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On Summable Bases in Banach Spaces

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Abstract

We introduce the notion of summable bases that naturally generalizes the notion of unconditional sequence bases for Banach spaces. We shall be particularly interested in some classical results on sequences and series in separable Banach spaces that carry over or naturally extend to the case of non-separable Banach spaces.

Keywords: Unconditional bases; unconditional convergence; Dvoretski-Rogers Theorem; bounded approximation; finite dimensional decomposition

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1 Introduction

The preponderant roles played by sequence bases in the study of the structure of Banach spaces have been extensively studied and several results have been established by several authors (see for example, [1],[2],[3] and references therein) many of which are believed to be in their final forms. Undoubtedly, the most useful and widely studied special case of such sequence bases is that of unconditional bases. The property that a basis is a sequence is quite irrelevant since the definition of unconditional basis applies to any arbitrary countable family of elements of the Banach space. The primary aim of this note is to introduce the more general notion of unconditional uncountable bases which we shall simply term as summable basis. Such a generalization allows us to extend some results and techniques of unconditional bases to the wider class of non separable Banach spaces.

The paper is organized as follows. In Section 2, we introduce and discuss the notion of summability of functions taking values in Banach spaces. Several results related to unconditional convergence of series are trivially generalized to the setting of the newly introduced notion. In Section 3, we show that the class of Banach space valued summmable functions can be given the structure of a Banach space. An extended version of the Dvortski-Rogers theorem [4] is obtained as an application. In the final Section 4, we introduce and study the notion of summable bases and related properties. Examples of Banach spaces having summable bases but failing to have sequence bases will be given. We also give an extension of the notion of unconditional finite dimensional decomposition.

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2 Definition of Summability

For any study of unconditional bases in Banach spaces, a featured role must be reserved for the unconditional convergence of series. A series $\sum_n x_n$ of elements of a Banach space X is to be unconditionally convergent if the series $\sum_n y_n$ converges whenever the sequence $n \mapsto y_n$ is a rearrengement of the sequence $n \mapsto x_n$. There are several equivalent formulations of such a definition (see for example [5]). In this section, we introduce the notion of summability of function that generalizes the notion of unconditonal convergence of series.

In what follows, X is a normed vector space, Ω is an infinite set, 2^{Ω} (resp. $2^{|\Omega|}$) denotes the set of all subsets (resp. finite subsets) of Ω. Given a function $f : \Omega \to X$, we associate the set function

$$\sigma_f : 2^{|\Omega|} \to X : \ A \mapsto \sigma_f(A) = \sum_{a \in A} f(a).$$

Since $2^{|\Omega|}$ is directed by containment $\supset$, the function σ_f is a net. We denote by

$$\sum_{\Omega} f = \lim_{A \in (2^{|\Omega|}, \supset)} \sigma_f(A)$$

whether or not such limit exists. Note that by the property of net-limit, there can only exist at most one such limit. For more details on net-limit, we refer the reader to [6].

We introduce the following definition.

Definition 2.1. Let X be a normed vector space. A function $f : \Omega \to X$ is said to be **summable** if the limit $\sum_{\Omega} f = \lim_{A \in (2^{|\Omega|}, \supset)} \sigma_f(A)$ exists in X.

In other words, for every $\varepsilon > 0$, there exists $A_0 \in 2^{|\Omega|}$ such that for every $A \in 2^{|\Omega|}$, $A \supset A_0$, we have $\left\|\sigma_f(A) - \sum_{\Omega} f\right\| < \varepsilon$.

It is not difficult to see that when Ω consists of the positive integrers, then f is summable if and ony if the series $\sum_{i=1}^{\infty} f(i)$ is unconditionally convergent.

We also have the following proposition:

Proposition 2.1. *If $f : \Omega \to X$ is summable, then for every $\varepsilon > 0$, there exists $A_0 \in 2^{|\Omega|}$ such that $\left\|\sum_{b \in B} f(b)\right\| < \varepsilon$ for every $B \in 2^{|\Omega|}$ that does not intersect A_0,*

Proof. Fix $\varepsilon > 0$. Let $A_0 \in 2^{|\Omega|}$ be such that for every $2^{|\Omega|} \ni A \supset A_0$, $\left\|\sigma_f(A) - \sum_{\Omega} f\right\| < \varepsilon/2$. Fix such a finite subset A. Then we also have for every $B \in 2^{|\Omega|}$ that does not intersect A_0, $A \cup B \supset A_0$, and therefore $\left\|\sigma_f(A \cup B) - \sum_{\Omega} f\right\| < \varepsilon/2$. It follows that

$$\left\|\sum_{b \in B} f(b)\right\| = \|\sigma_f(A \cup B) - \sigma_f(A)\| \le \left\|\sigma_f(A \cup B) - \sum_{\Omega} f\right\| + \left\|\sigma_f(A) - \sum_{\Omega} f\right\| < \varepsilon.$$

The proof is complete. □

Definition 2.2. Let$X$ be a normed vector space. A function $f : \Omega \to X$ is said to satisfy the **Cauchy summability criterion** if for every $\varepsilon > 0$, there exists a finite subset A_0 of $\mathbb{N}$ such that

$$\left\|\sum_{n \in A} f(n) - \sum_{n \in B} f(n)\right\| < \epsilon$$

whenever $A, B \in 2^{|\Omega|}$, $A, B \supset A_0$.

