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Vanishing g-Bochner Tensor of Viasman-Gray Manifold

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Abstract

The aim of this paper is to study the geometric properties of vanishing g-Bochner tensor of Viasman-Gray manifold. The necessary conditions for which the Viasman-Gray manifold is a manifold of vanishing g-Bochner tensor have been found. Finally, an application of vanishing g-Bochner tensor of Viasman-Gray manifold has been given.

Keywords: Almost hermitian manifold; viasman-gray manifold; g-bochner tensor.

Mathematics Subject Classification: 53C55, 53B35

1 Introduction

The almost Hermitian manifold is one of the important fundamental concepts of modern mathematics and its applications. The almost Hermitian structures give many examples of differential geometrical structures. These different structures have many applications in mathematics and theatrical physics.

In 1980 appeared an important study of classification of almost Hermitian manifold by Gray and Hervella [1], when they succeed to classify the almost Hermitian structure according to their geometrical structure properties. In particular, they are studied the action of the unitary group $U(n)$on the space W of all tensors of type (3,0). This action implies four irreducible components; $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$, so there are sixteen different subspaces from these four classes. Each class corresponds to a different type of almost Hermition manifold. Gray and Hervalla determined

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the conditions for each one of these classes belongs to the types of almost Hermition manifold. One of the important class is the Viasman-Gray manifold (*VG*-Manifold) which denoted by $W_1 \oplus W_4$, where W_1 is the nearly Kahler manifold (*NK*-manifold) and W_4 is the locally conformal Kahler manifold. Most of the studies of these kinds were done by using a Kozal's operator method [2].

There is an important and interesting method by using the G-structure space that does not depend on manifold itself but on a principle subfiber bundle of all complex frames which is called the adjoined G-structure space. This method were found by Kirichenko [3], who opened new prospects to study the geometrical properties of such manifolds.

In this paper we shall use the adjoined G-structure space method to study some of the geometrical properties of the *VG*-Manifold. In particular, we shall study the geometrical properties of the generalized Bochner (g-Bochner) tensor of *VG*-Manifold. There are many authors studied the g-Bochner tensor. In 2003, Pranovic [4] studied this tensor on general almost Hermitian manifold. In 2011, Abood [5] studied the geometric meaning of vanishing g-Bochner tensor of *NK*- manifold. The present paper deals with this tensor when it acts on more interesting class of almost Hermitian manifold which is *VG*- Manifold.

2 Preliminaries

Let M be 2*n*- dimensional smooth manifold, $C^{\infty}(M)$ be an algebra of smooth functions on M, $X(M)$ be a Lie algebra of vector fields on M. An almost Hermitian structure (AH −structure) [6] on M is a pair of tensors $\{J, g =<.,.>\}$, where J is an almost complex structure, $g =<.,.>$ is a Riemannian metric, such that $< JX, JY >=< X, Y >$; $X, Y \in X(M)$. A smooth manifold M with AH −structure is called an almost Hermitian manifold (AH −manifold). A fundamental form Ω is 2-form which is defined as $(X, Y) =< X, JY >$.

In the tangent space $T_P(M)$ there exists a basis of the form $\{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}\}$. Its corresponding frame is $\{p, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}} \dots, \varepsilon_{\hat{n}}\}$. Suppose that the indexes i, j, k, l in the range $1, 2, \dots, 2n$ while the indexes $a, b, c. d, e, f, g, h$ in the range $1, 2, \dots, n$. Denote $\hat{a} = a + n$.

Kirichenko [3] proved that the giving an AH −structure on M is equivalent to the giving an G-structure in the principle fiber bundle of all complex frames of manifold M with G-structure group is the unitary group $U(n)$. This principle fiber bundle is called an adjoined G-structure. In the space of the adjoined G-structure, the following matrices are respectively the components of the almost complex structure J, Riemannian metric g and the Kahler form Ω

$$(J_j^i) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, (g_{ij}) = \begin{pmatrix} o & I_n \\ I_n & o \end{pmatrix}, (\Omega_{ij}) = \begin{pmatrix} 0 & -\sqrt{-1}I_n \\ \sqrt{-1}I_n & 0 \end{pmatrix} \quad (2.1)$$

where I_n is the identity matrix of order *n*.

Recall that [1] an *AH*- structure (J, $g =<>$) is called astructure of class $W_1 \oplus W_4$ or Viasman– Gray structure (*VG* –structure) if satisfies the following condition:

$$\nabla(F)(X, Y) = \frac{-1}{2(n-1)}\{< X, Y > \delta F(Y) - < X, Y > \delta F(X) - < JX, JY > \delta F(JX)\},$$

where ∇ is the Riemannian connection of the metric g, $F(X, Y) =< JX, Y >$ is the Kahler form, δ is a codrivative and $X, Y \in X(M)$.

A manifold *M* with *VG* -structure is called a Viasman-Gray manifold (*VG*-manifold). For each *AH*-manifold, in particular for *VG*- manifold defined a Lie form by the formula

$\alpha = \frac{1}{n-1}\delta F \circ J$,

where δ represents the coderivative. If F is $r-$form, then its coderivative is $(r-1)$ $-$form, and its dual is a vector which is called a Lie vector.

