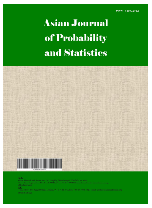

Asian Journal of Probability and Statistics

7(3): 14-27, 2020; Article no.AJPAS.57892
ISSN: 2582-0230

On the Extended Generalized Inverse Exponential Distribution with Its Applications

Sule Ibrahim[1*], Bello Olalekan Akanji[1] and Lawal Hammed Olanrewaju[2]

[1]Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.
[2]Department of Mathematics, Faculty of Sciences, Obafemi Awolowo University, Ile-Ife, Nigeria.

Authors' contributions

This work was carried out in collaboration among all authors. Author SI designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BOA managed the analyses of the study. Author LHO managed the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2020/v7i330184
Editor(s):
(1) Dr. Belkacem Chaouchi, Khemis Miliana University, Algeria.
Reviewers:
(1) Dalia Ahmed Eisa, Egypt.
(2) Deepak Kapgate, Nagpur University, India.
Complete Peer review History: http://www.sdiarticle4.com/review-history/57892

Original Research Article

Received: 03 April 2020
Accepted: 10 June 2020
Published: 19 June 2020

Abstract

We propose a new distribution called the extended generalized inverse exponential distribution with four positive parameters, which extends the generalized inverse exponential distribution. We derive some mathematical properties of the proposed model including explicit expressions for the quantile function, moments, generating function, survival, hazard rate, reversed hazard rate and odd functions. The method of maximum likelihood is used to estimate the parameters of the distribution. We illustrate its potentiality with applications to two real data sets which show that the extended generalized inverse exponential model provides a better fit than other models considered.

Keywords: Breast cancer; demography and insurance; inverse exponential; memorylessness; survival times.

1 Introduction

In the practice of distribution theory, the development of new statistical models is a prominent research topic. The literature is filled with such distributions that are very worthwhile in predicting and modeling real

**Corresponding author: E-mail: ibrahimsule76@yahoo.com;*

world phenomena. A number of classical distributions have been used comprehensively over the past decades for modeling data in several applied areas including bio-medical analysis, reliability engineering, economics, forecasting, astronomy, demography and insurance.

One of the preferred areas of research in the field of distribution is that of generating new distributions starting with a baseline distribution by adding one or more additional parameters. A generalized distribution may be important because it is connected with other special distributions in interesting ways (via transformations, limits, conditioning, etc.). In some cases, a parametric family may be important because it can be used to model a wide variety of random phenomena. In many cases, a special parametric family of distributions will have one or more distinguished standard members, corresponding to specified values of some of the parameters. Usually the standard distributions will be mathematically simpler, and often other members of the family can be constructed from the standard distributions by simple transformations on the underlying standard random variable. An incredible variety of special distributions have been studied over the years, and new ones are constantly being added to the literature. Some recent families of distributions are the generalized Burr-G family of distributions by Nasir et al. [1], extended Weibull G family by Korkmaz [2], Ampadu-G class of distributions by Ahmad [3], A new Weibull-X family of distributions by Ahmad et al. [4], Kumaraswamy odd Burr G family of distributions by Nasir et al. [5], the Marshal-Olkin Odd Lindley-G family of distributions by Jamal et al. [6], The Exponentiated Kumaraswamy-G family of distributions by Silva et al. [7], the Topp Leone exponentiated G family of distributions by Ibrahim et al. [8].

Due to simple mathematical usage and interesting properties, the Exponential distribution is a widely used model in life-testing experiments. As a result of that, it has been generalized by several researchers. Despite the usage of Exponential distribution in Poisson processes, reliability engineering and its attractive properties, the fact that the Exponential distribution has a constant failure rate (Lemonte, [9]) is a disadvantage because for that singular reason, the distribution becomes unsuitable for modeling real life situations with bathtub and inverted bathtub failure rates. This is actually a serious short-coming of the Exponential distribution. Also, the memorylessness is rarely obtainable in real life phenomena. To make up for these limitations, Keller and Kamath [10] came up with a modified version of the Exponential distribution, this modification resulted into the Inverse Exponential distribution and it has also been studied in some details by Lin et al. [11].

Gupta & Kundu [12], generalized the Exponential distribution by appending the shape parameter, and named the distribution as the generalized Exponential distribution. Generalized Inverted Exponential distribution was first introduced by Abouammoh and Alshingiti [13]. This distribution originated from the exponentiated Frechet distribution (Nadarajah and Kotz [14]).