We notice that for every $A, B \in 2^{|\Omega|}$,

$$\left\| \sum_{n \in A} f(n) - \sum_{n \in B} f(n) \right\| = \left\| \sum_{n \in A \triangle B} f(n) \right\|$$

where $A \triangle B$ is the symmetric difference of A and B. Clearly, $A \triangle B \in 2^{|\Omega|}$. The following fact is then easily derived.

Proposition 2.2. *A function $f : \Omega \to X$ satisfies the Cauchy summability criterion if and only if for every $\varepsilon > 0$ there exists a finite subset A_0 of Ω such that $\|\sum_{b \in B} f(b)\| < \varepsilon$ for every $B \in 2^{|\Omega|}$ that does not intersect A_0.*

We have already seen in Proposition 2.1 that every summable function satisfies the Cauchy summability criterion. As expected, the converse is also true if X is a Banach space. This follows from the general well known fact that for nets taking values in a Banach space, the Cauchy net condition is equivalent to the net convergence (see for example [6]). Clearly, the Cauchy summability condition introduced in Definition 2.2 corresponds exactly to the Cauchy criterion for the net $2^{|\Omega|} \ni A \mapsto \sigma_f(A) \in X$. Therefore we have:

Proposition 2.3. *Let X be a Banach space. A function $f : \Omega \to X$ satisfies the Cauchy summability criterion if and only if it is summable.*

The following are other useful characterizations of summability for Banach space valued functions.

Theorem 2.1. *Let X be a Banach space and let $f : \Omega \to X$. The following conditions are equivalent:*

1. *f is summable.*
2. *For any injection $\omega : \Gamma \to \Omega$, the function $\gamma \mapsto f(\omega(\gamma))$ is summable.*
3. *For every $\epsilon : \Omega \to \{-1, 1\}$, the function $\omega \mapsto \epsilon(\omega)f(\omega)$ is summable.*
4. *For every bounded function $\phi : \Omega \to \mathbb{K}$, the function $\omega \mapsto \phi(\omega)f(\omega)$ is summable.*

Proof. It is clear that $4 \Rightarrow 3 \Rightarrow 1$ and also $2 \Rightarrow 1$. To see that $1 \Rightarrow 2$, suppose f is summable and let $\varepsilon > 0$. Then there exists a finite subset A_0 of Ω such that $\|\sigma_f(A)\| < \varepsilon$ whenever $A \in 2^{|\Omega|}$ is disjoint from A_0. Let $\omega : \Gamma \to \Omega$ be an injective mapping. By injectivity of ω, we can choose $B_0 \in 2^{|\Gamma|}$ so that $\omega(B_0) \supset A_0$. Again by injectivity of ω, if $B \cap B_0 = \emptyset$, then $\omega(B) \cap \omega(B_0) = \emptyset$ and therefore $\omega(B) \cap A_0$. It follows that whenever $B \in 2^{|\Omega|}$ is disjoint from A_0 we have

$$\left\| \sum_{\gamma \in B} f(\omega(\gamma)) \right\| = \|\sigma_f(\omega(B))\| < \varepsilon.$$

Hence, the function $\gamma \mapsto f(\omega(\gamma))$ is summable. We have established that $1 \Rightarrow 2$. To show that $2 \Rightarrow 3$, let $\Gamma_1 = \epsilon^{-1}(1)$ and $\Gamma_{-1} = \epsilon^{-1}(-1)$. Then $\Omega = \Gamma_1 \cup \Gamma_{-1}$ and $\Gamma_1 \cap \Gamma_{-1} = \emptyset$. Let $\omega_1 : \Gamma_1 \to \Omega$ and $\omega_{-1} : \Gamma_{-1} \to \Omega$ be respectively, the canonical injection respectively of Γ_1 and Γ_{-1} into Ω. Then by 2

$$\gamma \in \Gamma_1 \mapsto f(\omega_1(\gamma)) = f(\gamma)$$

$$\gamma \in \Gamma_{-1} \mapsto f(\omega_{-1}(\gamma)) = f(\gamma)$$

are both summable. It follows that

$$\gamma \mapsto \epsilon(\gamma)f(\gamma) = 1_{\Gamma_1}(\gamma)f(\gamma) + 1_{\Gamma_{-1}}(\gamma)f(\gamma)$$

is summable.

