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Symmetric Rings with Involutions

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Abstract

Two fairly useful notions to support some commutativity conditions for non commutative rings are symmetry and reversibility. Our aim in this note is to study $*$- symmetric rings, where $*$ is an involution on the ring. A ring *R* with involution $*$ is called $*$- symmetric if for any elements *a,b,c*∈*R*, *abc*=0 ⇒ acb^*=0. Every $*$- symmetric ring with 1 is symmetric but the converse need not be true in general, even for the commutative rings. We discussed some characterizations in which these two notions and the notions of reversibility and $*$- reversibility coincide. We have extended $*$- symmetric rings to factor polynomial rings that are isomorphic to rings of Barnett matrices.

Keywords: $*$-symmetric rings; $*$-reversible rings; $*$-rigid rings.

1 Introduction

In ring theory several notions were introduced to facilitate some commutativity conditions for non-commutative rings. Among them two fairly useful notions are symmetry and reversibility. In [1] Lembak defined that a ring R with 1 is symmetric if for any elements $a, b, c \in R, abc = 0 \Rightarrow acb = 0$ and in [2] a ring R is called reversible if for any pair of elements $a, b \in R, ab = 0$, then $ba = 0$. On the other hand Shin in [3] used the same definition for rings without 1. For rings with 1, if $abc = 0 \Rightarrow acb = 0$, then it also implies that $bca = bac = cab = cba = 0$. While for rings without 1, the other equalities may not occur. For instance, for a ring without 1, if (i) $abc = 0 \Rightarrow acb = 0,$ there is no guaranty that any one of the remaining four, $bca, bac, cab, cba,$ is zero and same is the case for (ii) $abc = 0 \Rightarrow bac = 0$. Such rings may be termed as right and left symmetric rings, respectively

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(see details in [4]). Straight forward calculations show that if $1 \in R$, and if R satisfies either (i) or (ii), then R is symmetric. Moreover, if R (even without 1) satisfies both (i) and (ii) then R is symmetric. Note one more interesting fact that there are rings they do satisfy (iii) $abc = 0 \Rightarrow bac = cab = 0$, but they are neither right nor left symmetric, so are not symmetric as well. For instance, see Example 2.10 below. Conversely, a right or left symmetric ring does not satisfy the condition (iii). Every symmetric ring holds (iii) but conversely (iii) does not give a symmetric ring even though if the ring is with 1. On the other hand every reversible ring satisfies (iii) and if $1 \in R$, and R satisfies (iii), then R is reversible.

One may also be interested to investigate such properties for rings with involutions. In [5], it is defined that a ring R with an involution $*$ is called $*$-reversible, if for any pair of elements $a, b \in R, ab = 0$, then $\mathrm{b}a^* = 0$. We extend this definition for $*$-symmetric rings on the same lines:

Definition 1.1. A ring R with involution $*$ is called $*$-symmetric if for any elements $a, b, c \in R$, $abc = 0 \Rightarrow acb^* = 0$.

As in the case of $*$- reversibility, there is no ambiguity between left and right $*$-reversible rings [5], same is the case for left and right $*$-symmetric rings. Quick calculations reveals that if for any elements $a, b, c \in R,\ abc = 0 \Rightarrow acb^* = 0$, then $b^*ac = 0$. Moreover, with the same condition one may also get $bca = 0$ and $cab = 0$. Every $*$-symmetric ring which is left or right symmetric or possesses 1, is symmetric, but the converse in general need not be true. We pose in the section of examples that there are non $*$-symmetric commutative rings with 1 as well.

For an endomorphism α on a ring R, left (or right) α- symmetric rings were introduced and studied in [6], but definitely, these are different than $*$-symmetric rings, as a right $\alpha -$ symmetric ring is, in general, different than its left version (see examples in [6]).

In the study of rings with involutions many interesting and useful results can be obtained by involving rigidity: A ring R with the involution $*$ is called $*$-rigid, if for any $a \in R, aa^* = 0$, then $a = 0$. In literature, the terms isotropic and anisotropic are also used for such involutions (see [7]).

Along with $*$-rigidity we will study some properties of $*$-symmetric rings, along with reduced and semicommutative rings, etc. A ring R is reduced if it has no non-zero nilpotent elements and it is called semi-commutative [8] if for any pair of elements $a, b \in R,\ ab = 0$, then $aRb = 0$.