It is known [7] that the structure equations of Riemannian connection of the *VG*-structure in the adjoined *G*-structure space have the forms

1) $d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c;$

2) $d\omega_a = -\omega^b_a \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c;$

3) $d\omega^a_b = \omega^a_c \wedge \omega^c_b + \left(2B^{adh}B_{hbc} + A^{ad}_{bc}\right)\omega^c \wedge \omega_d + \left(B^{ah}{}_{[c}B_{d]bh} + A^a_{bcd}\right)\omega^c \Lambda \omega^d \ + + \Big(B_{bh}{}^{[c}B^{d]ah} +$
Abacdωc⋀ωd,

where $\{\omega^i\}$ are the components of the solder form, $\{\omega^i_j\}$ are the components of the connection form for Riemannian metric, $\omega_a = \omega^{\hat{a}}$ and $\{A^{ad}_{bc}, A^a_{bcd}, A^{acd}_b\}$ are some functions on adjoined G $-$structure space.The functions $\{A^{ad}_{bc}\}$ defined a tensor field on the manifold M, this tensor field is called a tensor of holomorphic sectional curvature. The tensors $\{B^{abc}\}$ and $\{B_{abc}\}$ are called the structure tensors while the tensors $\{B^{ab}{}_c\}$ and $\{B_{ab}{}^c\}$ are called the virtual tensors. It is obvious their conjugate are $\overline{B}^{abc} = B_{abc}$ and $\overline{B}^{ab}{}_c = B_{ab}{}^c$

Remark 2.1 [8]. By the Banaru's classification of AH $-$manifold, the *VG*- manifold satisfies the following properties:

$B^{abc} = -B^{bac};\ B_{abc} = -B_{bac};\ B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c;\ B_{ab}{}^c = \alpha_{[a}\delta^c_{b]}$,

where $\{\alpha_a, \alpha^a \equiv \alpha_{\hat{a}}\}$ are the components of the Lie form α.

The Table 2.1 gives the classification of Gray-Hervalla and the corresponding Banaru's classification for the classes that we care studied:

Table 2.1. Gray-Hervalla and Banaru Classification

Classification of Gray-Hervalla	**Classification of Banaru**
Nearly Kahler manifold (W_1)	
$\nabla_X(J)(X,Y) = 0$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_c = 0$
Locally Conformal Kahler Manifold (W_4)	
$\nabla_X(F)(Y,Z) = \frac{-1}{2(n-1)}\{< X,Y > \delta F(Z) - < X,Z > \delta F(Y) - <$ *X,JY>δFJZ+<X,JZ>FJY*	$B^{abc} = 0, \quad B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c$
Viasman-Gray manifold ($W_1 \oplus W_4$)	
$\nabla(F)(X,Y) = \frac{-1}{2(n-1)}\{< X,Y > \delta F(Y) - < X,Y >> \delta F(X) - <$ *JX,JY>δF*JX	$B^{abc} = -B^{bac}, \quad B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c$

Definition 2.2 [9]. A Riemannian curvature tensor *R* for smooth manifold *M* is an 4-covariant tensor $R: T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \mathbb{R}$ which is defined by

$R(X,Y,Z,W) = g(R(Z,W)Y,X),$

where $R(X,Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)Z;\ \ X,Y,Z,W \in T_p(M)$ and satisfies the following properties:

1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$;
2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$;
3) $R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$;
4) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

Lemma 2.3 [7]. The components of Riemannian curvature tensor R of VG −-Manifold in the adjoined G −structure space are given as follows:

1) 1) $R_{abcd} = 2(B_{ab[cd]} + \alpha_{[a}B_{b]cd})$;
2) $R_{\hat{a}bcd} = 2A^{a}_{bcd}$;
3) $R_{\hat{a}\hat{b}cd} = 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]})$;
4) 4) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}_{\ \ c}B_{hb}^{\ \ \ d}$,

where $\{\alpha^{a}_{\ b}, \alpha_{a}^{\ b}, \alpha_{ab}, \alpha^{ab}\}$ are some functions on adjoined G −structure space such that $d\alpha_a + \alpha_b\omega^b_a = \alpha^b_a\omega_b + \alpha_{ab}\omega^b$ and $d\alpha^a - \alpha^b\omega^a_b = \alpha^a_b\omega^b + \alpha^{ab}\omega_b$.

Lemma 2.4 [10]. In the adjoined G −structure space, an *AH*- manifold is manifold of class

R_1 if and only if, $R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0$;
R_2 if and only if, $R_{abcd} = R_{\hat{a}bcd} = 0$;
R_3 (*RK*-manifold) if and only if, $R_{abcd} = 0$.