$3 \Rightarrow 4$ We give the proof for real case. The changes for complex spaces are straightforward. Fix $A \in 2^{|\Omega|}$. Pick an $x* \in X^*$ so that

$$\sum_{\omega \in A} \phi(\omega) x^*(f(\omega)) = \left\| \sum_{\omega \in A} \phi(\omega) f(\omega) \right\|.$$

Let $\epsilon : \Omega \to \{-1, 1\}$ be defined by $\epsilon(\omega) = 1$ if $x^*(f(\omega)) \geq 0$ and $\epsilon(\omega) = -1$ if $x^*(f(\omega)) < 0$. Then

$$\left\| \sum_{\omega \in A} \phi(\omega) f(\omega) \right\| \leq \sum_{\omega \in A} |\phi(\omega)| \, |x^*(f(\omega))| \leq \sup_{\omega \in \Omega} |\phi(\omega)| \sum_{\omega \in A} \epsilon(\omega) x^*(f(\omega)) \tag{2.1}$$

$$\leq \sup_{\omega \in \Omega} |\phi(\omega)| \, x* \left(\sum_{\omega \in A} \epsilon(\omega) f(\omega) \right) \leq \sup_{\omega \in \Omega} |\phi(\omega)| \left\| \sum_{\omega \in A} \epsilon(\omega) f(\omega) \right\| \tag{2.2}$$

The desired result follows. This completes the proof. □

Corollary 2.2. *The restriction of summable function defined on set Ω to any subset of Ω is summable.*

Theorem 2.3. *Let X be a Banach space and let $f : \Omega \to X$ be summable function, then for any injection $\varpi : \Gamma \to \Omega$,*

$$\sum_{\Omega} f = \sum_{\Gamma} f \circ \varpi.$$

Proof. Assume that $\varepsilon > 0$. Choose $B_1 \in 2^{|\Gamma|}$ such that

$$\left\| \sum_{\Gamma} f \circ \varpi - \sum_{\gamma \in B_1} f(\varpi(\gamma)) \right\| < \frac{\varepsilon}{3}.$$

Choose $A_1 \in 2^{|\Omega|}$ such that $A_1 \supset \varpi(B_1)$ and

$$\left\| \sum_{\Omega} f - \sum_{\omega \in A_1} f(\omega) \right\| < \frac{\varepsilon}{3}.$$

By injectivity, we can choose $B_2 \in 2^{|\Gamma|}$ such that $\varpi(B_2) \supset A_1$ and

$$\left\| \sum_{\Gamma} f \circ \varpi - \sum_{\gamma \in B_2} f(\varpi(\gamma)) \right\| < \frac{\varepsilon}{3}.$$

Choose $A_2 \in 2^{|\Omega|}$ such that $A_2 \supset \varpi(B_2)$ and

$$\left\| \sum_{\Omega} f - \sum_{\omega \in A_2} f(\omega) \right\| < \frac{\varepsilon}{3}.$$

Continuing in this way, we construct two sequences $n \mapsto A_n$ and $n \mapsto B_n$ such that

$$\varpi(B_{n+1}) \supset A_n \supset \varpi(B_n)$$

$$\left\| \sum_{\Gamma} f \circ \varpi - \sum_{\gamma \in B_n} f(\varpi(\gamma)) \right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\| \sum_{\Omega} f - \sum_{\omega \in A_n} f(\omega) \right\| < \frac{\varepsilon}{3}. \tag{2.3}$$

Now we let $H = \bigcup_{n \in \mathbb{N}} B_n$ and define $\varpi' : H \to \Omega$ by $\varpi'(\eta) = \varpi(\eta)$. By our hypothesis, the function $\eta \mapsto f(\varpi'(\eta))$ is also summable. On the other hand, it follows from (2.3) that

$$\left\| \sum_{\Gamma} f \circ \varpi - \sum_{\gamma \in B_n} f(\varpi'(\gamma)) \right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\| \sum_{\Omega} f - \sum_{\gamma \in B_n} f(\varpi'(\gamma)) \right\| < \frac{\varepsilon}{3}.$$

By the uniqueness of limit, we must have $\sum_{\Gamma} f \circ \varpi = \sum_{\Omega} f$ as to be shown. □

We end this section by noticing that the above defined summability property corresponds exactely to the notion of integrability introduced in [7] with respect to the size function $\sigma : 2^{\Omega} \to [0, \infty]$ defined by

$$\sigma(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset. \end{cases}$$

3 Spaces of Summable Functions

We shall denote by $\Sigma(\Omega, X)$ the set of all X-valued summable functions defined on the set Ω. It follows immediately from the linearity of net-limits that $\Sigma(\Omega, X)$ is a vector space. It is also clear that if the limit $\sum_{\Omega} f = \lim_{A \in (2^{|\Omega|}, \supset)} \sigma_f(A)$ exists in X, then

$$\|f\|_{\Sigma} := \sup\left\{ \|\sigma_f(A)\|_X : A \in 2^{|\Omega|} \right\} < \infty.$$

It then follows from the linearity of the function $f \mapsto \sigma_f(A)$ and the properties of the supremum that the map $f \mapsto \|f\|_{\Sigma}$ defines a norm on $\Sigma(\Omega, X)$.

