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Tuba's Representation of the Pure Braid Group on Three Strands

Noura Y. Maanna[1] and Mohammad N. Abdulrahim[1*]

[1] *Department of Mathematics, Beirut Arab University, Lebanon.*

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Abstract

We consider Tuba's representation of the pure braid group, P_3, defined by the map $\psi : P_3 \longrightarrow GL(V)$, where V is an algebraically closed field. We then specialize the indeterminates used in defining the representation to non- zero complex numbers. Our objective is to find necessary and sufficient conditions that guarantee the irreducibility of Tuba's representations of the pure braid group P_3 with dimensions $d = 2$ and $d = 3$.

Keywords: Braid group; pure braid group
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1 Introduction

Emil Artin, in 1920's, pointed out that the braids with n strings form the nth braid group, denoted by B_n. There exists an obvious surjective group homomorphism $\pi : B_n \longrightarrow S_n$. The kernel of π is referred to as the pure braid group P_n with n generators. In 2001, I.Tuba and H.Wenzl defined a representation of B_3, namely $\rho : B_3 \longrightarrow GL(V)$, which is simple on the dimensional vector space V over an algebraically closed field F. The generators of B_3 $(\sigma_1 and\ \sigma_2)$ have lower and upper triangular forms respectively. Tuba and Wenzl gave a complete classification of simple representations of the braid group B_3, for dimensions $d \leq 5$ (see [1]). This was done by assuming a certain triangular form of the matrices of the generators of B_3. The matrix coefficients of the generators σ_1 and σ_2 are determined by the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and possibly some scalar, up to certain renormalization. The representation of B_3 with dimension $d = 2$ is irreducible if and only if $-\lambda_1^2 + \lambda_1\lambda_2 - \lambda_2^2 \neq 0$. For dimension $d = 3$, the representation is irreducible if and only if $\left(\lambda_m^2 + \lambda_k\lambda_n\right)\left(\lambda_n^2 + \lambda_k\lambda_m\right) \neq 0$, where $k \neq m \neq n \in \{1, 2, 3\}$.

Albeverio has found a class of representations of B_3 in every dimension n, which depends on n parameters [2]. The author in that work uses a deformation of pascal's triangle connected with q-shifted factorials to obtain the representations, and this generalizes the work of Tuba and Wenzl who

*Corresponding author: E-mail: mna@bau.edu.lb

classified all simple representations of B_3 for dimensions $d \leq 5$ [1] . This is also a generalization of the results of Humphries, who constructed the representations of the braid group B_3 in arbitrary dimension using the classical pascal triangle [3]. Le Bruyn in [4] proved that all the components of $n-$dimensional simple representations of B_3 are densely parametrized by rational quiver varieties and the explicit parametrizations are given for $n < 12$. Then Le Bruyn in [5] extended all this by establishing such parametrizations for all finite dimensions n, which also generalizes the work of Tuba and Wenzl.

In our work, we mainly consider the irreducibility criteria of Tuba's representation of the normal subgroup of the braid group with n strings, namely the pure braid group P_3, with dimensions 2 and 3 (see [1]). Our main results are Theorem 5.1 and Theorem 6.1, which determine necessary and sufficient conditions for the irreducibility of Tuba's representations of P_3 with dimensions 2 and 3 respectively.

2 Preliminaries

Definition 2.1. [6] The braid group on n strings, B_n , is the abstract group with presentation $B_n = \{\sigma_1, \ldots, \sigma_{n-1};\ \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, ...n-2$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1\}$.

The generators σ_1, ..., σ_{n-1} are called the standard generators of B_n.

Definition 2.2. [6] The pure braid group,P_n, is defined as the kernel of the homomorphism $B_n \longrightarrow S_n$, defined by $\sigma_i \longrightarrow (i, i+1)$, $1 \leq i \leq n-1$. It has the following generators:

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},\ \ 1 \leq i, j \leq n$$

Definition 2.3. A representation is a map $\gamma : G \longrightarrow GL(V)$, where G is a group and $GL(V)$ is the group of $n \times n$ invertible matrices over the algebraically closed field V.

Definition 2.4. A representation $\gamma : G \longrightarrow GL(V)$ is said to be irreducible if it has no non trivial proper invariant subspaces.

3 Tuba's Representation of $\mathrm{B_3}$

Imre Tuba and Hans Wenzl gave a complete classification of all simple resentations of B_3 with dimensions $d \leq 5$ by assuming a certain triangular form for the invertible $d \times d$ matrices A and B of the generators of B_3 that satisfy $ABA = BAB$. In particular, they proved that a simple $d-$dimensional representation $\varphi : B_3 \rightarrow GL(V)$ is determined, up to isomorphism, by the eigenvalues $\lambda_1, \ldots, \lambda_d$ ofthe images of the generators σ_1 and σ_2. For more details, see [1].

Below,we write the explicit matrices in the cases $d = 2$ and $d = 3$.

Proposition 3.1. *[1,p.499] Tuba's representation of B_3 is defined as follows:*

For $d = 2$

$$\sigma_1 \to \begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{pmatrix}.$$

For $d = 3$

$$\sigma_1 \to \begin{pmatrix} \lambda_1 & \lambda_1\lambda_3\lambda_2^{-1} + \lambda_2 & \lambda_2 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} \lambda_3 & 0 & 0 \\ -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -\lambda_1\lambda_3\lambda_2^{-1} - \lambda_2 & \lambda_1 \end{pmatrix}.$$

Here $\lambda_1,\ \lambda_2$ *and* λ_3 *are indeterminates.*

We then specialize the indeterminates to non- zero complex numbers. Hence, we have the following proposition.

Proposition 3.2. *[1,p.503] Tuba's representation of* B_3 *is irreducible if and only if*

(i) $-\lambda_1^2 + \lambda_1\lambda_2 - \lambda_2^2 \neq 0$ *for dimension* $d = 2$,

(ii) $\left(\lambda_m^2 + \lambda_k\lambda_n\right)\left(\lambda_n^2 + \lambda_k\lambda_m\right) \neq 0$ *for dimension* $d = 3$.

Here $k \neq m \neq n \ \in \ \{1, 2, 3\}$.