In Section 2 we will list several examples and counter examples and in Section 3 some elementary properties are investigated along with $*$-rigid, reduced, and $*$- Armendariz rings. Some extensions of $*$-symmetric rings for polynomial rings in the form of rings of Barnett matrices are discussed in Section 4.

All rings considered here may not necessarily be with 1. If a ring possesses a 1, then we will specifically mention it. By $(R,*)$, we mean a ring R with an involution $*$. If an involution is induced by an involution $*$ on the elements of R, then the induced involution will also be referred to as $*$. For instance, if $(R,*)$ is a ring with involution $*$, then $(R[x],*)$ is the polynomial ring with the involution $*$ defined on the elements $p(x) = a^0 + a_i x + \cdots + a_n x^{\mathrm{n}}$, by $(p(x))^* = a_0^* + a_i^* x + \cdots + a_n^* x^{\mathrm{n}}$.

A few facts about $*$- symmetric rings that we have mentioned above are listed as under:

Lemma 1.2. Let $(R,*)$ be a $*$-symmetric ring.

(1) If for any $a, b, c \in R,\ abc = 0$, then b^*ac =0, cb^*a=0, $bca = 0$, and $cab = 0$.
(2) If R is left or right symmetric, then R is symmetric.
(3) If $1 \in R$, then R is symmetric.

Remarks 1.3. From above observations we have following conclusions:

(a) In case R is a ring with 1, then
- (a_1) If R is right or left symmetric, then R is symmetric.
- (a_2) If R is symmetric then R is reversible.
- (a_3) If R is $*$- symmetric then R is $*$- reversible.

(b) In case R is a ring without 1, then
- (b_1) If R is right and left symmetric, then $R$is symmetric.
- (b_2) If R is symmetric then R may not be reversible.
- (b_3) If R is $*$- symmetric then R may not be $*$- reversible.

2 Examples

2.1 Every Commutative Ring is Symmetric and Reversible. In General it May not be $*$-Symmetric or $*$ -Reversible with Some Involution $*$.

For example, for any prime p, consider the ring ($\mathbb{Z}_p \oplus \mathbb{Z}_p$,+,·) with component-wise addition and multiplication. Clearly $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is symmetric and reversible.

Define the exchange involution $*$ on $\mathbb{Z}_p \oplus \mathbb{Z}_p$ by $(a,b)^* = (b,a)$, $\forall\, (a,b) \in \mathbb{Z}_p \oplus \mathbb{Z}_p$. For any $a,b,c,d \in \{1,\cdots,p-1\}$, consider the following products of non-zero elements of $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

$$(a,0)(0,b)(c,d) = 0,$$
$$(a,0)(c,d)(0,b)^* = (a,0)(c,d)(b,0) = (ab,0) \neq 0.$$

Hence $\mathbb{Z}_p \oplus \mathbb{Z}_p$ with the exchange involution $*$ is neither $*$-symmetric nor $*$-reversible. It is also not $*$-rigid.

2.2 Let R be a Ring and Set

$$LT_2(R) = \left\{\begin{bmatrix} a & 0 \\ b & a \end{bmatrix} : a,b \in R\right\}.$$

Note that R is commutative if and only if $LT_2(R)$ is commutative. If $LT_2(R)$ is a domain, then $LT_2(R)$ becomes symmetric.

Let R be commutative. Define an involution on $LT_2(R)$ by setting

$$\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}^* = \begin{bmatrix} a & 0 \\ -b & a \end{bmatrix}, \qquad \forall a,b \in R.$$

If R is a domain, then simple calculations show that $LT_2(R)$ is $*$-symmetric but not $*$- rigid.

If R is not a domain then the situation may change. For instance, take $R = \mathbb{Z}_{27}$. Let $t = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$.

Then $t^3 = 0$, but $t^2 t^* \neq 0$. So there is no chance for ($\mathbb{Z}_{27}$,$*$) to be $*$-symmetric, $*$-reversible, or $*$-rigid.