The generalized Inverted Exponential distribution on a convenient structure of the distribution function provides many practical applications, including, in horse racing, queue theory, modeling wind speeds. Oguntunde and Adejumo [15] have explored the statistical properties of the Generalized Inverted Generalized Exponential distribution and its parameters were estimated at both censored and uncensored cases using the method of maximum likelihood estimation (MLE). Dey et al. [16] presents some estimation and prediction of unknown parameters based on progressively censored generalized Inverted Exponential data.

The probability density and cumulative density function of generalized inverted exponential distribution with shape parameter β and scale parameter λ, are given respectively as

$$H(x;\beta,\lambda) = 1 - \left(1 - e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda} \tag{1}$$

and

$$h(x;\beta,\lambda)=\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1},\ x\ \geq 0. \tag{2}$$

This paper aims to introduce an extended version of the generalized inverse exponential distribution called extended generalized inverse exponential distribution using the Topp Leone exponentiated-G family of distribution because it is relatively new, it does not involve any special function and it has only two extra shape parameters, this way, the resulting compound distribution will not have too many parameters and would be easier to handle. The rest of the paper is outline as follows. In section 2, the extended generalized inverse exponential distribution is defined. Linear representation of the new model is presented in section 3. Section 4 provides the statistical properties of the new model. In section 5, the distribution of order statistics is presented. The maximum likelihood estimation is discussed in section 6. Section 7 presented the application of the new model to real data sets. Finally, concluding remark is given in section 8.

2 The Extended Generalized Inverse Exponential (EGIEx) Distribution

For an arbitrary baseline cumulative distribution function (cdf) $H(x,\varphi)$, the TLEx-G family with two extra positive shape parameters α and θ has cdf and probability density function (pdf) for (x > 0) given by

$$F(x;\alpha,\theta,\varphi)=\{1-[1-H(x,\varphi)^{\alpha}]^2\}^{\theta} \tag{3}$$

and

$$f(x;\alpha,\theta,\varphi)=2\alpha\theta h(x;\ \varphi)H(x;\ \varphi)^{\alpha-1}[1-H(x;\ \varphi)^{\alpha}]\{1-[1-H(x;\ \varphi)^{\alpha}]^2\}^{\theta-1} \tag{4}$$
$$x>0,\qquad \alpha,\qquad \theta,\qquad \varphi>0$$

respectively.

Where $h(x;\varphi)=\frac{dH(x;\,\varphi)}{dx}$ is the baseline pdf, α and θ are positive shape parameters.

The cdf of the new model is derived by substituting (1) into (3) as

$$F(x;\alpha,\theta,\beta,\lambda)=\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2\right\}^{\theta} \tag{5}$$

And the pdf corresponding to (5) is given as

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2\right\}^{\theta-1} \tag{6}$$

where $x\ \geq 0$, $\beta>0$ is the scale parameter and $\alpha,\lambda,\theta>0$ are the shape parameters respectively.

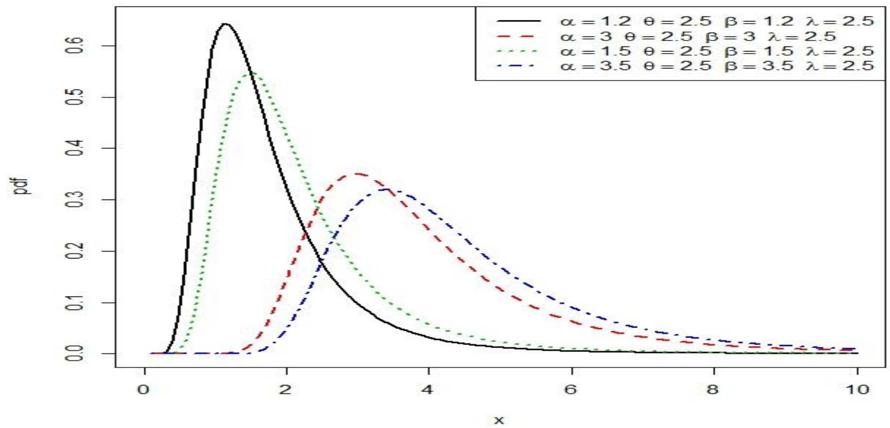

Fig. 1. Plots of pdf of extended generalized inverse exponential distribution at different parameter values