Theorem 3.1. *If X is a Banach space, the space $\Sigma(\Omega, X)$ is complete when endowed with the norm $f \mapsto \|f\|_{\Sigma}$.*

Proof. Let $n \mapsto f_n$ be a Cauchy sequence in $\Sigma(\Omega, X)$. Fix $\epsilon > 0$, and let $N_\epsilon > 0$ be such that for $m, n > N_\epsilon$ in $\mathbb{N}$,

$$\|f_n - f_m\|_{\Sigma} = \sup\left\{ \|\sigma_{f_n - f_m}(A)\| : A \in 2^{|\Omega|} \right\} < \epsilon. \tag{3.1}$$

In particular, if we consider the singleton $\{\omega\} \in 2^{|A|}$, then for $m, n > N_\epsilon$ in $\mathbb{N}$,

$$\|f_n(\omega) - f_m(\omega)\| < \epsilon.$$

We infer that the sequence $n \mapsto f_n(\omega)$ is Cauchy in X. Since X is a Banach space, we can define a function

$$\omega \in \Omega \mapsto X \ni f(\omega) = \lim_{n \to \infty} f_n(\omega).$$

On the other hand, since $f_n, f_m \in \Sigma(\Omega, X)$, there exist $A_n, A_m \in 2^{|\Omega|}$ such that

$$\left\| \sigma_{f_n}(A) - \sum_{\Omega} f_n \right\| \vee \left\| \sigma_{f_m}(A) - \sum_{\Omega} f_m \right\| < \epsilon \text{ whenever } A \supset A_n \cup A_m,$$

Combining these inequality with (3.1), it follows that for $m, n > N_\epsilon$ in $\mathbb{N}$ and for every $A \supset A_n \cup A_m$, we have

$$\left\| \sum_{\Omega} f_n - \sum_{\Omega} f_m \right\| \leq \left\| \sigma_{f_n}(A) - \sum_{\Omega} f_n \right\| + \|\sigma_{f_n - f_m}(A)\| + \left\| \sigma_{f_m}(A) - \sum_{\Omega} f_m \right\| < 3\epsilon.$$

This proves that the sequence $n \mapsto \sum_{\Omega} f_n$ is Cauchy in X, and thus converges to, say $a \in X$.

Now fix $A \in 2^{|\Omega|}$. Since for each $\omega \in A$, $f(\omega) = \lim_{n \to \infty} f_n(\omega)$, there exists $N_\omega > N_\epsilon$ such that for$m, n > N_\omega$ in $\mathbb{N}$,

$$\|f_n(\omega) - f_m(\omega)\| \leq \frac{\epsilon}{|A|}$$

where $|A|$ is the number of elements in A. It follows that

$$\|\sigma_{f_n - f_m}(A)\| \leq \sum_{\omega \in A} \|f_n(\omega) - f_m(\omega)\| \leq \epsilon.$$

If we let $m \to \infty$, we obtain $\|\sigma_{f_n - f}(A)\| \leq \epsilon$. Since $a = \lim_{m\to\infty} \sum_\Omega f_m$, there exists $N > \sup\{N_\omega : \omega \in A\}$ such that $\left\|\sum_\Omega f_m - a\right\| < \epsilon$ whenever$m > N$. Thus for $n, m > N$,

$$\|\sigma_f(A) - a\| \leq \|\sigma_{f_m - f}(A)\| + \left\|\sigma_{f_m}(A) - \sum_\Omega f_m\right\| + \left\|\sum_\Omega f_m - a\right\| < 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows that $f \in \Sigma(\Omega, X)$ and that $\sum_\Omega f = a$.

We say that a function $f : \Omega \to X$ is norm-summable if the scalar function $\|f\|\,\Omega \to [0,\infty)$ is summable. It is an obvious consequence of the triangle inequality and the Cauchy criterion that in a Banach space every norm-summable function is summable. It turns out that the converse is also true.

Theorem 3.2. *If every norm-summable function with values in a normed space X is summable, then X is a Banach space.*

Proof. Let $n \to x_n$ be a Cauchy sequence in X. Let $k \mapsto x_{n_k}$ be a subsequence such that $\|x_{n_k}\| < 2^{-1}$ and that for all $k \geq 2$, we have $\left\|x_{n_k} - x_{n_{k-1}}\right\| < 2^{-k}$. It follows that the function $f : k \mapsto x_{n_k} - x_{n_{k-1}}$ is norm-summable. By our hypothesis, it is summable. This implies that the sequences $k \mapsto \sum_{i=1}^k f(i) = x_{n_k}$ converges to an element x in X. Therefore the sequence $n \mapsto x_n$ also converges to x. This completes the proof. ☐

We shall denote by $\Sigma^1(\Omega, X)$ the vector space equipped with the norm

$$\|f\|_{\Sigma^1} = \sup\left\{\sum_{a\in A} \|f(a)\| : A \in 2^{|\Omega|}\right\}.$$

It is then easy to verify that for the case $X = \mathbb{K}$, the map $T : \ell^\infty(\Omega, \mathbb{K}) \to \left(\Sigma^1(\Omega, \mathbb{K})\right)^*$ defined by $T\phi(f) = \lim_{A\in(2^{|\Omega|},\supset)} \sum_{a\in A} \phi(a)f(a)$ is a linear isometry. Furthermore, if $\xi \in \left(\Sigma^1(\Omega, \mathbb{K})\right)^*$, then the function $\phi : \Omega \to \mathbb{K}$ defined by $\phi(\omega) = \xi(1_{\{\omega\}})$ is an element of $\ell^\infty(\Omega, \mathbb{K})$ and $T\phi = \xi$. We infer that $\left(\Sigma^1(\Omega, \mathbb{K})\right)^* \cong \ell^\infty(\Omega, \mathbb{K})$.