2.3 Let us Consider the Ring of Strictly Upper Triangular Matrices of Order Four in the Form

$$SUT_4(R) = \left\{ \begin{bmatrix} 0 & a & a_{13} & a_{14} \\ 0 & 0 & a & a_{24} \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a, a_{13}, a_{14}, a_{24} \in R \right\}$$

Where R is any ring. If $*$ is an involution on *R*, then with the induced involution (SUT_4(R),$*$) is $*$-symmetric if and only if $(R,*)$ is $*$-symmetric.

2.4 All Domains with Some Involution $*$ are $*$-symmetric.

For instance, the ring of real quaternions $\mathbb{H}$ is $*$-symmetric with the natural involution $*$ defined on its elements by $(a + bi + cj + dk)^* = a - bi - cj - dk$. The involution $*$ as defined on $\mathbb{H}$ is an $*$-rigid involution, as if $x = a + bi + cj + dk \in \mathbb{H}$, with $xx^* = 0$. Then

$$xx^* = |x^2| = a^2 + b^2 + c^2 + d^2 = 0 \Longrightarrow x = 0.$$

2.5 For the Non-commutative Quaternion Algebra $\mathbb{F}(Q_8)$ (see [7; p.25]) Over any Field $\mathbb{F}$ with Char($\mathbb{F}$) ≠ 2 and with a Basis {*1,i,j,k*},

$$\mathbb{F}(Q_8) \coloneqq \{x = a + bi + cj + dk\colon\ a, b, c, d \in \mathbb{F}, i^2, j^2 \in \mathbb{F}^{\times}, ij = k = -ji\},$$

and with the involution defined by $x^* = a - bi - cj - dk$ is not $*$-symmetric, in general. For instance, if $\mathbb{F} = \mathbb{F}_3$, and $i^2 = j^2 = 2$, then, $i(1 + i + j)(1 + 2i + 2j) = 0$ but $i(1 + i + j)((1 + i + j \neq 0$.

Similarly $(1 + i + j)(1 + i + k) = 0$ but $(1 + i + k)(1 + i + j) = i + j + k \neq 0$. From these computations we conclude that $\mathbb{F}_3(Q_8)$ is neither symmetric, nor reversible and nor $*$-reversible. Also $(1 + i + j)(1 + 2i + 2j) = 0$ and $(1 + i + j) \neq 0$ implies that $\mathbb{F}_3(Q_8)$ is not $*$-rigid.

2.6 Now Consider $Q_8 = \{1, x_{-1}, x_i, x_{-i}, x_j, x_{-j}, x_k, x_{-k}\}$

The group ring $\mathbb{Z}_2(Q_8)$, as discussed in [9; Example 7], is reversible and is not symmetric. Let us define an involution on its elements,

$$x = a_1 + a_2x_{-1} + a_3x_i + a_4x_{-i} + a_5x_j + a_6x_{-j} + a_7x_k + a_8x_{-k}, \forall a_i \in \mathbb{Z}_2$$

by

$$x^* = a_1 + a_2x_{-1} + a_3x_{-i} + a_4x_i + a_5x_{-j} + a_6x_j + a_7x_{-k} + a_8x_k, \qquad \forall a_i \in \mathbb{Z}_2.$$

Then $xx^* = 0$ if and only if $\sum_{i=1}^{8} a_i^2 = 0$. This holds even though $x \neq 0$. For instance, if $x = 1 + x_i + x_j + x_k$, then one calculates that $xx^* = 0$. Hence $\mathbb{Z}_2(Q_8)$ is not $*$-reversible and it is not rigid under $*$ as well. Hence it is not $*$-symmetric. In fact, $\mathbb{Z}_2(Q_8)$ cannot be $*$- symmetric for any involution $*$ on it (because of Lemma 1.2(3). Because $\mathbb{Z}_2(Q_8)$) is reversible and has identity, it satisfies condition (iii).