3 Main Results

Definition 3.1 [4]. The generalized Bochner (g-Bochner) tensor is a tensor of type (4,0) which is defined as the form:

$$\begin{aligned} GB(X,Y,Z,W) = {} & (HR)(X,Y,Z,W) - \tfrac{1}{2(n+2)}\{g(X,W)r(HR)(Y,Z) + g(Y,Z)r(HR)(X,W) - \\ & g(X,Z)\,r(HR)(Y,W) - g(Y,W)\,r(HR)(X,Z) + \Omega(X,W)\,r(HR)(JY,Z) + \\ & \Omega(Y,Z)\ r(HR)(JX,W) - \Omega(X,Z)r(HR)(JY,W) - \Omega(Y,W)r(HR)(JX,Z) - \\ & 2\Omega(X,Y)\ r(HR)(JZ,W) - 2\Omega(Z,W)\ r(HR)(JX,Y)\}\tfrac{K(HR)}{4(n+1)(n+2)}\{g(X,W)\,g(Y,Z) - \\ & g(X,Z)\,g(Y,W) + \Omega(X,W)\Omega(Y,Z) - \Omega(X,Z)\,\Omega(Y,W) - 2\Omega(X,Y)\,\Omega(Z,W)\}, \end{aligned}$$

where, $(HR)(X,Y,Z,W)$ is a generalized holomorphic (g-holomorphic) curvature tensor and is given as the form:

$$\begin{aligned} (HR)(X,Y,Z,W) = \frac{1}{16}\{ & 3[R(X,Y,Z,W) + R(JX,JY,Z,W) + R(X,Y,JZ,JW) + R(JX,JY,JZ,JW)] \\ & - R(X,Z,JW,JY) - R(JX,JZ,W,Y) - R(X,W,JY,JZ) - R(JX,JW,Y,Z) \\ & + R(JX,Z,JW,Y) + R(X,JZ,W,JY) + R(JX,W,Y,JZ) + R(X,JW,JY,Z)\}, \end{aligned}$$

$r(HR)$ is a generalized Ricci(g-Ricci) tensor which is defined as follows:

$$r(HR)_{ij} = (HR)^{k}_{ijk}\ ,$$

and $K(HR)$ is a generalized scalar(g-scalar) curvature tensor and is given as the form

$$K(HR) = g^{ij}r(HR)_{ij}\ .$$

The components of the g-Bochner curvature tensor for *AH*- manifold are given as the form

$$GB_{ijkl} = (HR)_{ijkl} - \frac{1}{2(n+2)} - \{g_{il}r(HR)_{jk} + g_{jk}r(HR)_{il} - g_{ik}r(HR)_{jl} - g_{jl}\, r(HR)_{ik}$$
$$+\sqrt{-1}\,\Omega_{il}r(HR)_{Jjk} + \sqrt{-1}\,\Omega_{jk}r(HR)_{Jil} - \sqrt{-1}\,\Omega_{ik}r(HR)_{Jjl} - \sqrt{-1}\,\Omega_{jl}r(HR)_{Jik}$$
$$-2\sqrt{-1}\,\Omega_{ij}r(HR)_{Jkl} - 2\sqrt{-1}\,\Omega_{kl}\ \ r(HR)_{Jij}\} + \frac{\mathrm{K(HR)}}{4(\mathrm{n}+1)(\mathrm{n}+2)}\{g_{il}\,g_{jk} - g_{ik}g_{jl}$$
$$+\Omega_{il}\,\Omega_{jk} - \Omega_{ik}\Omega_{jl} - 2\Omega_{ij}\Omega_{kl}\}.$$

Proposition 3.2. The components of the g-holomorphic curvature tensor for *VG* – manifold are given as the following forms:

1) $(HR)_{abcd} = (HR)_{\hat{b}bcd} = (HR)_{ab\hat{c}d} = (HR)_{a\hat{b}cd} = (HR)_{abc\hat{d}} = (HR)_{\hat{a}\hat{b}cd} = 0;$
2) $(HR)_{\hat{a}b\hat{c}d} = -\frac{1}{2}\left(2A^{ac}_{bd} - B^{ah}{}_{b}B_{hd}{}^{c} - B^{ah}{}_{d}B_{hb}{}^{c}\right);$
3) $(HR)_{\hat{a}bc\hat{d}} = \frac{1}{2}(2A^{ad}_{bc} - B^{ah}{}_{b}B_{hc}{}^{d} - B^{ah}{}_{c}B_{hb}{}^{d}).$

And the others are conjugate to the above components.

Proof.

1) For $i = a, j = b, k = c, l = d$, then

$$(HR)_{abcd} = \frac{1}{16}\{3[R_{abcd} - R_{abcd} - R_{abcd} + R_{abcd}] + R_{acdb} + R_{acdb} + R_{adbc} + R_{adbc} - R_{acdb} - R_{acdb} - R_{adbc} - R_{adbc}\} = 0$$

2) For $i = \hat{a}, j = b, k = \hat{c}, l = d$, then

$$(HR)_{\hat{a}b\hat{c}d} = \frac{1}{16}\{3[R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d}] + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}} + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}}\} = \frac{1}{2}(R_{\hat{a}b\hat{c}d} - R_{\hat{a}db\hat{c}})$$

By using Definition 2.2 we get

$$(HR)_{\hat{a}b\hat{c}d} = -\frac{1}{2}(R_{\hat{a}bd\hat{c}} + R_{\hat{a}db\hat{c}})$$

Therefore, for *VG*-manifold we have

$$(HR)_{\hat{a}b\hat{c}d} = -\frac{1}{2}(2A^{ac}_{bd} - B^{ah}{}_{b}B_{hd}{}^{c} - B^{ah}{}_{d}B_{hb}{}^{c})$$

By the same way, we can compute the other components.