4 Tuba's Representation of P_3

Let P_3 be the pure braid group on three strings. Applying Tuba's representation on the normal subgroup of the braid group, namely the pure braid group, we get the following representations of dimensions $d = 2$ and $d = 3$.

Definition 4.1. Tuba's representation of the pure braid group P_3 is defined as follows.

For dimension $d = 2$

$$A_{12} = \begin{pmatrix} \lambda_1^2 & \lambda_1^2 + \lambda_1\lambda_2 \\ 0 & \lambda_2^2 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \lambda_2^2 & 0 \\ -\lambda_1^2 - \lambda_1\lambda_2 & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \lambda_2^2 + \lambda_1^2 + \lambda_1\lambda_2 & \lambda_2^2 + \lambda_1\lambda_2 \\ -\lambda_1^2 - \lambda_1\lambda_2 & -\lambda_1\lambda_2 \end{pmatrix}.$$

For dimension $d = 3$

$$A_{12} = \begin{pmatrix} \lambda_1^2 & \frac{(\lambda_1+\lambda_2)\left(\lambda_2^2+\lambda_1\lambda_3\right)}{\lambda_2} & (\lambda_1+\lambda_2)(\lambda_2+\lambda_3) \\ 0 & \lambda_2^2 & \lambda_2(\lambda_2+\lambda_3) \\ 0 & 0 & \lambda_3^2 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} \lambda_3^2 & 0 & 0 \\ -\lambda_2(\lambda_2+\lambda_3) & \lambda_2^2 & 0 \\ (\lambda_1+\lambda_2)(\lambda_2+\lambda_3) & \frac{-(\lambda_1+\lambda_2)\left(\lambda_2^2+\lambda_1\lambda_3\right)}{\lambda_2} & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \frac{K}{\lambda_2} & \frac{L}{\lambda_1\lambda_2^2} & \frac{\lambda_3(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)}{\lambda_1} \\ \frac{-\lambda_1(\lambda_2+\lambda_3)(\lambda_2+\lambda_3+\lambda_1)}{\lambda_3} & \frac{-M}{\lambda_2} & -\lambda_2(\lambda_2+\lambda_3) \\ \frac{\lambda_1(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)}{\lambda_3} & \frac{(\lambda_1+\lambda_2)\left(\lambda_2^2+\lambda_1\lambda_3\right)}{\lambda_2} & \lambda_2^2 \end{pmatrix}.$$

Here the constants K, L and M are given by

$$K = \lambda_1^2(\lambda_2+\lambda_3) + \lambda_1(\lambda_2+\lambda_3)^2 + \lambda_2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right),$$

$$L = \lambda_3(\lambda_1+\lambda_2)(\lambda_2+\lambda_3+\lambda_1)\left(\lambda_2^2+\lambda_1\lambda_3\right),$$

and

$$M = \lambda_1^2\lambda_3 + \lambda_3^2(\lambda_2+\lambda_3) + \lambda_1(\lambda_2+\lambda_3)^2.$$

5 Irreducibility of Tuba's Representation of the Pure Braid Group P_3 with Dimension $d = 2$

We find a sufficient condition for the irreducibility of the complex specialization of Tuba's representation of the pure braid group P_3 with dimension $d = 2$.

Theorem 5.1. *Tuba's representation* $\varphi : P_3 \to GL_2(\mathbb{C})$ *is irreducible if and only if* $\lambda_1 \neq -\lambda_2$ *and* $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 \neq 0$.

Proof. We show that if $\lambda_1 \neq -\lambda_2$ and $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 \neq 0$ then ϕ is irreducible. We consider two cases of whether or not λ_1 and λ_2 are equal.

- Assume that $\lambda_1 \neq \lambda_2$.

The images of the generators of P_3 under φ have distinct eigenvalues. Let us diagonalize the matrix A_{12} by an invertible matrix I given by

$$I = \begin{pmatrix} 1 & \frac{-\lambda_1}{\lambda_1-\lambda_2} \\ 0 & 1 \end{pmatrix}.$$

So, we get

$$I^{-1}A_{12}I = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}.$$

Suppose, to get contradiction, that this representation is reducible. That is, there exists a non trivial proper invariant subspace S of dimension 1.

The subspace S has to be one of the following subspaces $\langle e_1 \rangle$ or $\langle e_2 \rangle$.

Case 1 $S = \langle e_1 \rangle$. Since $e_1 \in S$, it follows that

$$\left(I^{-1}A_{23}I\right) e_1 = \begin{pmatrix} \frac{\lambda_2\left(\lambda_2^2+\lambda_1^2\right)}{-\lambda_1+\lambda_2} \\ -\lambda_2\left(\lambda_1+\lambda_2\right) \end{pmatrix} \in S.$$ This implies that $(\lambda_1 + \lambda_2) = 0$,

a contradiction.

Case 2 $S = \langle e_2 \rangle$. Since $e_2 \in S$, it follows that

$$\left(I^{-1}A_{23}I\right) e_2 = \begin{pmatrix} \frac{\lambda_1\left(\lambda_1^3+\lambda_2^3\right)}{(\lambda_1-\lambda_2)^2} \\ \frac{\lambda_1\left(\lambda_1^2+\lambda_2^2\right)}{(\lambda_1-\lambda_2)} \end{pmatrix} \in S.$$

This implies that $\left(\lambda_1^3 + \lambda_2^3\right) = 0$. That is, $\lambda_1 = -\lambda_2$ or $\lambda_1 = \frac{1}{2}\lambda_2\left(1 \pm i\sqrt{3}\right)$, which lead to a contradiction. Therefore, there is no non-trivial proper invariant subspace in the case $\lambda_1 \neq \lambda_2$.

- Assume that $\lambda_1 = \lambda_2$. Then,

$$A_{12} = \begin{pmatrix} \lambda_1^2 & 2\lambda_1^2 \\ 0 & \lambda_1^2 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \lambda_1^2 & 0 \\ -2\lambda_1^2 & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} 3\lambda_1^2 & 2\lambda_1^2 \\ -2\lambda_1^2 & -\lambda_1^2 \end{pmatrix}.$$

Suppose, to get contradiction, that this representation is reducible. That is, there exists a proper non trivial invariant subspace S of dimension 1.

The subspace S has to be one of the following subspaces $\langle e_1\rangle$, $\langle e_2\rangle$ or $< e_1 + \alpha e_2 >$, where $\alpha \in \mathbb{C}^*$.