2.7 [10; Example 2] Consider the Group Ring $\mathbb{C}[S_3]$, where $S_3 = \{1, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$ is the Symmetric Group on Three Letters. $\mathbb{C}[S_3]$ Adhere to an Involution $*$ Defined by

$$\left(\sum_{g \in S_3} r_g g\right)^* = \left(\sum_{g \in S_3} r_g g^{-1}\right).$$

Assume that

$$\alpha = \frac{1}{6}\left(\sum_{g \in S_3} r_g g\right), \beta = \frac{1}{6}(1 + \sigma + \sigma^2 - \tau - \tau\sigma - \tau\sigma^2), \gamma = \frac{1}{3}(2 - \sigma - \sigma^2).$$

Then $\{\alpha, \beta, \gamma\}$ form a complete set of orthogonal elements of C[S$_3$], so

$$\mathbb{C}[S_3] = \mathbb{C}[S_3]\alpha \oplus \mathbb{C}[S_3]\beta \oplus \mathbb{C}[S_3]\gamma.$$

Note that $\mathbb{C}[S_3]\alpha$ and $\mathbb{C}[S_3]\beta$ are anisotropic and $*$-reversible while $\mathbb{C}[S_3]\gamma$ is isotropic and not $*$-reversible, so the ring is neither $*$-rigid nor $*$-reversible. Hence it is not $*$- symmetric.

2.8 The left Klein 4-rings V_{2^n} as Defined in [4] are Left Symmetric

These are neither symmetric, nor reversible. It is not possible to define an involution on them, so there is no chance for them to be $*$-symmetric or $*$-reversible. Similarly, $V_{2^n}^{op}$ are only right symmetric.

2.9 Let $(R_1, *_1)$ and $(R_2, *_2)$ be Two Rings. A Natural Way to Induce an involution on their Direct Sum is Following:

Let $a_1 \in R_1$ and $a_2 \in R_2$. Then the induced involution on $R_1 \oplus R_2$ is defined by

$$(a_1, a_2)^* = (a_1^{*1}, a_2^{*2})$$

If $\{(R_i, *_i): i \in I\}$ is an indexed family of rings, then the induced involution $*$ on $\oplus_i \in I, R_i$ is defined as above. Hence we conclude that: $\forall i \in I, R_i\ is\ *_i$-symmetric if and only if $\oplus_i \in I, R_i$ is $*$-symmetric.

2.10 The Ring $\mathbb{Z}_2(Q_8)$ as Discussed in Example 2.6. is a Reversible Ring with 1, and is Neither Symmetric nor $*$-Symmetric for any Involution $*$. The Strictly Upper Triangular Matrix Ring $SUT_3(\mathbb{Z}_2)$ as Defined by

$$SUT_3(\mathbb{Z}_2) = \left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{Z}_2\right\}$$

is noncommutative and is without 1. But clearly it is symmetric and $*$-symmetric for any involution $*$. But both rings satisfy the condition (iii) that if for any elements *a,b,c*∈*R*, *abc*=0, then *cab*=0 and *bac*=0. So the direct sum

$(SUTM_3(\mathbb{Z}_2)\oplus\mathbb{Z}_2(Q_8)\ ,+,\cdot)$

is a ring with + and defined component wise. This ring clearly satisfies $abc = 0 \Rightarrow cab = bac = 0$. It is neither symmetric nor reversible. It is also not $*$-symmetric and $*$- reversible for any involution $*$ defined on it.

3 Some Elementary Results

Proposition 3.1. For a ring R with involution $*$ the following hold:
(i) If R is a reduced and $*$-symmetric, then R is $*$-reversible.
(ii) If R is $*$-reversible, then R is symmetric if and only if R is $*$-symmetric.
(iii) R is $*$-rigid and $*$-symmetric if and only if R is reduced and $*$-reversible.
(iv) R is $*$-rigid and semi-commutative if and only if R is semi prime and $*$-symmetric.

Proof:- (i) Let $ab = 0$. Then

$$aba^* = 0 \Rightarrow b^*aa^* = 0 \Rightarrow b^*aa^*b^* = 0 \Rightarrow b^*ab^*a = 0 \Rightarrow (b^*a)^2 = 0,$$

which implies that $b^*a = 0,$ hence R is $*$-reversible.

(ii) Let *R* be a symmetric ring. Let for any $a, b, c \in R, abc = 0$. Then

$$(ac)b = b(ac)^* = acb^* = 0.$$

Hence R is $*$-symmetric.

Conversely, let R be a $*$-symmetric ring. Then by the same argument as above, let $abc = 0,$ then $(ac)b^* = 0.$ By $*$-reversibility of R,$bac = 0.$ Also $b^*(ac)^* = b^*c^*a^* \Rightarrow acb = 0.$

By repeating same steps we get $bca = 0$ and $cab = 0,$ and hence $cba = 0.$ Thus R is a symmetric ring.