Lemma 3.3. (i) In the adjoined G-structure space, the components of the g-Ricci tensor of the *VG*-manifold are given by the following forms:

1) $r(HR)_{ab} = 0;$
2) $r(HR)_{\hat{a}b} = -\frac{1}{2}\left(2A^{ac}_{cb} - B^{ah}{}_{b}B_{hc}{}^{c} - B^{ah}{}_{c}B_{hb}{}^{c}\right)$

(ii) The component of the g-scalar curvature tensor for *VG*-manifold is given by the following form

$$K(HR) = 2r(HR)_{\hat{a}a}$$

Proof. (i) 1) For $i = a\,,\ j = b\,,\ k = c$, we have

$$r(HR)_{ab} = (HR)^{k}_{abk} = (HR)^{c}_{abc} + (HR)^{\hat{c}}_{ab\hat{c}} = 0$$

2) For $i = \hat{a},\ j = b\,,\ k = c$, we have

$$r(HR)_{\hat{a}b} = (HR)^{k}_{\hat{a}bk} = (HR)^{c}_{\hat{a}bc} + (HR)^{\hat{c}}_{\hat{a}b\hat{c}}$$

According to the Proposition 3.2, we get

$$r(HR)_{\hat{a}b} = -\frac{1}{2}(R_{\hat{a}cb\hat{c}} + R_{\hat{a}\,bcc})$$

Making use of the Lemma 2.3, it follows that

$$r(HR)_{\hat{a}b} = -\frac{1}{2}\left(2A^{ac}_{cb} - B^{ah}{}_{b}B_{hc}{}^{c} - B^{ah}{}_{c}B_{hb}{}^{c}\right)$$

By the same technique we can compute the other components.

(ii) $K(HR) = g^{ab}r(HR)_{ab} + g^{\hat{a}b}r(HR)_{\hat{a}b} + g^{a\hat{b}}r(HR)_{a\hat{b}} + g^{\hat{a}\hat{b}}r(HR)_{\hat{a}\hat{b}}$

$$K(HR) = 2g^{\hat{a}b}r(HR)_{\hat{a}b} = 2\delta^{b}_{a}r(HR)_{\hat{a}b}$$

Therefore, we have $K(HR) = 2r(HR)_{\hat{a}a}$ □

Lemma 3.4. In the adjoined *G*-structure space, the components of the g-Bochner curvature tensor for *VG*- manifold are given as the following forms:

1) $GB_{abcd} = GB_{\hat{a}bcd} = GB_{a\hat{b}cd} = GB_{ab\hat{c}d} = GB_{abc\hat{d}} = 0;$

2) $GB_{\hat{a}\hat{b}cd} = -\frac{1}{2(n+2)}\{-2A^{ba}_{ac} - 2A^{ab}_{bd} + 2A^{ba}_{ad} + 2A^{ab}_{cb} + B^{bh}{}_{c}B_{ha}{}^{a} + B^{bh}{}_{a}B_{hc}{}^{a} + B^{ah}{}_{d}B_{hb}{}^{b} + B^{ah}{}_{b}\,B_{hd}{}^{b}$
$-B^{bh}{}_{d}\,B_{ha}{}^{a} - B^{bh}{}_{a}B_{hd}{}^{a} - B^{ah}{}_{c}B_{hb}{}^{b} - B^{ah}{}_{b}B_{hc}{}^{b}\}$;

3) $GB_{\hat{a}b\hat{c}d} = -\frac{1}{2}\left(2A^{ac}_{db} - B^{ah}{}_{d}B_{hb}{}^{c} - B^{ah}{}_{b}B_{hd}{}^{c}\right) + \frac{1}{2(n+2)}\{-2A^{ac}_{da} + B^{ah}{}_{a}\,B_{hd}{}^{c} + B^{ah}{}_{d}B_{ha}{}^{c} - 2A^{ac}_{bc} + B^{ah}{}_{b}B_{hc}{}^{c} + B^{ah}{}_{c}B_{hb}{}^{c}\} + \frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}^{ac}_{db}\}$;

4) $GB_{\hat{a}bc\hat{d}} = \frac{1}{2}\{\left(2A^{ad}_{cb} - B^{ah}{}_{c}B_{hb}{}^{d} - B^{ah}{}_{b}B_{hc}{}^{d}\right) - \frac{1}{2(n+2)}\{-2A^{ad}_{ca} + B^{ah}{}_{a}B_{hc}{}^{d} + B^{ah}{}_{c}\,B_{ha}{}^{d} - 2A^{ad}_{bd} + B^{ah}{}_{b}B_{hd}{}^{d} + B^{ah}{}_{d}B_{hb}{}^{d}\} - \frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}^{ad}_{bc}\}$.

And the others are conjugate to the above components, where $\hat{\delta}^{ad}_{bc} = \delta^{a}_{c}\delta^{b}_{d} + \delta^{a}_{d}\delta^{b}_{c}$.

Proof .