Case 3 $S = \langle e_1\rangle$. Since $e_1 \in S$, it follows that $A_{13}e_1 = \begin{pmatrix} 3\lambda_1^2 \\ \\ -2\lambda_1^2 \end{pmatrix} \in S$, a contradiction.

Case 4 $S = \langle e_2\rangle$. Since $e_2 \in S$, it follows that $A_{13}e_2 = \begin{pmatrix} 2\lambda_1^2 \\ \\ -\lambda_1^2 \end{pmatrix} \in S$, a contradiction.

Case 5 $S = \langle e_1 + \alpha e_2\rangle$. Since $(e_1 + \alpha e_2) \in S$, it follows that $A_{12}(e_1 + \alpha e_2) = \begin{pmatrix} \lambda_1^2 + 2\alpha\lambda_1^2 \\ \\ \alpha\lambda_1^2 \end{pmatrix} \in S$,

a contradiction.

We now show that if $\lambda_1 = -\lambda_2$ or $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 = 0$ then the representation is reducible.

Assume that $\lambda_1 = -\lambda_2$. Then

$$A_{12} = \begin{pmatrix} \lambda_1^2 & 0 \\ \\ 0 & \lambda_2^2 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \lambda_2^2 & 0 \\ \\ 0 & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \lambda_1^2 & 0 \\ \\ 0 & \lambda_1^2 \end{pmatrix}.$$

In this case, we have complete reducibility.

Assume that $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 = 0$, then the reducibility follows directly from that on the braid group, B_3 (see [6]).

□

6 Irreducibility of Tuba's Representation of the Pure Braid Group P_3 with Dimension $d = 3$

As for dimension $d = 2$, we find a sufficient condition that guarantees the irreducibility of Tuba's representation of P_3 with dimension 3.

Proposition 6.1. *Tuba's representation* $\varphi : P_3 \to GL_3(\mathbb{C})$ *is irreducible if* $\lambda_i \neq -\lambda_j$ *and* $\left(\lambda_m^2 + \lambda_k\lambda_n\right)\left(\lambda_n^2 + \lambda_k\lambda_m\right) \neq 0$ *for all* $i \neq j$ *and* $m \neq n \neq k \in \{1, 2, 3\}$.

Proof. We consider two cases of whether or not $\lambda_i \neq \lambda_j$ for all $i \neq j$.

- **Suppose that $\lambda_i \neq \lambda_j$ for all i and j.**

Let us diagonalize the matrix A_{12} by an invertible matrix T given by

$$T = \begin{pmatrix} 1 & \frac{\lambda_2^2+\lambda_1\lambda_3}{\lambda_2(-\lambda_1+\lambda_2)} & \frac{-\lambda_1\lambda_3-\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(-\lambda_2+\lambda_3)} \\ 0 & 1 & \frac{-\lambda_2}{\lambda_2-\lambda_3} \\ 0 & 0 & 1 \end{pmatrix}.$$

So, we get

$$(T^{-1}A_{12}T) = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}.$$

Suppose, to get contradiction, that this representation $P_3 \longrightarrow GL_3(\mathbb{C})$ is reducible. That is, there exists a proper non- zero invariant subspace, S, of dimension 1 or dimension 2.

Assume that the dimension of S is 1.

The subspace S has to be one of the following subspaces: $\langle e_1\rangle, \langle e_2\rangle$ or $\langle e_3\rangle$.

Case 6 $S = \langle e_1\rangle$. Since $e_1 \in S$, it follows that

$$\left(T^{-1}A_{23}T\right)e_1 = \begin{pmatrix} \frac{\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3)+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)} \\ \frac{\lambda_2(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{-\lambda_3+\lambda_2} \\ (\lambda_1+\lambda_2)(\lambda_2+\lambda_3) \end{pmatrix} \in S.$$

This implies that $(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)=0$, a contradiction.

Case 7 $S=\langle e_2\rangle$. Since $e_2\in S$, it follows that

$$\left(T^{-1}A_{23}T\right)e_2=\begin{pmatrix}\frac{-(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1\lambda_3+\lambda_2^2)\left(\lambda_2\lambda_3+\lambda_1^2\right)}{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)\lambda_2}\\ \\ y\\ \\ \frac{(\lambda_1\lambda_3+\lambda_2^2)(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)}{\lambda_2(-\lambda_1+\lambda_2)}\end{pmatrix}\in S.$$

This implies that $(\lambda_1\lambda_3+\lambda_2^2)=0$ or $(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)=0$, a contradiction.

Here, $y=\frac{-\lambda_1^3\lambda_3+\lambda_2^2\lambda_3^2+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_3\lambda_1\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}$.

Case 8 $S=\langle e_3\rangle$. Since $e_3\in S$, it follows that

$$\left(T^{-1}A_{13}T\right)e_3=\begin{pmatrix}\frac{(\lambda_2\lambda_3+\lambda_1^2)\left(\lambda_1\lambda_3+\lambda_2^2\right)(\lambda_1+\lambda_2)}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_2)\lambda_1}\\ \\ \frac{\left(\lambda_2\lambda_1+\lambda_3^2\right)(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)^2}\\ \\ z\end{pmatrix}\in S.$$

This implies that $(\lambda_2\lambda_3+\lambda_1^2)\left(\lambda_2\lambda_1+\lambda_3^2\right)=0$ or $(\lambda_1+\lambda_2)=0$, a contradiction.

Here, $z=\frac{\lambda_1^3\lambda_2+\lambda_2^2\lambda_3^2+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_2\lambda_1\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$.

Assume that the dimension of S is 2.

The subspace S has to be one of the following subspaces $\langle e_1,e_2\rangle,\langle e_1,e_3\rangle$ or $\langle e_2,e_3\rangle$.