It is worth noticing that when $\Omega = \mathbb{N}$, then $\Sigma^1(\mathbb{N}, \mathbb{K}) = \ell^1$, and more generally, $\Sigma^1(\mathbb{N}, X) = \ell^1(X)$ whenever X is a finite dimensional Banach space. We end this section with an extension of the Dvoretski-Rogers Theorem.

Theorem 3.3. *Let X be an infinite dimensional Banach space and Ω an infinite set. Then there exists a function $f : \Omega \to X$ which is summable but not norm-summable.*

Proof. Since X is infinite dimensional, by the Dvoretski-Rogers theorem there exists a sequence $n \mapsto x_n$ of elements of$X$ such that the series $\sum_n x_n$ converges, say to x, while $\sum_n \|x_n\| = \infty$. We write Ω as disjoint union of countably many sets U_n. Define

$$f(t) = \begin{cases} x_n & \text{if } t \in U_n,\ n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma_{\|f\|}(A) = \sum_{n_i} \|x_{n_i}\| = \infty$, that is, $\|f\| \notin \Sigma(\Omega, [0,\infty))$. On the other hand, given $\epsilon > 0$, there exists a finite subset $N_\epsilon \subset \mathbb{N}$ such that

$$\left\|\sum_{n\leq K\cup N_\epsilon} x_n - x\right\| < \epsilon \tag{3.2}$$

for every finite subset K of $\mathbb{N}$. By the Cauchy criterion, there exists $N_0 \geq \max N_\epsilon$ such that for every finite subset N of $\mathbb{N} \setminus \{1, 2, \ldots, N_0\}$, we have

$$\left\| \sum_{n \in N} x_n \right\| < \epsilon. \tag{3.3}$$

Now let $A_0 = \{n = 1, \ldots, N_\epsilon\}$. Then for every finite subset $A \supset A_0$ there exists a finite subset $\{n_1, n_2, \ldots, n_k\}$ of $\mathbb{N} \setminus \{1, 2, \ldots, N_\epsilon\}$, such that $\sigma_f(A) = \sum_{n \leq N_\epsilon} x_n + \sum_{j=1}^{k} x_{n_j}$. It follows that

$$\|\sigma_f(A) - x\| \leq \left\| \sum_{n \leq N_\epsilon} x_n - x \right\| + \left\| \sum_{j=1}^{k} x_{n_j} \right\|. \tag{3.4}$$

By combining (3.2), (3.3), and (3.4), we conclude that

$$\|\sigma_f(A) - x\| < 2\epsilon.$$

This shows that $f \in \Sigma(\Omega, X)$. □

4 Summable Bases

Recall that a sequence $n \mapsto x_n$ in a Banach space X is called a Schauder basis if for each $x \in X$ there is a unique sequence of scalar $n \mapsto \lambda_n$ such that $x = \lim_n \sum_{i=1}^{n} \lambda_i x_i$. On the other hand, if $f : \Omega \to X$ is a summable function then Property 4. of Theorem 2.1 ensures us that

$$\sum_{\Omega} \phi f = \lim_{A \in (2^{|\Omega|}, \supset)} \sum_{a \in A} \phi(a) f(a) \tag{4.1}$$

represents an element of X for every bounded function $\phi : \Omega \to \mathbb{K}$.

Let S_f denotes the linear space of all scalar functions $\phi : \Omega \to \mathbb{K}$ for which the limit in (4.1) represents an element of X. Then the expression

$$\|\phi\|_{S_f} := \|\phi f\|_{\Sigma} = \sup \left\{ \|\sigma_{\phi f}(A)\|_X : A \in 2^{|\Omega|} \right\}$$

evidently defines a seminorm on S_f. It is then quickly seen that

$$\left\| \sum_{\Omega} \phi f \right\|_X \leq \|\phi\|_{S_f}.$$

We claim that $(S_f, \|\cdot\|_{S_f})$ is complete. Let $n \mapsto \phi_n$ be a Cauchy sequence in S_f. Since

$$|\phi_n(\omega) - \phi_m(\omega)| \, \|f(\omega)\| \leq \|\phi_n - \phi_m\|_{S_f},$$

for each $\omega \in \Omega$, the sequence $n \mapsto \phi_n(\omega)$ is Cauchy in X. The space X being a Banach space, we can define a function $\omega \mapsto \phi(\omega) = \lim_n \phi_n(\omega)$. Given $\epsilon > 0$, let $N > 0$ be so chosen that