1) For $i = a\,,\ j = b\,,\ k = c$, and $l = d$, we have

$$GB_{abcd} = (HR)_{abcd} - \frac{1}{2(n+2)}\{g_{ad}r(HR)_{bc} + g_{bc}r(HR)_{ad} - g_{ac}\ r(HR)_{bd} - g_{bd}r(HR)_{ac} + \sqrt{-1}\Omega_{ad}\, r(HR)_{bc} + \sqrt{-1}\Omega_{bc}\ r(HR)_{ad} - \sqrt{-1}\Omega_{ac}r(HR)_{bd} - \sqrt{-1}\Omega_{bd}\ r(HR)_{ac} - 2\sqrt{-1}\Omega_{ab}r(HR)_{cd} - 2\sqrt{-1}\Omega_{cd}r(HR)_{ab}\} + \frac{K(HR)}{4(n+1)(n+2)}\{g_{ad}\,g_{bc} - g_{ac}\,g_{bd} + +\ \Omega_{ad}\Omega_{bc} - \Omega_{ac}\Omega_{bd} - 2\Omega_{ab}\Omega_{cd}\}$$

Making use of the matrices (2.1), consequently we get

$$GB_{abcd} = 0$$

Therefore, we have $GB_{abcd} = GB_{\hat{a}bcd} = GB_{a\hat{b}cd} = GB_{ab\hat{c}d} = GB_{abc\hat{d}} = 0$.

2) For i $= \hat{a}$, $j = \hat{b}$, $k = c$, and $l = d$, we have

$$GB_{\hat{a}\hat{b}cd} = (HR)_{\hat{a}\hat{b}cd} - \frac{1}{2(n+2)}\{g_{\hat{a}d}r(HR)_{\hat{b}c} + g_{\hat{b}c}r(HR)_{\hat{a}d} - g_{\hat{a}c}\ r(HR)_{\hat{b}d} - g_{\hat{b}d}\,r(HR)_{\hat{a}c} - \sqrt{-1}\,\Omega_{\hat{a}d}r(HR)_{\hat{b}c} - \sqrt{-1}\,\Omega_{\hat{a}c}r(HR)_{\hat{b}d} + \sqrt{-1}\,\Omega_{\hat{b}d}\ r(HR)_{\hat{a}c} + \sqrt{-1}\,\Omega_{\hat{b}c}\,r(HR)_{\hat{a}d} - 2\sqrt{-1}\,\Omega_{\hat{a}\hat{b}}r(HR)_{cd} + 2\sqrt{-1}\,\Omega_{cd}r(HR)_{\hat{a}\hat{b}}\} + \frac{K(HR)}{4(n+1)(n+2)}\{g_{\hat{a}d}g_{\hat{b}c} - g_{\hat{a}c}g_{\hat{b}d} + \Omega_{\hat{a}d}\Omega_{\hat{b}c} - \Omega_{\hat{a}c}\Omega_{\hat{b}d} - 2\Omega_{\hat{a}\hat{b}}\Omega_{cd}\}$$

Making use of the matrices (2.1), it follows that

$$GB_{\hat{a}\hat{b}cd} = -\frac{1}{(n+2)}\{g_{\hat{a}d}r(HR)_{\hat{b}c} + g_{\hat{b}c}r(HR)_{\hat{a}d} - g_{\hat{a}c}r(HR)_{\hat{b}d} - g_{\hat{b}d}\,r(HR)_{\hat{a}c}\}$$

According to the Lemma (3.3) and the matrices (2.1), we get

$$GB_{\hat{a}\hat{b}cd} = -\frac{1}{2(n+2)}\{-2A^{ba}_{ac} - 2A^{ab}_{bd} + 2A^{ba}_{ad} + 2A^{ab}_{cb} + B^{bh}{}_{c}B_{ha}{}^{a} + B^{bh}{}_{a}B_{hc}{}^{a} + B^{ah}{}_{d}B_{hb}{}^{b} + B^{ah}{}_{b}\,B_{hd}{}^{b} - B^{bh}{}_{d} - B^{bh}{}_{a}B_{hd}{}^{a} - B^{ah}{}_{c}B_{hb}{}^{b} - B^{ah}{}_{b}B_{hc}{}^{b}\}$$

3) $$GB_{\hat{a}b\hat{c}d} = (HR)_{\hat{a}b\hat{c}d} - \frac{1}{2(n+2)}\{g_{\hat{a}d}r(HR)_{b\hat{c}} + g_{b\hat{c}}r(HR)_{\hat{a}d} - g_{\hat{a}\hat{c}}\ r(HR)_{bd} - g_{bd}\,r(HR)_{\hat{a}\hat{c}} + \sqrt{-1}\,\Omega_{\hat{a}d}\ \ r(HR)_{b\hat{c}} - \sqrt{-1}\ \Omega_{\hat{a}\hat{c}}r(HR)_{bd} + \sqrt{-1}\,\Omega_{bd}\,r(HR)_{\hat{a}\hat{c}} + \sqrt{-1}\Omega_{b\hat{c}}\,r(HR)_{\hat{a}d} + 2\sqrt{-1}\,\Omega_{\hat{a}b}\,r(HR)_{\hat{c}d} + 2\sqrt{-1}\,\Omega_{\hat{c}d}r(HR)_{\hat{a}b}\} + \frac{K(HR)}{4(n+1)(n+2)}\{g_{\hat{a}d}g_{b\hat{c}} - g_{\hat{a}\hat{c}}g_{bd} + \Omega_{\hat{a}d}\,\Omega_{b\hat{c}} - \Omega_{\hat{a}\hat{c}}\Omega_{bd} - 2\Omega_{\hat{a}b}\Omega_{\hat{c}d}\ \}$$