Case 9 $S=\langle e_1,e_2\rangle$. Since $(\alpha e_1+\beta e_2)\in S$ for every α,β, it follows that $\left(T^{-1}A_{23}T\right)(\alpha e_1+\beta e_2)=$

$$\begin{pmatrix}-\beta\frac{(\lambda_2\lambda_3+\lambda_1^2)\left(\lambda_2\lambda_1+\lambda_3^2\right)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)^2\lambda_2}+\alpha x\\ \\ \alpha\frac{(\lambda_3+\lambda_2)(\lambda_1+\lambda_3)\lambda_2}{\lambda_2-\lambda_3}-\beta t\\ \\ \alpha(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)+\beta\frac{(\lambda_1+\lambda_2)\left(\lambda_{1+}\lambda_3\right)\left(\lambda_2^2+\lambda_1\lambda_3\right)}{\lambda_2(-\lambda_1+\lambda_2)}\end{pmatrix}\in S.$$

This implies that $(\lambda_1+\lambda_2)(\lambda_{1+}\lambda_3)\left(\lambda_2^2+\lambda_1\lambda_3\right)=0$ and $(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)=0$, a contradiction.

Here, $x = \frac{\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3)+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_2\lambda_3\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$

and

$$t = \frac{\lambda_1^3\lambda_3+\lambda_2^2\lambda_3^2+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_3\lambda_1\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}.$$

Case 10 $S = \langle e_1, e_3\rangle$. Since $(\alpha e_1 + \beta e_3) \in S$ for every α, β, it follows that $\left(T^{-1}A_{23}T\right)(\alpha e_1 + \beta e_3) =$

$$\begin{pmatrix} \beta\frac{(\lambda_2\lambda_3+\lambda_1^2)\left(\lambda_1\lambda_2+\lambda_3^2\right)(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)^2} + \alpha s \\ \\ \alpha\frac{(\lambda_3+\lambda_2)(\lambda_1+\lambda_3)\lambda_2}{\lambda_2-\lambda_3} + \beta\frac{(\lambda_1+\lambda_2)\left(\lambda_1+\lambda_3\right)\left(\lambda_3^2+\lambda_1\lambda_2\right)\lambda_2}{(-\lambda_3+\lambda_1)(\lambda_2-\lambda_3)^2} \\ \\ \alpha\left(\lambda_1+\lambda_2\right)\left(\lambda_2+\lambda_3\right)+\beta r \end{pmatrix} \in S.$$

This implies that $(\lambda_3 + \lambda_2)(\lambda_1 + \lambda_3)\lambda_2 = 0$ and $(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)({\lambda_3}^2 + \lambda_1\lambda_2)\lambda_2 = 0$, a contradiction. The constants are given by

$$s = \frac{(\lambda_1\lambda_2\lambda_3(\lambda_2+\lambda_3)+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_2\lambda_3\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$

and

$$r = \lambda_1^2 + \frac{(\lambda_1+\lambda_2)^2\lambda_3(\lambda_3+\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} + \frac{\left(\lambda_1\lambda_3+\lambda_2^2\right)(\lambda_1+\lambda_2)}{(\lambda_2-\lambda_3)}.$$

Case 11 $S = \langle e_2, e_3\rangle$. Since $(\alpha e_2 + \beta e_3) \in S$, for every α, β, it follows that $\left(T^{-1}A_{23}T\right)(\alpha e_2 + \beta e_3) =$

$$\begin{pmatrix} \beta f - \alpha\frac{\left(\lambda_1\lambda_3+\lambda_2^2\right)(\lambda_1+\lambda_3)(\lambda_2\lambda_3+\lambda_1^2)(\lambda_2+\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)^2\lambda_2} \\ \\ \beta\frac{(\lambda_1+\lambda_2)\left(\lambda_1+\lambda_3\right)\left(\lambda_3^2+\lambda_1\lambda_2\right)\lambda_2}{(-\lambda_3+\lambda_1)(\lambda_2-\lambda_3)^2} - \alpha h \\ \\ \beta r + \alpha\frac{(\lambda_1+\lambda_2)\left(\lambda_1+\lambda_3\right)\left(\lambda_2^2+\lambda_1\lambda_3\right)}{\lambda_2(-\lambda_1+\lambda_2)} \end{pmatrix} \in S.$$

This implies that $(\lambda_2\lambda_3 + \lambda_1^2)\left(\lambda_1\lambda_2 + \lambda_3^2\right)(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3) = 0$

and

$$\left(\lambda_1\lambda_3 + \lambda_2^2\right)(\lambda_1 + \lambda_3)(\lambda_2\lambda_3 + \lambda_1^2)(\lambda_2 + \lambda_3) = 0.$$

Here, the constants are given by

$$h = \frac{\lambda_1^3\lambda_3+\lambda_2^2\lambda_3^2+\lambda_1^2\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_3\lambda_1\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}$$

and

$$f = \frac{(\lambda_2\lambda_3+\lambda_1^2)\left(\lambda_1\lambda_2+\lambda_3^2\right)(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)^2}.$$

Therefore, there is no non trivial invariant subspace of dimensions 1 and 2. This implies that, the representation is irreducible for $\lambda_i \neq \lambda_j$ for all distinct integers $i, j, k \in \{1, 2, 3\}$.

- **Suppose that $\lambda_i = \lambda_j$ for some i and j.**

Case 12 $\lambda_1 = \lambda_3$. Then

$$A_{12} = \begin{pmatrix} \lambda_1^2 & \frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2} & (\lambda_2+\lambda_1)^2 \\ 0 & \lambda_2^2 & \lambda_2\left(\lambda_2+\lambda_1\right) \\ 0 & 0 & \lambda_1^2 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ -\lambda_2\left(\lambda_2+\lambda_1\right) & \lambda_2^2 & 0 \\ (\lambda_2+\lambda_1)^2 & -\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2} & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \frac{2\lambda_1^3+4\lambda_2\lambda_1^2+2\lambda_1\lambda_2^2+\lambda_2^3}{\lambda_2} & \frac{(\lambda_2+\lambda_1)(2\lambda_1+\lambda_2)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2^2} & (\lambda_2+\lambda_1)^2 \\ -\left(\lambda_2+\lambda_1\right)\left(\lambda_2+2\lambda_1\right) & -\frac{(\lambda_1^3+(\lambda_2+\lambda_1)(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2))}{\lambda_2} & -\lambda_2(\lambda_2+\lambda_1) \\ (\lambda_2+\lambda_1)^2 & \frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2} & \lambda_2^2 \end{pmatrix}.$$

