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Some Notes on the Distribution of Mersenne Primes

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Abstract

Mersenne primes are a special kind of primes, which are always an important content in number theory. The study of Mersenne primes becomes one of hot topics of the nowadays science. It has not settled that whether there exist infinite Mersenne primes. And several of conjectures on the distribution of it provided by scholars. Starting from the Mersenne primes known about, in this paper we study the distribution of Mersenne primes and argued against some suppositions by data analyzing.

Keywords: Mersenne Primes, Distribution, Zhou Conjecture, Number Theory

1. Introduction

In 300 B.C , Euclid, famed ancient Greek mathematician proved that there are infinitude prime numbers by contradiction, and raised that a small amount of prime numbers could be expressed in the formulation of $2^p - 1$ (p is a prime number). After that, many famous mathematicians including mathematical masters like Fermat, Descartes, Leibniz, Goldbach, Euler, Gauss all researched into the prime numbers of this kinds of special formulation. But M. Mersenne, the French mathematician in 17th century and founder of Institute of France, was the first one to research into the prime number of $2^p - 1$ formulation deeply and systematically. These kinds of formulation like $2^p - 1$ is named as “Mersenne number” and expressed as M_p to commemorate him. If a Mersenne number is a prime number, then is named as “Mersenne prime”. The Mersenne prime seems like very simple, but the calculation of it is very complex. As the increase of index p, the calculation increases more complex. It not only need advanced theory and practiced skills, but arduous calculation is needed to validate whether a Mersenne number is prime or not. People have only discovered 47 Mersenne primes in the past 2300 years [1].

It should be noted that we have only settled the sequence of first 39 primes, while the last 8 primes left. That is, there is no other prime number p which makes $2^p - 1$ to be a prime number in the range of $2 \le p \le 13\ 466\ 917$. And we still cant assure that whether exists other prime number p which makes $2^p - 1$ to be prime number in the range of $20\ 996\ 011 \le p \le 43\ 112\ 609$. Though we have not discovered other prime p which makes $2^p - 1$ to be a prime number by checked the range at least once, but twice could confirm the seating arrangement [2].

It is not known whether the set of Mersenne primes is infinite. But researching on the attribution of Mersenne Primes is very important for the seeking of new Mersenne primes and exploring whether exist infinitude Mersenne primes. From the known Mersenne primes, the distribution of these special kinds of prime number is either sparse or dense in the positive integer, so very anomalous. In the long-term exploring, mathematicians have advanced some kind of suppositions. For example, mathematicians like Shanks from England, Bertrand from France, Ramanujan from India, Gilles from America and Brillhart from Germany have all supposed the distribution of Mersenne primes. The common point of their supposition is that they all presented as asymptotic expression.

In 1992, Haizhong Zhou [3], famed Chinese mathematician and linguist advanced the precise formulation of the Mersenne prime distribution: If $2^{2^n} < p < 2^{2^{n+1}}$ (n = 0, 1, 2, 3,…), then the amount of Mersenne primes is $2^{n+1} - 1$. The accurate and beautiful expression is made by him. Zhou conjecture has not been proved or disproved, and becomes a well-known mathematical problem [4].

Some researchers have raised the suppositions for the distribution of Mersenne primes. In this article we would argue against the deduction and raise different views by data analyzing for several suppositions.

2. Questions and View Points

2.1. Questions

In 1995, based on Zhou conjecture, Suwen Chen tried to make a further discussion on the distribution of Mersenne primes [5]. In the paper he defined that: Sequence $\boldsymbol{R}$ consists of primes p which makes $2^p - 1$ to be prime and numbers like 2^{2^k} (k = 0, 1, 2, …) in ascending, and the item No. n marked $\boldsymbol{R}(n)$; $\boldsymbol{P}$ is the sequence consists of primes p which makes $2^p - 1$ primes in ascending, while the item No. n marked $\boldsymbol{P}(n)$; $\boldsymbol{Q}$ is the sequence consists of numbers like $\boldsymbol{Q}(n) = 2^{2^n}$ (n = 0, 1, 2, 3, …). By analyzing the first 35 items settled by $\boldsymbol{R}(n)$, conjectures proposed like this:

CONJECTURE 1

1) $\boldsymbol{R}(2^{n+1}) = \boldsymbol{Q}(n) = 2^{2n}$;

2) $n/2 - 1 < \log_2 \boldsymbol{R}(n) < n/2 + 1$;

3) $\lim\limits_{n\to\infty} \frac{\log_2 \boldsymbol{R}(n)}{n} = \frac{1}{2}$

Then, raise doubts on item 2). we shall list the related data as following for convenience.