Making use of the matrices (2.1), it follows that

$$GB_{\hat{a}b\hat{c}d} = (HR)_{\hat{a}b\hat{c}d} + \frac{1}{(n+2)}\{\delta^{a}_{b}r(HR)_{\hat{c}d} + \delta^{c}_{d}r(HR)_{\hat{a}b}\} + \frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}^{ac}_{bd}\}$$

According to the Lemmas (3.2), (3.3) and the matrices (2.1) we get

$$GB_{\hat{a}b\hat{c}d} = -\frac{1}{2}\left(2A^{ac}_{db} - B^{ah}{}_{d}B_{hb}{}^{c} - B^{ah}{}_{b}B_{hd}{}^{c}\right) + \frac{1}{2(n+2)}\{-2A^{ac}_{da} + B^{ah}{}_{a}B_{hd}{}^{c} + B^{ah}{}_{d}B_{ha}{}^{c} - 2A^{ac}_{bc} + B^{ah}{}_{b}B_{hc}{}^{c} + B^{ah}{}_{c}B_{hb}{}^{c}\} + \frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}^{ac}_{db}\}$$

By the same manner we can get the other component.

Theorem 3.5. If *M* is *VG*- manifold with zero g-Bochner curvature tensor, then *M* is a manifold of class R_1 if and only if, *M* is a manifold of zero g- Ricci tensor.

Proof. Suppose that *M* is *VG*-manifold with zero g-Bochner curvature tensor.

Making use of proposition 3.2 and Lemma 3.4 we have

$$\frac{1}{2}(R_{\hat{a}b\hat{c}d} - R_{\hat{a}db\hat{c}}) + \frac{1}{(n+2)}\{\delta^{a}_{b}r(HR)_{\hat{c}d} + \delta^{c}_{d}r(HR)_{\hat{a}b}\} + \frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}^{ac}_{bd}\} = 0$$

According to the Definition 2.2, we obtain

$$-\frac{1}{2}(2R_{\hat{a}bd\hat{c}}-R_{\hat{a}\hat{c}db})+\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

Since *M* is manifold of class R_1 , so by the Lemma 2.4 and Lemma 2.3 we get

$$-A_{bc}^{ad}-B^{adh}B_{hbc}+B^{ah}{}_{c}B_{hb}{}^{d}+\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

Symmetrizing by (h,b) and antisymmetrizing by the indexes (a,d) and (h,b), it follows that

$$\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

Contracting by the indices (a,b) and (c,d) we obtain

$$2\{\delta_a^a r(HR)_{\hat{a}d}+\delta_d^d r(HR)_{\hat{a}a}\}+nK(HR)=0$$
$$4n\{r(HR)_{\hat{a}a}\}+nK(HR)=0$$

Making use of the Lemma 3.3, it follows that

$$r(HR)_{\hat{a}a}=0$$

This shows *M* is a manifold of zero g- Ricci tensor.

The opposite side is simple so we omit the proof. □

Definition 3.6 [11]. A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij}=eg_{ij}$, where e is an Einstein constant.

Similar to the above definition we can put the following definition:

Definition 3.7. A Riemannian manifold is called an generalized Einstein manifold, if the g- Ricci tensor satisfies the equation $r(HR)_{ij}=Heg_{ij}$, where He is a generalized Einstein constant.

Definition 3.8 [10]. An *AH*-manifold has *J*-invariant Ricci tensor if $\circ\, r=r\circ J$.

Lemma 3.9 [10]. An *AH*-manifold has *J*-invariant Ricci tensor if and only if, in the adjoined G-structure space the equality $r_b^{\hat{a}}=0$ holds.

Remark 3.10. It is easy to find that the definition 3.8 and Lemma 3.9 hold for the generalized Ricci tensor.

Now we are in position to give an application of vanishing g-Bochner tensor of *VG* manifold.

Theorem 3.11. Suppose that *M* is *VG*-manifold with a zero g-Bochner curvature tensor and *J*-invariant g-Ricci tensor, if *M* is a flat manifold, then *M* is a generalized Einstein manifold.

Proof: According to the Lemma 3.4 we have

$$GB_{\hat{a}b\hat{c}d}=(HR)_{\hat{a}b\hat{c}d}+\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}$$

Suppose that *M* is flat manifold with a zero g-Bochner curvature tensor and *J*-invariant g-Ricci tensor. So, we obtain

$$\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

Contracting the above equation by indexes (d,a) we get

$$\frac{1}{(n+2)}\{\delta_b^d r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{dc}\}=0$$

Consequently, according to *J*-invariant g-Ricci tensor we deduce

$$r(HR)_{ij}=He\delta_j^i\ ,$$

where $He=\frac{-K(HR)}{4}$ is the generalized Einstein constant.