(iii) Let R be a $*$-rigid and $*$-symmetric ring. If $a^3 = 0$, then

$$a^*aa^* = 0 \Rightarrow a^*aa^*a = (a^*a)^*(a^*a) = 0 \Rightarrow a^*a = 0 \Rightarrow a = 0.$$

Now let $ab = 0.$ Then

$$(ba^*)(ba^*)^* = (ba^*)(ab^*) = (ba^*)(b^*a^*) = 0 \Rightarrow ba^* = 0.$$

Conversely, let R be $*$-reversible and reduced. If for any $a \in R, aa^* = 0,$ then

$$aa = 0 \Rightarrow a = 0.$$

Now let, for any $a, b, c \in R, if\ abc = 0$, then

$$cb^*a^* = 0 \Rightarrow (cb^*acb^*)a^* = 0 \Rightarrow acb^*acb^* = 0 \Rightarrow acb^* = 0.$$

(iv) Let R be $*$-rigid and semi-commutative. Then for any $a \in R, aRa = 0$

$$\Rightarrow aa^*a = 0 \Rightarrow aa^*aa^* = (aa^*)^*aa^* = 0 \Rightarrow aa^* = 0 \Rightarrow a = 0.$$

Now we want to prove that R is $*$-symmetric. Let for any $a, b, c \in R, abc = 0$. Then

$$c^*(b^*a^*) = 0 \Rightarrow c^*R(b^*a^*) = 0 \Rightarrow c^*(ab)b^*a^* = 0 \Rightarrow (c^*ab)(b^*a^*c) = 0 \Rightarrow b^*a^*c = 0.$$

Again by semi-commutivity and $*$-rigidity,

$$(b^*c^*a)(a^*cb) = 0 \Rightarrow b^*c^*a = 0 \Rightarrow (a^*cb^*)(bc^*a) = 0$$
$$\Rightarrow bc^*a = 0 \Rightarrow (bc^*a^*)(acb^*) = 0 \Rightarrow acb^* = 0.$$

Conversely, let R be $*$-symmetric and semiprime. If for any $a \in R,\ aa^* = 0$, then

$$aa^*r = 0 \Rightarrow ara = 0 \Rightarrow a = 0.$$

Now to prove R is semi-commutative, let for any $a, b \in R, ab = 0$, then $\forall r \in R$,

$$b^*a^*r = 0 \Rightarrow ab^*r = 0 \Rightarrow ab^*rab^* = 0 \Rightarrow ab^* = 0 \Rightarrow ab^*r = 0 \Rightarrow arb = 0.$$

■

Corollary 3.2. Let R be a $*$-rigid ring. Then the following are equivalent.

(1) R is $*$-symmetric
(2) R is symmetric
(3) R is $*$-reversible
(4) R is reversible

(1)⇒ (2) By Lemma 3.1(i) R is reduced, hence symmetric.
(2)⇒ (3) Let for any $a, b \in R, ab = 0$. Then

$$abb^* = 0 \Rightarrow ab^*b = 0 \Rightarrow ab^*ba^* = 0 \Rightarrow (ab^*)(ab^*)^* = 0 \Rightarrow ab^* = 0.$$

Hence R is $*$-reversible.

(3)⇒ (4) Trivial.
(4)⇒ (1) Any reversible ring is semi-commutative, hence by Lemma 3.1(iv) it is $*$-symmetric.

Proposition 3.3. Let *R* be a ring with 1 and with an involution $*$. If *e* is a central idempotent, then eR and $(1-e)R$ are $*$-symmetric if and only if R is $*$-symmetric.