$$\|\phi_n - \phi_m\|_{S_f} < \epsilon$$

whenever $n, m > N$. Thus when $n, m > N$,

$$\left\| \sum_{a \in A} (\phi_n(a) - \phi_m(a)) f(a) \right\| < \epsilon$$

for each $A \in 2^{|\Omega|}$. Letting $n \to \infty$, we have for $m > N$

$$\left\| \sum_{a \in A} (\phi(a) - \phi_m(a)) f(a) \right\| < \epsilon.$$

Since $\phi_m \in S_f$, there exists $A_0 \in 2^{|\Omega|}$ such that for $B \in 2^{|\Omega|}$, B disjoint from A_0,

$$\left\| \sum_{b \in B} \phi_m(b) f(b) \right\| < \epsilon.$$

It follows that for $m > N$

$$\left\| \sum_{b \in B} \phi(b) f(b) \right\| \leq \left\| \sum_{b \in B} (\phi(b) - \phi_m(b)) f(b) \right\| + \left\| \sum_{b \in B} \phi_m(b) f(b) \right\| < 2\epsilon.$$

Thus $\phi \in S_f$ and $\phi = \lim_n \phi_n$. This proves our claim.

If we assume that $f(\omega) \neq 0$ for all $\omega \in \Omega$, then $\|\cdot\|_{S_f}$ is a norm on X and hence $(S_f, \|\cdot\|_{S_f})$ is a Banach space; the linear operator $T : (S_f, \|\cdot\|_{S_f}) \to (X, \|\cdot\|_X)$ given by $T\phi = \sum_\Omega \phi$ is one-to-one. In what follows, we shall always assume that $f(\omega) \neq 0$ for all $\omega \in \Omega$.

The above discussions prompt us to introduce the following definition:

Definition 4.1. Let X be a Banach space. A function $f : \Omega \to X$ is called a **summable basis** for X if for every $x \in X$, there exists a unique function $\phi : \Omega \to \mathbb{K}$ such that

$$x = \lim_{A \in (2^{|\Omega|}, \supset)} \sum_{a \in A} \phi(a) f(a).$$

A function f that is a summable basis for the closed linear span of its range is called a **basic function**.

For case of $\Omega = \mathbb{N}$, the notions of basic function and summable basis exactly correspond to the notion of respectively unconditional basic sequence and unconditional basis. It is worth remarking that as opposed to the particular case of sequence basis, the existence of a summable basis does not require the separability of the Banach space in consideration. However, many results related to sequence bases carry over by obvious mimicries to the setting of summable bases.

It follows from the foregoing discussion and from the open mapping theorem that if f is a summable basis for X, then the linear operator

$$T : (S_f, \|\cdot\|_{S_f}) \to (X, \|\cdot\|_X)$$

is an isomorphism. Consequently,

Proposition 4.1. *Let X be a Banach space with summable basis $f : \Omega \to X$. Then for every $A \in 2^{|\Omega|}$, the natural linear projection $P_A : X \to X$, defined by*

$$P_A(\sum_\Omega \phi f) = \sum_{a \in A} \phi(a) f(a)$$

is continuous.

A noteworthy corollary is the following.

Proposition 4.2. *Let X be a Banach space with summable basis $f : \Omega \to X$. Then the coefficient functionals $\{f^*_\omega : \omega \in \Omega\}$ defined by $f^*_\omega(\sum_\Omega \phi f) = \phi(\omega)$ are continuous linear functionals.*

It follows that if $f : \Omega \to X$ is a summable basis for X, then for each $x \in X$, we have

$$x = \sum_{\Omega} f^*_{(\cdot)}(x) f = \lim_{A \in (2^{|\Omega|}, \supset)} \sum_{a \in A} f^*_{(a)}(x) f(a).$$

The following is a basic test for checking whether a function f is a basic function.

Proposition 4.3. *Let X be a Banach space, let $f : \Omega \to X$ be a summable function. Then the following are equivalent*

1 *f is a basic function.*

2 *There exists a constant $K > 0$ so that for every bounded function $\phi : \Omega \to \mathbb{K}$, for every pair A, B of finite subsets of Ω, $A \supset B$ implies*

$$\|\sigma_{\phi f}(B)\| \leq K \|\sigma_{\phi f}(A)\|.$$

The constant K shall be called the **summable basis constant**.

Proof. Assume that f is a basis function for $Y = \overline{\text{span}}\{f(\omega) : \omega \in \Omega\}$. Then for every $A \in 2^{|\Omega|}$, the natural projection P_A is continuous on Y. Since $\sup\{\left\| P_A \sum_{\Omega} \phi f \right\| : A \in 2^{|\Omega|}\} = \|\phi\|_{S_f} < \infty$, it follows from the Banach-Steinhaus theorem that $\sup\left\{\|P_A\| : A \in 2^{|\Omega|}\right\} = K < \infty$. Thus if $A \supset B$ in $2^{|\Omega|}$ and $\sum_{\Omega} \phi f \in S_f$, then

$$\|\sigma_f(B)\| = \left\| P_B(\sum_{\Omega} \phi f) \right\| = \left\| P_B P_A(\sum_{\Omega} \phi f) \right\| = \|P_B(\sigma_f(A))\| \leq \|P_B\| \|(\sigma_f(A))\|.$$

The desired inequality follows.