Therefore, according to the Remark 3.10 and Definition 3.7 we have M is generalized Einstein manifold.

Theorem 3.12. Suppose that M is *VG*-manifold with zero g-Bochner curvature tensor and *J*-invariant g-Ricci tensor. Then $A_{db}^{dc}=c\delta_b^c$ if and only if, *M* is generalized Einstein manifold.

Proof: Suppose that M is *VG*-manifold with the zero g-Bochner curvature tensor. By using Lemma 3.4 we get

$$-\frac{1}{2}\left(2A_{db}^{ac}-B_{\ d}^{ah}B_{hb}^{\ c}-B_{\ b}^{ah}B_{hd}^{\ c}\right)+\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

By Symmetrization and antisymmetrizing the above equation by the indices (*h*,*a*) we get:

$$-(A_{db}^{ac})+\frac{1}{(n+2)}\{\delta_b^a r(HR)_{\hat{c}d}+\delta_d^c r(HR)_{\hat{a}b}\}+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

Suppose that M is a generalized Einstein manifold, so we have

$$-(A_{db}^{ac})+\frac{He}{(n+2)}(\delta_b^a\delta_d^c+\delta_d^c\delta_b^a)+\frac{K(HR)}{2(n+1)(n+2)}\{\hat{\delta}_{bd}^{ac}\}=0$$

Contracting by indexes (d,a), it follows that

$$-\left(A_{db}^{dc}\right)+\frac{He}{(n+2)}\left(\delta_b^d\delta_d^c+\delta_d^c\delta_b^d\right)+\frac{K(HR)}{2(n+2)}\delta_b^c=0$$

$$A_{db}^{dc}=\left(\frac{4He+K(HR)}{2(n+2)}\right)\delta_b^c$$

Therefore, we get

$$A_{db}^{dc}=c\delta_b^c$$

Conversely, by the symmetrization and antisymmetrizing to equation $GB_{\hat{a}b\hat{c}d} = 0$ by the indices (*h*, *a*) and contracting by indexes (d, a), we get

$$-\left(A_{db}^{dc}\right) + \frac{1}{(n+2)}\left\{\delta_b^d r(HR)_{\hat{c}d} + \delta_d^c r(HR)_{\hat{a}b}\right\} + \frac{K(HR)}{2(n+1)(n+2)}\left\{\hat{\delta}_{bd}^{dc}\right\} = 0$$

$$-\left(A_{db}^{dc}\right) + \frac{2r(HR)_{\hat{c}b}}{(n+2)} + \frac{K(HR)\delta_b^c}{2(n+2)} = 0$$

$$\frac{2r(HR)_{\hat{c}b}}{(n+2)} = \left(\frac{4He+K(HR)}{2(n+2)}\right)\delta_b^c - \frac{K(HR)\delta_b^c}{2(n+2)}$$

Thus, we have

$$r(HR)_b^c = He\delta_b^c$$

According to the Lemma 3.9, consequently we get M is generalized Einstein manifold.

Definition 3.13 [4]. Suppose that $\lambda(X,Y,Z,W) = R(X,Y,Z,W) - R(X,Y,JZ,JW)$. Consider the following tensor $\lambda(X,Y) = \lambda(X,Y,Y,X)$. We say that an *AH*- manifold *M* is of a constant type at *p* ∈ M provided that for all *X*∈ $T_p(M)$, $\lambda(X,Y) = \lambda(X,Z)$.

If the equation $\lambda(X,Y) = \lambda(X,Z)$ holds for all *p* ∈ M, then the manifold *M* is called of a pointwise constant type, if $\lambda(X,Y) = \lambda(X,Z)$ is constant function, then *M* is called of a global constant type.

The above definition remains true if we use the g-Bochner tensor.

Theorem 3.14. If *M* is *VG*- manifold of g-Bochner curvature tensor, then *M* is a manifold of constant type.

Proof: Suppose that M is *VG*-manifold of g-Bochner curvature tensor.

By using the definition 3.13, we have

$$\lambda(X,Y) = GB(X,Y,Y,X) - GB(X,Y,JY,JX)$$

Firstly, we compute the tensor $GB(X,Y,Y,X)$

In the adjoined *G*-structure space, the components $GB(X,Y,Y,X)$= $GB_{ijkl}X^iY^jY^kX^l$ can be written as follows:

