix exponential. Advances in Applied Mathematics and Mechanics. 2009;1:573-580.
16. Wen-Xiu Ma, Boris Shekhman. Do the chain rules for matrix functions hold without commutativity? .Linear and Multilinear Algebra. 2010;58(1):79-87.
17. Trenogin VA, Khasseinov KA. Dual problems to abstract nonlocal problems. Science conference of DE, Turkey, Fetchie; 16-23 June 2001.

18. Khasseinov KA. Initial and multi-point problems for LDE and characteristic of Riccati type. Synopsis of thesis for degree of a candidate of physic-mathematical sciences. Moscow. 1984;114.

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