Conversely, assume that 2 holds. It follows that for $\omega \in A \in 2^{|\Omega|}$

$$|\phi(\omega)| \|f(\omega)\| \leq K \left\| \sum_{a \in A} \phi(a) f(a) \right\| \leq K \left\| \sum_{\Omega} \phi f \right\|_{S_f}.$$

Therefore, $\sum_{\Omega} \phi f = 0$ implies $\phi = 0$. This proves the uniqueness of the function ϕ such that

$$\sum_{\Omega} \phi f = \lim_{A \in (2^{|\Omega|}, \supset)} \sum_{a \in A} \phi(a) f(a).$$

Condition 2 also ensures us that for each $A \in 2^{|\Omega|}$, the projection given by

$$P_A(\sum_{\omega \in \Omega} \lambda_\omega f) = \sum_{a \in A} \lambda_a f(a)$$

is a bounded linear operator from span $\{f(\omega) : \omega \in \Omega\}$ onto itself. It follows that each P_A has a continuous extension to Y still denoted by P_A. Let $x \in Y$ and fix $\epsilon > 0$. Then there is $x_A = \sum_{a \in A} \lambda_a f(a)$ for some $A \in 2^{|\Omega|}$ and where $\lambda_a \in \mathbb{K}$, such that $\|x - x_A\| < \epsilon$. Then for every $A \subset B \in 2^{|\Omega|}$, we have

$$\|x - P_B(x)\| \leq \|x - x_A\| + \|x_A - P_B x_A\| + \|P_B(x_A - x)\| \leq (1 + K)\epsilon.$$

It follows that

$$x = \lim_{B \in (2^{|\Omega|}, \supset)} P_B(x) = \lim_{B \in (2^{|\Omega|}, \supset)} \sum_{b \in B} f^*_{(b)}(x) f(b).$$

Example 4.1. *The indicator functions of finite subsets of a set* Ω *form a summable basis for the Banach space* $\Sigma(\Omega, \mathbb{K})$.

Proof. Let $f : 2^{|\Omega|} \to \Sigma(\Omega, \mathbb{K})$ be defined by $f(A) = 1_A$, the indicator function of a finite subset A in Ω. Let $x \in \Sigma(\Omega, \mathbb{K})$. Let $M = \sum_\Omega x$. The function $\phi : 2^{|\Omega|} \to \mathbb{K}$ defined by $\phi(A) = \frac{1}{M} \sum\limits_{a \in A} x(a)$ is bounded. It follows from the property 4. of Theorem 2.1 that the function $A \mapsto \phi(A)f(A)$ is summable and it is easily checked that

$$x = \lim_{\Gamma \in \left(2^{\left|2^{|\Omega|}\right|}, \supset\right)} \sum_{A \in \Gamma} \phi(A)f(A).$$

The proof is complete. □

For the next example, let (Ω, μ) be a finite measure space. We denote by Π the set of all subsets of Ω of positive measure. Define the mesh or the norm of $\Gamma \in 2^{|\Pi|}$ to be $\|\Gamma\| = \max\{\mu(I_i) : I_i \in \Gamma\}$. If $\Gamma, \Delta \in \Pi(A, \Sigma)$, we say that Δ is a refinement of Γ and we write $\Delta \succ \Gamma$ if $\|\Delta\| \leq \|\Gamma\|$ and $\bigsqcup \Gamma \subset \bigsqcup \Delta$. It is readily seen that that the set $2^{|\Pi|}$ is directed by the binary relation $\succ$.

Example 4.2. *If* (Ω, μ) *be a finite measure space, then the Lebesgue function spaces* $L^p(\Omega, \mu), 1 \leq p \leq \infty$ *have summable bases.*

Proof. We are going to show that the function $f : \Pi \to L^p(\Omega, \mu)$ defined by $f(A) = 1_A$ is a basis function for $L^p(\Omega, \mu)$. Let $x \in L^p(\Omega, \mu)$. The function $\phi : \Pi \to \mathbb{K}$ defined by $\phi(A) = \frac{1}{\mu(A)} \int_A x d\mu$ is bounded. It follows from Property 4. of Theorem 2.1 that the function $A \mapsto \phi(A)f(A)$ is summable and it is easily checked that

$$x = \lim_{\Gamma \in \left(2^{|\Pi|}, \succ\right)} \sum_{A \in \Gamma} \frac{1}{\mu(A)} \int_A x d\mu 1_A.$$

The proof is complete. □

It is worth remarking that $L^p(\Omega, \mu)$ may not be separable and then it fails to have a Schauder (sequence) basis. Evidently, every unconditional (sequence) basis is a summable basis. However, $L^1(\Omega, \mu)$ provides us with an example of a Banach space with a summable basis but which fails to have an unconditional (sequence) basis.