Suppose that there exists a non trivial invariant subspace S, in which $(\alpha e_1+\beta e_2+\gamma e_3) \in S, \quad \alpha, \beta, \gamma \neq 0$. It follows that

$$A_{12}(\alpha e_1+\beta e_2+\gamma e_3) = \begin{pmatrix} \alpha\lambda_1^2+\beta\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2}+\gamma\left(\lambda_2+\lambda_1\right)^2 \\ \beta\lambda_2^2+\gamma\lambda_2\left(\lambda_2+\lambda_1\right) \\ \gamma\lambda_1^2 \end{pmatrix} \in S \tag{6.1}$$

and

$$\lambda_1^2(\alpha e_1 + \beta e_2 + \gamma e_3) = \begin{pmatrix} \alpha\lambda_1^2 \\ \\ \beta\lambda_1^2 \\ \\ \gamma\lambda_1^2 \end{pmatrix} \in S. \tag{6.2}$$

Let us subtract equation (6.2) from equation (6.1). Then $A_{12}(\alpha e_1+\beta e_2+\gamma e_3)-\lambda_1^2(\alpha e_1+\beta e_2+\gamma e_3) =$

$$\begin{pmatrix} \beta\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2} + \gamma\left(\lambda_2+\lambda_1\right)^2 \\ \\ \beta(\lambda_2^2-\lambda_1^2) + \gamma\lambda_2\left(\lambda_2+\lambda_1\right) \\ \\ 0 \end{pmatrix} \in S. \tag{6.3}$$

For simplicity, we let $u = \beta\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2} + \gamma\left(\lambda_2+\lambda_1\right)^2$ and $w = \beta(\lambda_2^2-\lambda_1^2) + \gamma\lambda_2\left(\lambda_2+\lambda_1\right)$.

Again, we have

$$A_{12}(e_1u + e_2w) = \begin{pmatrix} \lambda_1^2 u + \frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2} w \\ w\lambda_2^2 \\ \\ 0 \end{pmatrix} \in S \tag{6.4}$$

and

$$\lambda_2^2(e_1u + e_2w) = \begin{pmatrix} \lambda_2^2 u \\ \\ \lambda_2^2 w \\ 0 \end{pmatrix} \in S. \tag{6.5}$$

Now, we subtract equation (6.5) from equation (6.4) to get

$$e_1(u(\lambda_1^2-\lambda_2^2) + w\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2}) \in S.$$

Here, we consider the following two cases.

1. If $u(\lambda_1^2-\lambda_2^2)+w\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2}\neq 0$ then $e_1\in S$. So $e_2\in S$ and $e_3\in S$. This implies that the invariant subspace S is the whole space, that is $S=\mathbb{C}^3$.

2. If $u(\lambda_1^2-\lambda_2^2)+w\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2}=0$, then $u(\lambda_1^2-\lambda_2^2)=-w\frac{(\lambda_2+\lambda_1)\left(\lambda_2^2+\lambda_1^2\right)}{\lambda_2}$. This implies that $-\gamma\lambda_2\left(\lambda_2+\lambda_1\right)^2\left(2\lambda_1^2\right)=0$, a contradiction.

Therefore, there is no non trivial proper invariant subspace in the case $\lambda_1=\lambda_3$.

Case 13 $\lambda_1=\lambda_2$. Then

$$A_{12}=\begin{pmatrix}\lambda_1^2 & 2\lambda_1\left(\lambda_1+\lambda_3\right) & 2\lambda_1\left(\lambda_1+\lambda_3\right)\\ 0 & \lambda_1^2 & \lambda_1\left(\lambda_1+\lambda_3\right)\\ 0 & 0 & \lambda_3^2\end{pmatrix},$$

$$A_{23}=\begin{pmatrix}\lambda_3^2 & 0 & 0\\ -\lambda_1\left(\lambda_1+\lambda_3\right) & \lambda_1^2 & 0\\ 2\lambda_1\left(\lambda_1+\lambda_3\right) & -2\lambda_1\left(\lambda_1+\lambda_3\right) & \lambda_1^2\end{pmatrix}$$

and

$$A_{13}=\begin{pmatrix}3\lambda_1^2+4\lambda_1\lambda_3+2\lambda_3^2 & \frac{2\lambda_3\left(2\lambda_1^2+3\lambda_1\lambda_3+\lambda_3^2\right)}{\lambda_1} & 2\lambda_3\left(\lambda_1+\lambda_3\right)\\ \frac{-\lambda_1(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3)}{\lambda_3} & -2\lambda_1^2-4\lambda_1\lambda_3-\lambda_3^2 & -\lambda_1\left(\lambda_1+\lambda_3\right)\\ \frac{2\lambda_1^2(\lambda_1+\lambda_3)}{\lambda_3} & 2\lambda_1\left(\lambda_1+\lambda_3\right) & \lambda_1^2\end{pmatrix}.$$

Suppose that there exists an invariant subspace S, then $\alpha e_1+\beta e_2+\gamma e_3\in S,\ \alpha,\beta,\gamma\neq 0$. It follows that $A_{12}(\alpha e_1+\beta e_2+\gamma e_3)\in S$ and $A_{23}(\alpha e_1+\beta e_2+\gamma e_3)\in S$. Let us take the case where $A_{12}(\alpha e_1+\beta e_2+\gamma e_3)\in S$. That is

$$A_{12}(\alpha e_1+\beta e_2+\gamma e_3)=\begin{pmatrix}\alpha\lambda_1^2+2\beta\lambda_1\left(\lambda_1+\lambda_3\right)+2\gamma\lambda_1\left(\lambda_1+\lambda_3\right)\\ \beta\lambda_1^2+\gamma\lambda_1\left(\lambda_1+\lambda_3\right)\\ \gamma\lambda_3^2\end{pmatrix}\in S \tag{6.6}$$

and

$$\lambda_3^2(\alpha e_1 + \beta e_2 + \gamma e_3) = \begin{pmatrix} \alpha\lambda_3^2 \\ \\ \beta\lambda_3^2 \\ \\ \gamma\lambda_3^2 \end{pmatrix} \in S. \tag{6.7}$$

Subtracting equation (6.7) from equation (6.6), we get :

$e_1 t + e_2 q \in S$, where $t = \alpha(\lambda_1^2 - \lambda_3^2) + 2\beta\lambda_1(\lambda_1 + \lambda_3) + 2\gamma\lambda_1(\lambda_1 + \lambda_3)$ and $q = \beta(\lambda_1^2 - \lambda_3^2) + \gamma\lambda_1(\lambda_1 + \lambda_3)$.