Proof: Suppose that *eR* and $(1-e)R$ are $*$-symmetric. Let $abc = 0\ for\ a, b, c \in R$. Then

$$0 = eabc\ = a(eb)c.$$

Similarly

$$(1-e)abc = 0 = a((1-e)b)c.$$

By hypothesis we get

$$0 = ac(eb)^* = (ac)eb^* = e(ac)b^*$$

and

$$0 = ac((1-e))^* = ac(1-e)^*b^* = ac(1-e)b^*$$

Thus

$$acb^* = eacb^* + (1-e)acb^* = 0,$$

and therefore R is $*$-symmetric. The converse is obvious.■

A ring R is called Armendariz in [11] if in the polynomial ring $R[x]$, $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) = 0$, then all products of the form $a_i b_j = 0, \forall i = 1, \cdots, m, j = 1, \cdots, n.$

Let us define that:

Definition 3.4. A ring R is called $*$- Armendariz if in the polynomial ring $R[x]$,

$$\left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = 0,$$

then all products of the form $b_j a_i^* = 0$ (Equivalently $a_i b_j^* = 0$) $\forall i = 1, \cdots, m, j = 1, \cdots, n.$

Lemma 3.5. Every $*$- Armendariz ring is $*$- symmetric and $*$- reversible.

Proof: Let $a, b, c \in R$ be such that $abc = 0$. Then

$$\begin{aligned} abxcx = (ab)cx^2 = 0 &\Rightarrow (ab)c^* = 0 \\ &\Rightarrow ax(bc^*)x = a(bc^*)^* x^2 = 0 \\ &\Rightarrow abc^* = 0. \end{aligned}$$

Hence R is $*$- symmetric. The other part is trivial. ■

Proposition 3.6. Let *R* be a reduced ring. Then the following are equivalent.

(i) R is$*$-Armendariz.
(ii) R is $*$- symmetric.
(iii) R is$*$-reversible.

Proof:
(i)⇒(ii) & (iii) hold by Lemma 3.5.
(ii)⇒(iii). If for any $a, b \in R, ab = 0$, then

$$a^* ab = 0 \Rightarrow a^* b a^* = 0 \Rightarrow b a^* b a^* = 0 \Rightarrow b a^* = 0.$$

Hence R is $*$-reversible.

Now we prove (iii)⇒(i). Let in $R[x]$, $p(x) = (\sum_{i=0}^{m} a_i x^i)$ and $q(x) = (\sum_{j=0}^{n} b_j x^j)$ with $p(x)q(x) = 0$.
Because R is reduced, so R is Armendariz. $\forall\, i = 1, \cdots, m$ and $\forall j = 1, \cdots, n, a_i b_j = 0$.
Because R is $*$- reversible, $\forall\, i = 1, \cdots, m$ and $\forall j = 1, \cdots, n$, $b_j a_i^* = 0$.
Hence R is $*$- Armendariz. ■

4 Extensions of $*$-Symmetric Rings

Let R be a ring and M an (R, R) −bimodule. The well-known trivial extension of *R* by *M* is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$. If we let $M = R$, then $T(R,R)$ is isomorphic to the factor ring $\frac{R[x]}{\langle x^2 \rangle}$ of the polynomial ring $R[x]$.

Let $*$ be an involution on a ring R. An induced involution, again denoted by $*$, on the trivial extension $T(R,R)$ of R, is given by:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a^* & b^* \\ 0 & a^* \end{pmatrix}.$$

Note that the ring in Example 2.2 is a $T(R,R)$ ring where we have a different involution.

For a $*$-symmetric ring $R, T(R,R)$ need not to be a $*$- symmetric ring as it is discussed in the next example.

Example 4.1. Consider the $*$-symmetric ring (Example 2.2)

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{Z} \right\},$$

where$*$ is defined by

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

For

$$A = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}, \quad B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\mathrm{C} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 01 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, \quad B^* = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in T(R,R).$$

$ABC = 0$ but $ACB^* \neq 0$. Thus $T(R,R)$ is not $*$-symmetric.

Proposition 4.2. Let R be a $*$-rigid ring. If R is a $*$-symmetric ring, then $T(R,R)$ is also a $*$-symmetric ring.