Attention to this data:

WHEN n = 20 and $\boldsymbol{R}(n)$ = 2 203, GET $\log_2\boldsymbol{R}(n)$ = 11.10525, $\boldsymbol{n/2}$ = 10.00000, $\boldsymbol{n/2}$ + 1 = 11.00000;

WHEN n = 37 and $\boldsymbol{R}(n)$ = 756 839, GET $\log_2\boldsymbol{R}(n)$ = 19.52963, $\boldsymbol{n/2}$ = 18.50000, $\boldsymbol{n/2}$ + 1 = 19.50000;

WHEN n = 41, $\boldsymbol{R}(n)$ = 2 976 221, GET $\log_2\boldsymbol{R}(n)$ = 21.50505, $\boldsymbol{n/2}$ = 20.50000, $\boldsymbol{n/2}$ + 1 = 21.50000;

WHEN n = 43, $\boldsymbol{R}(n)$ = 6 972 593, GET $\log_2\boldsymbol{R}(n)$ = 22.73326, $\boldsymbol{n/2}$ = 21.50000, $\boldsymbol{n/2}$ + 1 = 22.50000;

WHEN n = 44, $\boldsymbol{R}(n)$ = 13 466 917, GET $\log_2\boldsymbol{R}(n)$ = 23.68292, $\boldsymbol{n/2}$ = 22.00000, $\boldsymbol{n/2}$ + 1 = 23.00000

The location of Mersenne primes ignored for they have not been settled when 20 996 011 ≤ p ≤43 112 609. The data listed above satisfied the formula $\log_2\boldsymbol{R}(n) > \boldsymbol{n/2} + 1$, and item 2) in the conjecture follows the formula $\log_2\boldsymbol{R}(n) < \boldsymbol{n/2} + 1$ which contradicts the result $\log_2\boldsymbol{R}(n) > \boldsymbol{n/2} + 1$, so it concludes that the formula $\log_2\boldsymbol{R}(n) < \boldsymbol{n/2} + 1$ in conjecture $\boldsymbol{n/2} - 1 < \log_2\boldsymbol{R}(n) < \boldsymbol{n/2} + 1$ is wrong.

In Chen's paper, the author presupposed that $2^{\frac{n+3}{2}} < \boldsymbol{P}(n) < 2^{\frac{n+7}{2}}$, while $\boldsymbol{P}(31) \le \boldsymbol{P}(n) \le \boldsymbol{P}(58)$ according to conjecture $\boldsymbol{n/2} - 1 < \log_2\boldsymbol{R}(n) < \boldsymbol{n/2} + 1$. According to Chen's deduction, a great difference occurred on some numbers. It will be subscribed by the location-settled Mersenne primes as following.

WHEN $\boldsymbol{R}(32)$ = 756839, $2^{19.5} < 741456$, GET 756 839 < 741 456, impossible;

WHEN $\boldsymbol{R}(36)$ = 2976221, $2^{21.5} < 2965821$, GET 2 976221 < 2 965 821, impossible;

WHEN $\boldsymbol{R}(38)$ = 6 972 593, $2^{22.5} < 5931642$, GET 6 972 593 < 5 931 642, impossible;

2.2. Different Views

Scholars proposed many conjectures on how to confirm the quantity of Mersenne primes in certain range. In 1980, Lenstra and Pomerance [6] independently presupposed the quantity of Mersenne primes while less than x, which would be $(e^{\gamma}/\log 2)\log\log x$, as $\gamma = 0.5772$ is the Euler's constant. Based on that, Wagstaff presupposed a conjecture in 1983, as following:

CONJECTURE 2 [7]

1) IF the quantity of Mersenne primes which less than x is $\pi_M(x)$, Then

$$\pi_M(x) \approx \frac{e^{\gamma}}{\ln 2}\log\log x = (2.5695\cdots)\ln\ln x,$$

As γ is the Euler's constant;

2) the expect value of Mersenne primes M_q is about $e^{\gamma} = 1.7806\cdots$, while $x < q < 2x$;

3) the probability of M_q is a prime number is about

$$\frac{e^{\gamma}}{\ln 2}\cdot\frac{\ln aq}{\ln 2} = (2.5695\cdots)\frac{\ln aq}{q},$$

as $a = \begin{cases} 2 & q \equiv 3 (\mathrm{mod}\, 4) \\ 6 & q \equiv 1 (\mathrm{mod}\, 4) \end{cases}$.