$$GB_{ijkl}X^iY^jY^kX^l = GB_{abcd}X^aY^bY^cX^d + GB_{\hat{a}bcd}X^{\hat{a}}Y^bY^cX^d + GB_{a\hat{b}cd}X^aY^{\hat{b}}Y^cX^d + GB_{ab\hat{c}d}X^aY^bY^{\hat{c}}X^d$$
$$+ GB_{abc\hat{d}}X^aY^bY^cX^{\hat{d}} + GB_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{a}}Y^{\hat{b}}Y^{\hat{c}}X^{\hat{d}} + GB_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}Y^cX^d +$$
$$GB_{\hat{a}\hat{b}\hat{c}d}X^{\hat{a}}Y^{\hat{b}}Y^{\hat{c}}X^d + GB_{\hat{a}\hat{b}c\hat{d}}X^{\hat{a}}Y^{\hat{b}}Y^cX^{\hat{d}} + GB_{a\hat{b}\hat{c}\hat{d}}X^aY^{\hat{b}}Y^{\hat{c}}X^{\hat{d}} +$$
$$GB_{\hat{a}b\hat{c}\hat{d}}X^{\hat{a}}Y^bY^{\hat{c}}X^{\hat{d}} + GB_{ab\hat{c}\hat{d}}X^aY^bY^{\hat{c}}X^{\hat{d}} + GB_{\hat{a}bc\hat{d}}X^{\hat{a}}Y^bY^cX^{\hat{d}} +$$
$$GB_{a\hat{b}\hat{c}d}X^aY^{\hat{b}}Y^{\hat{c}}X^d \; GB_{\hat{a}b\hat{c}d}X^{\hat{a}}Y^bY^{\hat{c}}X^d + GB_{a\hat{b}c\hat{d}}X^aY^{\hat{b}}Y^cX^{\hat{d}}$$

Making use of the properties of g-Bochner tensor we get

$$GB(X,Y,Y,X) = 2GB_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}Y^cX^d \tag{3.1}$$

Now we need to compute the tensor $GB(X,Y,JY,JX)$.

In the adjoined G-structure space, the components $GB(X,Y,JY,JX)$= $GB_{ijkl}X^iY^j(JY)^k(JX)^l$can be written as follows:

$$
\begin{aligned}
GB_{ijkl}X^iY^j(JY)^k(JX)^l = {} & GB_{abcd}X^aY^b(JY)^c(JX)^d + GB_{\hat{a}bcd}X^{\hat{a}}Y^b(JY)^c(JX)^d + GB_{a\hat{b}cd}X^aY^{\hat{b}}(JY)^c(JX)^d \\
& + GB_{ab\hat{c}d}\,X^aY^b(JY)^{\hat{c}}(JX)^d + GB_{abc\hat{d}}X^aY^b(JY)^c(JX)^{\hat{d}} + GB_{\hat{a}\hat{b}\hat{c}\hat{d}}\,X^{\hat{a}}Y^{\hat{b}}(JY)^{\hat{c}}(JX)^{\hat{d}} \\
& + GB_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}(JY)^c(JX)^d + GB_{\hat{a}\hat{b}\hat{c}d}X^{\hat{a}}Y^{\hat{b}}(JY)^{\hat{c}}(JX)^d + GB_{\hat{a}\hat{b}c\hat{d}}X^{\hat{a}}Y^{\hat{b}}(JY)^c(JX)^{\hat{d}} \\
& + GB_{a\hat{b}\hat{c}\hat{d}}X^aY^{\hat{b}}(JY)^{\hat{c}}(JX)^{\hat{d}} + GB_{\hat{a}b\hat{c}\hat{d}}\,X^{\hat{a}}Y^b(JY)^{\hat{c}}(JX)^{\hat{d}} + GB_{ab\hat{c}\hat{d}}X^aY^b(JY)^{\hat{c}}(JX)^{\hat{d}} \\
& + GB_{\hat{a}bc\hat{d}}X^{\hat{a}}Y^b(JY)^c(JX)^{\hat{d}} + GB_{a\hat{b}\hat{c}d}X^aY^{\hat{b}}(JY)^{\hat{c}}(JX)^d + GB_{\hat{a}b\hat{c}d}X^{\hat{a}}Y^b(JY)^{\hat{c}}(JX)^d \\
& + GB_{a\hat{b}c\hat{d}}X^aY^{\hat{b}}(JY)^c(JX)^{\hat{d}}
\end{aligned}
$$

By suing the property $(JX)^a = \sqrt{-1}X^a$, $(JX)^{\hat{a}} = \sqrt{-1}X^{\hat{a}}$ and the properties of g-Bochner tensor we have

$$GB(X,Y,JY,JX) = -2GB_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}Y^cX^d \tag{3.2}$$

According to the definition 3.12 and the equations (3.1) and (3.2), it follows that

$$\lambda(X,Y) = 4GB_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}Y^cX^d \tag{3.3}$$

Using the same manner to compute the tensor $\lambda(X,Z) = GB(X,Z,Z,X) - GB(X,Z,JZ,JX)$, implies

$$\lambda(X,Z) = 4GB_{\hat{a}\hat{b}cd}X^{\hat{a}}Z^{\hat{b}}Z^cX^d \tag{3.4}$$

Combining the equations (3.3) and (3.4), we obtain

$$\lambda(X,Y) = \lambda(X,Z)$$

Therefore, by the definition 3.13 we get that *M* is a manifold of constant type.

4 Conclusion

This paper introduces the geometric meaning of vanishing generalized Bochner tensor of Viasman-Gray manifold. We found the necessary conditions in which the Viasman-Gray manifold has vanishing generalized Bochner tensor. The necessary condition for the existence of constant type of Viasman-Gray manifold has been found. Finally, we gave a theoretical physics application of vanishing generalized Bochner tensor of Viasman-Gray manifold.

Competing Interests

Authors have declared that no competing interests exist.

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