Proof: Let *ABC*=0 for,

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \qquad B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}, \qquad \mathrm{C} = \begin{pmatrix} e & f \\ 0 & e \end{pmatrix} \in T(R,R).$$

$$ace = 0; \tag{1}$$

and

$$acf + ade + bce = 0. \tag{2}$$

Note that R is reduced (by Proposition 3.1(iii)). So, for any $a, b \in R$ if $ab^2 = 0$ $(or, a^2 b = 0)$, then by Lemma 1.2,

$$bab = 0 \Rightarrow (ba)^2 = 0 \Rightarrow ba = 0 \Rightarrow ab = 0. \quad (3)$$

By (1) and Lemma 1.2,

$$acer = 0 \Rightarrow ac(er) = 0 \Rightarrow c(er)a = 0$$

and so

$$c(er)(ar) = 0 \Rightarrow (ar)c(er) = 0 \Rightarrow ar(er)c^* = 0 \Rightarrow arec^*r^* = 0 \Rightarrow arer^*c = 0.$$

From (2), we have.

$$acfe + ade^2 + bce^2 = 0.$$

By (1) and Lemma 1.2,

$$eac = 0 \Rightarrow eacf = 0 \Rightarrow (acfe)^2 = 0 \Rightarrow acfe = 0.$$

Thus by (2) and (3) we get

$$ade^2 + bce^2 = ade + bce = 0.$$

Hence

$$a^2de + abce = 0.$$

Again by (1) and Lemma 1.2,

$$bace = 0 \Rightarrow ebac = 0 \Rightarrow ebca^* = 0$$
$$\Rightarrow (bc)(a^*)e = 0 \Rightarrow a(bc)e = 0.$$

So

$$a^2de = 0 \Rightarrow ade = 0.$$

Hence by (2)

$$acf + bce = 0$$

If we multiply last equation on the left side by *a*, we get:

$$a^2cf + abce = 0,$$

since $abce = 0$, it follows that $a^2cf = acf = 0$, hence $bce = 0$.

From (2) $acf = 0$.

Finally, since R is $*$ symmetric, $aec^* = 0,\ afc^* = 0,\ aed^* = 0$ and $bec^* = 0$.

Hence we conclude that: $ACB^* = 0$, and therefore $T(R,R)$ is $*$-symmetric.

Abusing notations, let us continue to use the same involution $*$ and the term $*$-symmetric for the ring *R* and simultaneously for its extension rings.

Example 4.3. We verify that if R is reduced and $*$- symmetric, then the ring

$$S = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} \ : a, b, c \in R \right\}$$

is also $*$- symmetric.

Consider three elements

$$A = \begin{bmatrix} a & a_1 & a_2 \\ 0 & a & a_1 \\ 0 & 0 & a \end{bmatrix}, \quad B = \begin{bmatrix} b & b_1 & b_2 \\ 0 & b & b_1 \\ 0 & 0 & b \end{bmatrix}, \quad C = \begin{bmatrix} c & c_1 & c_2 \\ 0 & c & c_1 \\ 0 & 0 & c \end{bmatrix} \in S$$

If $ABC = 0$, then:

$(1) \quad abc = 0$
$(2) \quad abc_1 + ab_1c + a_1bc = 0$
$(3) \quad abc_2 + ab_1c_1 + a_1bc_1 + ab_2c + ab_2c + a_1b_1c + a_2bc = 0.$

Since R is $*$- symmetric, by (1) we get:

$$bca = cab = acb^* = b^*ac = 0.$$

Multiply (2) by ab

$$abc_1ab + ab_1cab + a_1bcab = 0 \Rightarrow abc_1ab = c_1(ab)^2 = 0 \Rightarrow c_1ab = abc_1 = ac_1b^* = 0.$$

This means that

$$ab_1c + a_1bc = 0 \Rightarrow bcab_1c + bca_1bc = 0 \Rightarrow a_1bc = a_1cb^* = 0 \;\&\; acb_1^* = 0.$$

We conclude that

$$(4) \qquad acb_1^* + ac_1b^* + a_1cb^* = 0.$$

Now multiply (3) by ba,

$$abc_2ba + ab_1c_1ba + a_1bc_1ba + ab_2cba + ab_2cba + a_1b_1cba + a_2bcba = 0.$$

This gives

$$ab(c_2b)a = 0 \Rightarrow abc_2 = ac_2b^* = 0.$$

Multiply the remaining terms in (3) by a from right and by c from left we get

$$cab_1c_1a + ca_1bc_1a + cab_2ca + cab_2ca + ca_1b_1ca + ca_2bca = 0$$

and simplify the terms we get:

$$cab_1c_1a + cab_2ca = 0 \Rightarrow b_2ca = 0 \Rightarrow acb_2^* = 0.$$

Same technique is continued until we get

$$ac_1b_2^* = a_1cb_1^* = ac_1b^* = 0.$$

Hence we conclude that,

$$a_1cb^* + ac_1b_1^* + a_1c_1b^* + acb_2^* + a_1cb_1^* + a_2cb_2^* = 0$$

so, $ACB^* = 0$, means that S is $*$- symmetric.