This conjecture explains the probability of M_q is a Mersenne prime in the precondition of q, and also pointed the quantity of Mersenne primes in certain range. It has been confirmed that there are 39 Mersenne primes when $p \le 13466917$. However, the number

$$\begin{aligned}\pi_M(x) &\approx \frac{e^{\gamma}}{\ln 2}\log\log 13466917 \\ &= (2.5695\cdots)\ln\ln 13466917 \approx 7.190360\end{aligned},$$

quite differs from the actual situation which could be thought that the conjecture 1) is not advisable. Based on conjecture 3), take $q = 521$ and $q = 257$ in Mersenne conjecture as an example, for 521*257 l(mod4), while $a = 6$, then we get

$$(2.5695\cdots)\frac{\ln aq}{q} = (2.5695\cdots)\frac{\ln(6\cdot 521)}{521} \approx 0.039691$$

$$(2.5695\cdots)\frac{\ln aq}{q} = (2.5695\cdots)\frac{\ln(6\cdot 257)}{257} \approx 0.073397$$

It's well known that M_{521} is a Mersenne prime, but M_{257}. It could meet a big mistake to some extent if we

Table 1. Relationship between *R*(*n*) and *n* [5].

n	*R*(*n*)	*P*, *Q*	$\log_2 R(n)$	*n*/2	*n*	*R*(*n*)	*P*, *Q*	$\log_2 R(n)$	*n*/2
1	2	***P***(1)	1.00000	0.50000	27	11 213	***P***(23)	13.45288	13.50000
2	2	***Q***(0)	1.00000	1.00000	28	19 937	***P***(24)	14.28316	14.00000
3	3	***P***(2)	1.58496	1.50000	29	21 701	***P***(25)	14.40547	14.50000
4	4	***Q***(1)	2.00000	2.00000	30	23 209	***P***(26)	14.50240	15.00000
5	5	***P***(3)	2.32193	2.50000	31	44 497	***P***(27)	15.44142	15.50000
6	7	***P***(4)	2.80735	3.00000	32	65 536	***Q***(4)	16.00000	16.00000
7	13	***P***(5)	3.70044	3.50000	33	86 243	***P***(28)	16.39612	16.50000
8	16	***Q***(2)	4.00000	4.00000	34	110 503	***P***(29)	16.75373	17.00000
9	17	***P***(6)	4.08746	4.50000	35	132 049	***P***(30)	17.01071	17.50000
10	19	***P***(7)	4.24793	5.00000	36	216 091	***P***(31)	17.72128	18.00000
11	31	***P***(8)	4.95420	5.50000	37	756 839	***P***(32)	19.52963	18.50000
12	61	***P***(9)	5.93074	6.00000	38	859 433	***P***(33)	19.71303	19.00000
13	89	***P***(10)	6.47573	6.50000	39	1 257 787	***P***(34)	20.26246	19.50000
14	107	***P***(11)	6.74147	7.00000	40	1 398 269	***P***(35)	20.41521	20.00000
15	127	***P***(12)	6.89968	7.50000	41	2 976 221	***P***(36)	21.50505	20.50000
16	256	***Q***(3)	8.00000	8.00000	42	3 021 377	***P***(37)	21.52677	21.00000
17	521	***P***(13)	9.02514	8.50000	43	6 972 593	***P***(38)	22.73326	21.50000
18	607	***P***(14)	9.24555	9.00000	44	13 466 917	***P***(39)	23.68292	22.00000
19	1 279	***P***(15)	10.32080	9.50000	45	20 996 011	***P***(?)	24.32361	22.50000
20	2 203	***P***(16)	11.10525	10.00000	46	24 036 583	***P***(?)	24.51873	23.00000
21	2 281	***P***(17)	11.15545	10.50000	47	25 964 951	***P***(?)	24.63006	23.50000
22	3 217	***P***(18)	11.65150	11.00000	48	30 402 457	***P***(?)	24.85768	24.00000
23	4 253	***P***(19)	12.05427	11.50000	49	32 582 657	***P***(?)	24.95760	24.50000
24	4 423	***P***(20)	12.11081	12.00000	50	37 156 667	***P***(?)	25.14712	25.00000
25	9 689	***P***(21)	13.24213	12.50000	50	42 643 801	***P***(?)	25.34583	25.50000
26	9 941	***P***(22)	13.27918	13.00000	52	43 112 609	***P***(?)	25.36161	26.00000

Note: The symbol ? means the location of those Mersenne primes haven't been settled.

judge from higher probability.

3. Conclusions

Starting from analyzing of the known Mersenne primes, different ideas proposed about the distributional conjectures of Mersenne primes, which would be beneficial to the studies on the quantity of Mersenne primes and the distribution of its index prime *p*.

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