Then

$$A_{12}(e_1 t + e_2 q) = \begin{pmatrix} \lambda_1^2 t + 2\lambda_1(\lambda_1 + \lambda_3) q \\ \\ \lambda_1^2 q \\ \\ 0 \end{pmatrix} \in S \tag{6.8}$$

and

$$\lambda_1^2(e_1 t + e_2 q) = \begin{pmatrix} \lambda_1^2 t \\ \\ \lambda_1^2 q \\ \\ 0 \end{pmatrix} \in S. \tag{6.9}$$

Let us subtract equation (6.9) from equation (6.8). Then we have the following equation

$A_{12}(e_1 t + e_2 q) - \lambda_1^2(e_1 t + e_2 q) = 2e_1\lambda_1(\lambda_1 + \lambda_3)^2(\beta(\lambda_1 - \lambda_3) + \gamma) \in S.$

Let us consider the following two cases:

1. If $2\lambda_1(\lambda_1 + \lambda_3)^2(\beta(\lambda_1 - \lambda_3) + \gamma) \neq 0$, then the invariant subpace S will be the whole space.
2. If $2\lambda_1(\lambda_1 + \lambda_3)^2(\beta(\lambda_1 - \lambda_3) + \gamma) = 0$, this implies that $\beta(\lambda_1 - \lambda_3) = -\gamma\lambda_1$.

On the other hand, we have $A_{23}(\alpha e_1+\beta e_2+\gamma e_3)\in S$. Then

$$A_{23}(\alpha e_1+\beta e_2+\gamma e_3)=\begin{pmatrix}\alpha\lambda_3^2\\ \\ -\alpha\lambda_1\left(\lambda_1+\lambda_3\right)+\beta\lambda_1^2)\\ \\ 2\alpha\lambda_1\left(\lambda_1+\lambda_3\right)-2\beta\lambda_1\left(\lambda_1+\lambda_3\right)+\gamma\lambda_1^2\end{pmatrix}\in S \tag{6.10}$$

and

$$\lambda_3^2\left(\alpha e_1+\beta e_2+\gamma e_3\right)=\begin{pmatrix}\alpha\lambda_3^2\\ \\ \beta\lambda_3^2\\ \\ \gamma\lambda_3^2\end{pmatrix}\in S. \tag{6.11}$$

Now, after subtracting equation (6.11) from equation (6.10), we have

$$e_2n+e_3p\in S.$$

Here the constants n and p are given by

$$n=-\alpha\lambda_1\left(\lambda_1+\lambda_3\right)+\beta(\lambda_1^2-\lambda_3^2),$$

$$p=2\alpha\lambda_1\left(\lambda_1+\lambda_3\right)-2\beta\lambda_1\left(\lambda_1+\lambda_3\right)+\gamma(\lambda_1^2-\lambda_3^2).$$

Then

$$A_{23}(e_2n+e_3p)=\begin{pmatrix}0\\ \lambda_1^2n\\ \\ -2\lambda_1\left(\lambda_1+\lambda_3\right)n+\lambda_1^2p\end{pmatrix}\in S \tag{6.12}$$

and

$$\lambda_1^2\left(e_2n+e_3p\right)=\begin{pmatrix}0\\ \\ \lambda_1^2n\\ \\ \lambda_1^2p\end{pmatrix}\in S. \tag{6.13}$$

We subtract equation (6.13) from equation (6.12). We then have that $e_3(-2\lambda_1(\lambda_1+\lambda_3)n) \in S$. Now, we consider the following two cases

1. If $-2\lambda_1(\lambda_1+\lambda_3)n \neq 0$, then $e_3 \in S$. So $e_2 \in S$, and $e_1 \in S$. This implies that the invariant subspace S will be the whole space, that is $S = \mathbb{C}^3$.

2. If $-2\lambda_1(\lambda_1+\lambda_3)n = 0$, then $\beta(\lambda_1 - \lambda_3) = \alpha\lambda_1$.

Under direct computations, we get $\alpha = -\gamma$. Let us substitute $\alpha = -\gamma$ in the following vector $e_2(n) + e_3(p)$. It follows that $e_3[\gamma(\lambda_1^2 + \lambda_3^2)] \in S$. Then $e_3 \in S,\ e_2 \in S$, and $e_1 \in S$ and so the invariant subspace S is the whole space, that is $S = \mathbb{C}^3$.

Case 14 $\lambda_2 = \lambda_3$. Then

$$A_{12} = \begin{pmatrix} \lambda_1^2 & (\lambda_2+\lambda_1)^2 & 2\lambda_2(\lambda_2+\lambda_1) \\ 0 & \lambda_2^2 & 2\lambda_2^2 \\ 0 & 0 & \lambda_2^2 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} \lambda_2^2 & 0 & 0 \\ -2\lambda_2^2 & \lambda_2^2 & 0 \\ 2\lambda_2(\lambda_2+\lambda_1) & -(\lambda_2+\lambda_1)^2 & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} 2\lambda_1^2 + 4\lambda_2\lambda_1 + 3\lambda_2^2 & \frac{(\lambda_2+\lambda_1)^2(2\lambda_2+\lambda_1)}{\lambda_1} & 2\lambda_1(\lambda_2+\lambda_1) \\ -2\lambda_1(2\lambda_2+\lambda_1) & -\lambda_1^2 - 4\lambda_2\lambda_1 - 2\lambda_2^2 & -2\lambda_2^2 \\ 2\lambda_1(\lambda_2+\lambda_1) & (\lambda_2+\lambda_1)^2 & \lambda_2^2 \end{pmatrix}.$$

Suppose that there exists an invariant subspace S. Then $e_1 \in S$ or $e_2 + ve_3 \in S$ for some $v \in \mathbb{C}^*$. Assume that $e_1 \in S$, it follows that $A_{23}e_1 \in S$ and $A_{13}e_1 \in S$. Now, we consider the case where $A_{23}e_1 \in S$. Then

$$A_{23}e_1 = \begin{pmatrix} \lambda_2^2 \\ -2\lambda_2^2 \\ 2\lambda_2(\lambda_2+\lambda_1) \end{pmatrix} \in S \tag{6.14}$$

and

$$-\lambda_2^2 e_1 = \begin{pmatrix} -\lambda_2^2 \\ 0 \\ 0 \end{pmatrix} \in S. \tag{6.15}$$