Theorem 4.4. Let R be a reduced ring and n any positive integer. If R is $*$-symmetric, then $R[x]/< x^n >$ is a $*$-symmetric ring where $< x^n >$ is the ideal generated by x^n.

Proof: Let $S = R[x]/\langle x^n \rangle$ For $n = 0$, $R[x] = S$. Because R is reduced, so R is Armendariz, and $R[x]$ is $*$-symmetric. $For\ n = 1$, $S = \frac{R[x]}{\langle x \rangle} \cong R$ is$*$-symmetric. $For\ n = 2\ or\ 3$, S is $*$- symmetric by Examples 2.2 & 4.4 and because of the fact that $R[x]/\langle x^n \rangle$ has the Barnett matrix representation even for non-commutative rings without 1,

$$\frac{R[x]}{\langle x^n \rangle} \cong \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \ddots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{bmatrix} : a_i, b, c \in R \right\}.$$

Now let

$$f = a_0 + a_1\bar{x} + \cdots + a_{n-1}\bar{x}^{n-1},\ \ g = b_0 + b_1\bar{x} + \cdots + b_{n-1}\bar{x}^{n-1},\ \ h = c_0 + c_1\bar{x} + \cdots + c_{n-1}\bar{x}^{n-1} \in S$$

Assume that $fgh = 0$.

Hence, $a_ib_ic_k\bar{x}^{i+j+k} = 0$ for all $i, j\ and\ k$, when $i + j + k \geq n$

Then we have

(1) $a_0b_0c_0 = 0$
(2) $a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 = 0$
(3) $a_0b_0c_2 + a_0b_1c_1 + a_0b_2c_0 + a_1bc_1 + a_1b_1c_0 + a_2bc_0 = 0$
$\vdots$

$(n-2)$ $a_0b_0c_{n-2} + a_0b_1c_{n-3} + \cdots a_{n-3}b_1c_0 + a_{n-2}b_0c_0 = 0$
$(n-1)$ $a_0b_0c_{n-1} + a_0b_1c_{n-2} + \cdots a_{n-2}b_0c_1 + a_{n-2}b_1c_0 + a_{n-1}b_0c_0 = 0$

By induction hypothesis, $a_ib_jc_k = 0$ for $i + j + k = 0, \ldots, (n-2)$ and from equ $(n-1) \times b_0c_0$ gives, $a_{n-1}b_0c_0 = 0$ and so $(n-1)$ becomes

$(n-1)'$ $a_0b_0c_{n-1} + a_0b_1c_{n-2} + \cdots a_{n-2}b_1c_0 = 0.$

If we multiply $(n-1)'$ by b_1c_0 (on the left side we get,):

$$b_1c_0a_0b_0c_{n-1} + b_1c_0a_0b_1c_{n-2} + \cdots + b_1c_0a_{n-2}b_1c_0 = 0.$$

Since $b_1c_0a_i = 0$, $i = 0, \cdots, n-1$, then $a_{n-2}b_1c_0 = 0$.

Again we multiply $\{(n-1)' \backslash a_{n-2}b_1c_0\}$ by b_0c_1, we find, $a_{n-2}b_0c_1 = 0$. Next we multiply $\{(n-1)' \backslash$ *an−2b1c0+an−2b0c1* by *b1c2* we get, *an−2b1c2=0.*

Continuing in this way finally we get

$$a_{n-2}b_1c_0 = a_{n-2}b_0c_1 = a_{n-2}b_1c_2 = a_0b_1c_{n-2} = \cdots = a_0b_0c_{n-1} = 0.$$

The rest is trivial.

5 Conclusion

In this work we have extended a study of symmetric rings to $*$-symmetric rings where $*$ is some involution on the ring. We have invertigated some basic properties and have posed several examples and counter examples.

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Competing Interests

Author has declared that no competing interests exist.

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