Recall that a Banach space X is said to be a π_λ-space (see for example [8],[9]) if there exists a family $\{X_\gamma : \gamma \in \Gamma\}$ of finite dimensional subspaces of X which satisfies

1. $\{X_\gamma : \gamma \in \Gamma\}$ is a net when directed by containment.
2. $\overline{\bigcup_{\gamma \in \Gamma} X_\gamma} = X$
3. For every $\gamma \in \Gamma$ there is a projection P_γ from X onto X_γ with $\|P_\gamma\| \leq \lambda$.

Proposition 4.4. *A Banach space with a summable basis is a* π_λ*-space for some* λ.

Proof. Let $f : \Omega \to X$ be a function basis for X. For every $A \in 2^{|\Omega|}$, we let $P_A : X \to X$ the projection $P_A \sum\limits_\Omega \phi f = \sigma_{\phi f}(A)$. Then clearly, the net $\left\{P_A X : A \in 2^{|\Omega|}\right\}$ satisfies 1.,2., and 3.. □

Since every π_λ-space has the bounded approximation property, we have

Corollary 4.3. *Every Banach space with a summable basis has the bounded approximation property.*

Recall that a Banach space X is said to have unconditional finite dimensional decomposition (uFDD) if there exists a sequence $n \mapsto X_n$ of finite dimensional subspaces of$X$ such that for every $x \in X$ there exists a unique sequence $n \mapsto x_n$, where $x_n \in X_n$ such that $x = \sum_n x_n$ unconditionally. We extend such a definition as follows.

Definition 4.2. We say that a Banach space X has the summable finite dimensional decomposition (**sFDD**) if there exists a net $\{X_\omega : \omega \in \Omega\}$ of finite dimensional subspaces of X such that for every $x \in X$ there exists a unique function $\Omega \ni \omega \mapsto x(\omega) \in X$, where $x(\omega) \in X_\omega$ such that $x = \sum_\Omega x(\omega) = \lim_{A \in (2^{|\Omega|}, \supset)} \sum_{a \in A} x(a)$.

Clearly if $f : \Omega \to X$ is a summable basis for the Banach space X, then $\{\text{span}\{f(\omega)\}, \omega \in \Omega\}$ is an sFDD for X. We end this paper by showing that every Banach space with an sFDD isomorphically embeds in a Banach space with a summable basis.

Theorem 4.4. *Every Banach space having the sFDD is isomorphic to a subspace of a Banach space with a summable basis.*

Proof. We have seen (Example 4.1) that $\left\{1_A : A \in 2^{|\Omega|}\right\}$ is a summable basis for the space $\sum(\Omega, \mathbb{K})$. Let K be its summable basis constant. Assume that $\{X_\omega : \omega \in \Omega\}$ is an sFDD for X. Let $X_0 = \text{span}\{X_\omega : \omega \in \Omega\}$. By the Hahn-Banach theorem, for every $x = \sum_{a \in A} x_a \neq 0$ in X_0, where $A \in 2^{|\Omega|}$, there exists an $x^* \in X^*$ such that $x^*(x) = \|x\|$. The scalar function

$$\begin{array}{cccl} \phi_x : & \Omega & \to & \mathbb{K} \\ & \omega & \mapsto & \begin{cases} x^*(x_\omega) & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases} \end{array}$$

is obviously bounded. It follows from property 4. of Theorem 2.1 that we can define a map $T : X_0 \to \sum(\Omega, \mathbb{K})$ by $T \sum_{a \in A} x_a = \phi_x 1_A$. We then notice that for every $B \subset A$

$$\left|\sum_{b \in B} x^*(x_b)\right| \leq K \left|\sum_{a \in A} x^*(x_a)\right| = K \left|x^*(\sum_{a \in A} x_a)\right| = K \|x\| \tag{4.2}$$

which implies $\left\|T \sum_{a \in A} x_a\right\|_\Sigma \leq K \|x\|$. Thus T extends to a bounded linear operator $\tilde{T}$ from X into $\sum(\Omega, \mathbb{K})$. On the other hand,

$$\|x\| = |x^*(x)| = \left|\sum_{a \in A} x^*(x_a)\right| \leq \left\|T \sum_{a \in A} x_a\right\|_\Sigma, \tag{4.3}$$

Equation (4.2) and Equation (4.3) show that $\tilde{T}$ is an isomorphism from X into $\sum(\Omega, \mathbb{K})$.

5 Conclusions

We have extended the notion of sequence basis to the natural generalization of function basis.

a In Section (2), the notion of summability of Banch space valued functions has been discussed.

b In Section (3), the Banach space structure of the space of Banach space valued summable functions has been studied. An extension of the Dvoretski-Rogers theorem has been established.

c In Section (4), we introduced and gave a comprehensive study of the notion of summable basis and related properties.

Competing Interests

The author declares that no competing interests exist.

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