Add equation (6.14) and equation (6.15) to get

$$-2\lambda_2^2 e_2 + 2\lambda_2 (\lambda_2 + \lambda_1) e_3 = \begin{pmatrix} 0 \\ -2\lambda_2^2 \\ 2\lambda_2 (\lambda_2 + \lambda_1) \end{pmatrix} \in S. \tag{6.16}$$

Now, we consider the case where $A_{13}e_1 \in S$, that is

$$A_{13}e_1 = \begin{pmatrix} 2\lambda_1^2 + 4\lambda_2\lambda_1 + 3\lambda_2^2 \\ -2\lambda_1 (2\lambda_2 + \lambda_1) \\ 2\lambda_1 (\lambda_2 + \lambda_1) \end{pmatrix} \in S \tag{6.17}$$

and

$$(-2\lambda_1^2 - 4\lambda_2\lambda_1 - 3\lambda_2^2)e_1 = \begin{pmatrix} -2\lambda_1^2 - 4\lambda_2\lambda_1 - 3\lambda_2^2 \\ 0 \\ 0 \end{pmatrix} \in S. \tag{6.18}$$

Now add equation (6.17) and equation (6.18) to get

$$-2\lambda_1 (2\lambda_2 + \lambda_1) e_2 + 2\lambda_1 (\lambda_2 + \lambda_1) e_3 = \begin{pmatrix} 0 \\ -2\lambda_1 (2\lambda_2 + \lambda_1) \\ 2\lambda_1 (\lambda_2 + \lambda_1) \end{pmatrix} \in S. \tag{6.19}$$

Let us multiply equation (6.19) by λ_2 and equation (6.16) by λ_1. Then, by simple calculations, we get that $(\lambda_2+\lambda_1)\, e_2 \in S$.

On the other hand, we have

$$A_{23}e_2 = \begin{pmatrix} 0 \\ \\ \lambda_2^2 \\ \\ -(\lambda_2+\lambda_1)^2 \end{pmatrix} \in S \tag{6.20}$$

and

$$-\lambda_2^2 e_2 = \begin{pmatrix} 0 \\ \\ -\lambda_2^2 \\ \\ 0 \end{pmatrix} \in S. \tag{6.21}$$

By adding the equations (6.20) and (6.21), we get that $-(\lambda_2+\lambda_1)^2\, e_3 \in S$. Then $e_3 \in S$. In this case, the invariant subspace is $S = \mathbb{C}^3$.

We now assume that $e_2 + ve_3 \in S$. Then $A_{23}\,(e_2+ve_3) \in S$, where $v \neq 0$. So, we have

$$A_{23}\,(e_2+ve_3) = \begin{pmatrix} 0 \\ \\ \lambda_2^2 \\ \\ -(\lambda_1+\lambda_2)^2 + v\lambda_1^2 \end{pmatrix} \in S \tag{6.22}$$

and

$$-\lambda_2^2(e_2+e_3) = \begin{pmatrix} 0 \\ \\ -\lambda_2^2 \\ \\ -v\lambda_2^2 \end{pmatrix} \in S. \tag{6.23}$$

By adding equation (6.22) and equation (6.23), we have $\left(\lambda_1^2 - \lambda_2^2\right) v = (\lambda_1+\lambda_2)^2$.

Let us substitute $\left(\lambda_1^2 - \lambda_2^2\right) v = (\lambda_1 + \lambda_2)^2$ in the vector $(e_2 + ve_3)$. This implies that $\left(\lambda_1^2 - \lambda_2^2\right) e_2 + (\lambda_1 + \lambda_2)^2 e_3 \in S$.

Then we have

$$A_{13}(\left(\lambda_1^2 - \lambda_2^2\right) e_2 + (\lambda_1 + \lambda_2)^2 e_3) = \begin{pmatrix} (\lambda_1 + \lambda_2)^4 \\ \\ -\lambda_1^2(\lambda_1^2 + 4\lambda_2\lambda_1 + 3\lambda_2^2) \\ \\ \lambda_1^2 (\lambda_1 + \lambda_2)^2 \end{pmatrix} \in S \tag{6.24}$$

and

$$-\lambda_1^2[\left(\lambda_1^2 - \lambda_2^2\right) e_2 + (\lambda_1 + \lambda_2)^2 e_3] = \begin{pmatrix} 0 \\ \\ -\lambda_1^2 \left(\lambda_1^2 - \lambda_2^2\right) \\ \\ -\lambda_1^2 (\lambda_1 + \lambda_2)^2 \end{pmatrix} \in S. \tag{6.25}$$

Adding the equations (6.24) and (6.25), we get the vector $(\lambda_1 + \lambda_2)^2 e_1 - 2\lambda_1^2 e_2 \in S$. Then

$$A_{12}[(\lambda_1 + \lambda_2)^2 e_1 - 2\lambda_1^2 e_2] = \begin{pmatrix} -\lambda_1^2 (\lambda_1 + \lambda_2)^2 \\ \\ -2\lambda_1^2\lambda_2^2 \\ \\ 0 \end{pmatrix} \in S. \tag{6.26}$$

Also we have

$$\lambda_2^2[-(\lambda_1 + \lambda_2)^2 e_1 + 2\lambda_1^2 e_2] = \begin{pmatrix} -\lambda_2^2 (\lambda_1 + \lambda_2)^2 \\ \\ 2\lambda_1^2\lambda_2^2 \\ \\ 0 \end{pmatrix} \in S. \tag{6.27}$$

We add equation (6.26) and equation (6.27) to get that $-(\lambda_1 + \lambda_2)^2 [\lambda_2^2 + \lambda_1^2] e_1 \in S$. This implies that $e_1 \in S$, $e_2 \in S$ and $e_3 \in S$. Then the invariant subspace will be the whole space, that is $S = \mathbb{C}^3$. So, there is no non trivial proper invariant subspace in the case $\lambda_2 = \lambda_3$.

□

Next, we find a necessary condition that guarantees the irreducibility of Tuba's representation P_3 with dimension 3.

Proposition 6.2. *Tuba's representation* $\varphi : P_3 \to GL_3(\mathbb{C})$ *is reducible if* $\lambda_i = -\lambda_j$ *or* $\left(\lambda_m^2 + \lambda_k\lambda_n\right)\left(\lambda_n^2 + \lambda_k\lambda_m\right) = 0$ *for some* $i \neq j$ *or* $m \neq n \neq k \in \{1, 2, 3\}$.

Proof. Suppose $\lambda_i = -\lambda_j$ for some $i \neq j$.

Case 15 $\lambda_1 = -\lambda_2$. Then

$$A_{12} = \begin{pmatrix} \lambda_2^2 & 0 & 0 \\ 0 & \lambda_2^2 & \lambda_2\left(\lambda_2 + \lambda_3\right) \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \lambda_3^2 & 0 & 0 \\ -\lambda_2\left(\lambda_2 + \lambda_3\right) & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \eta & 0 & 0 \\ -\lambda_2\left(\lambda_2 + \lambda_3\right) & -[\lambda_2\lambda_3 + \lambda_2\left(\lambda_2 + \lambda_3\right) - \left(\lambda_2 + \lambda_3\right)^2] & -\lambda_2\left(\lambda_2 + \lambda_3\right) \\ 0 & 0 & \lambda_2^2 \end{pmatrix}.$$

Here, $\eta = \lambda_2\left(\lambda_2 + \lambda_3\right) + \left(\lambda_2 + \lambda_3\right)^2 + \left(\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2\right)$. In this case, there exists an invariant subspace of dimension 1 and it is generated by e_2.

Case 16 $\lambda_2 = -\lambda_3$. Then

$$A_{12} = \begin{pmatrix} \lambda_1^2 & \left(\lambda_2 + \lambda_1\right)\left(\lambda_2 - \lambda_1\right) & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} \lambda_2^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & -\left(\lambda_2 + \lambda_1\right)\left(\lambda_2 - \lambda_1\right) & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \lambda_2^2 & \left(\lambda_2 + \lambda_1\right)\left(\lambda_2 - \lambda_1\right) & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & \left(\lambda_2 + \lambda_1\right)\left(\lambda_2 - \lambda_1\right) & \lambda_2^2 \end{pmatrix}.$$

In this case, there exists an invariant subspace of dimension 1 that is generated by e_1.

Case 17 $\lambda_1 = -\lambda_3$. Then

$$A_{12} = \begin{pmatrix} \lambda_1^2 & \tau & (\lambda_1+\lambda_2)(\lambda_2-\lambda_1) \\ 0 & \lambda_2^2 & \lambda_2(\lambda_2-\lambda_1) \\ 0 & 0 & \lambda_1^2 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ -\lambda_2(\lambda_2-\lambda_1) & \lambda_2^2 & 0 \\ (\lambda_1+\lambda_2)(\lambda_2-\lambda_1) & -\tau & \lambda_1^2 \end{pmatrix}$$

and

$$A_{13} = \begin{pmatrix} \lambda_2^2 & -\tau & \lambda_1^2-\lambda_2^2 \\ \lambda_2(\lambda_2-\lambda_1) & 2\lambda_1^2-\lambda_2^2 & -\lambda_2(\lambda_2-\lambda_1) \\ \left(\lambda_1^2-\lambda_2^2\right) & \tau & \lambda_2^2 \end{pmatrix}.$$

Here $\tau = \frac{(\lambda_1+\lambda_2)\left(\lambda_2^2-\lambda_1^2\right)}{\lambda_2}$. Let us consider the following assumptions.

1. Assume that $\lambda_2 = \lambda_1$, then $\lambda_2 = -\lambda_3$. The invariant subspace is of dimension $d = 1$ and is generated by e_1 (see Case 16).

2. Assume that $\lambda_2 = \lambda_3$, then $\lambda_2 = -\lambda_1$. The invariant subspace is of dimension $d = 1$ and is generated by e_2 (see Case 15).

3. Assume that $\lambda_2 \neq \lambda_1$ and $\lambda_2 \neq \lambda_3$. Now we can diagonalize A_{12} by the invertible matrix χ given by

$$\chi = \begin{pmatrix} 0 & 1 & -\frac{-\lambda_2-\lambda_1}{\lambda_2} \\ -\frac{\lambda_2}{\lambda_2+\lambda_1} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, we get

$$(\chi^{-1}A_{12}\chi) = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix}.$$

Now, we conjugate A_{23} and A_{13} by the matrix χ, it follows that

$$(\chi^{-1}A_{23}\chi) = \begin{pmatrix} \lambda_2^2 & -\lambda_1^2+\lambda_2^2 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$$

and

$$(\chi^{-1}A_{13}\chi) = \begin{pmatrix} \lambda_1^2 & \lambda_1^2-\lambda_2^2 & 0 \\ \lambda_1(-\lambda_2+\lambda_1(1+\lambda_2^2)) & \lambda_2^2 & -\frac{\lambda_1(-\lambda_2^2+\lambda_1\lambda_2^3+\lambda_1^2(1+\lambda_2^2))}{\lambda_2} \\ 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

In this case, there exists an invariant subspace of dimension 2 that is generated by $< e_1, e_2 >$. Therefore, this representation is reducible for $\lambda_i = -\lambda_j$.

□

Therefore, we have determined a necessary and sufficient condition for the irreducibility of the pure braid group P_3 for dimension $d = 3$. Hence, we have the following theorem.

Theorem 6.1. *Tuba's representation* $\varphi : P_3 \to GL_3(\mathbb{C})$ *is irreducible if and only if* $\lambda_i \neq -\lambda_j$ *and* $\left(\lambda_m^2 + \lambda_k\lambda_n\right)\left(\lambda_n^2 + \lambda_k\lambda_m\right) \neq 0$ *for all* $i \neq j$ *and* $m \neq n \neq k \in \{1, 2, 3\}$.

7 Conclusion

Theorem 5.1 and Theorem 6.1 determine necessary and sufficient conditions for the irreducibility of the complex specialization of Tuba's representations of the pure braid group P_3 with dimensions 2 and 3.

Competing Interests

The authors declare that no competing interests exist.

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