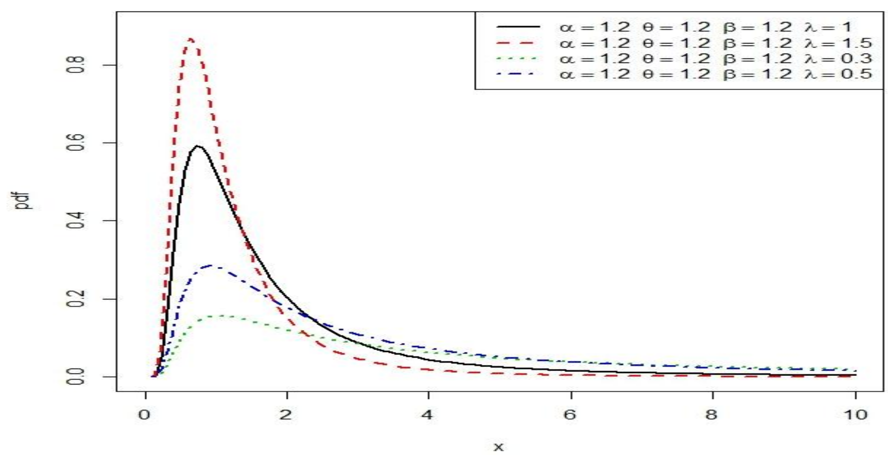

Fig. 2. Plots of pdf of extended generalized inverse exponential distribution with parameter λ values varies and other parameters fixed at 1.2

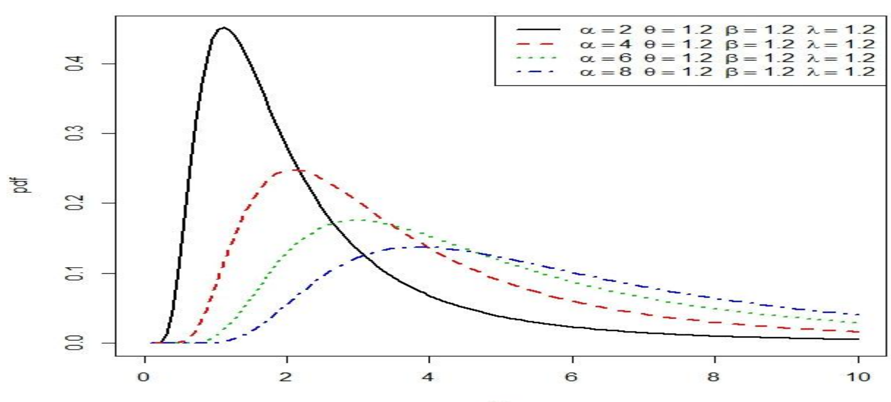

Fig. 3. Plots of pdf of extended generalized inverse exponential distribution with parameter α values varies and other parameters fixed at 1.2

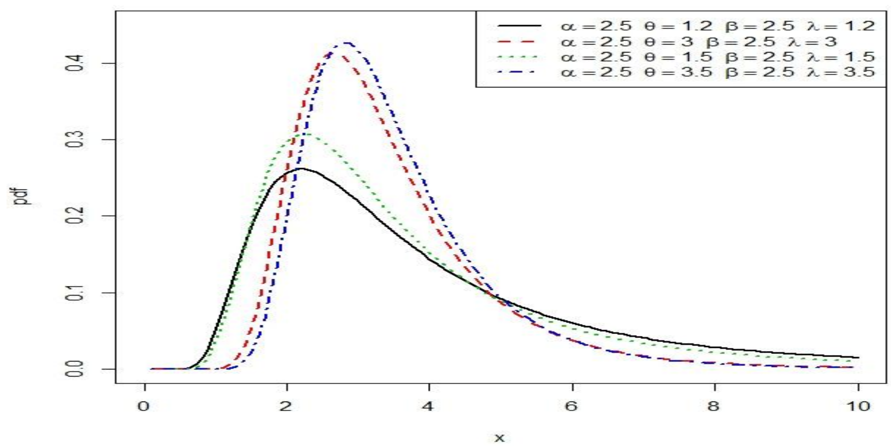

Fig. 4. Plots of pdf of extended generalized inverse exponential distribution with parameter θ and λ values varies and other parameters fixed at 2.5

3 Linear Representation of the New Model

Using the series expansion

$$(1-y)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\theta)}{i!\Gamma(\theta-i)} y^i \tag{7}$$

$$\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\,\Gamma(\theta-i)}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2i}$$

$$\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2i+1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(2(i+1))}{j!\,\Gamma(2(i+1)-j)}\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha j}$$

$$\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha(j+1)-1} = \sum_{i=0}^{\infty}\frac{(-1)^k\Gamma(\alpha(j+1))}{k!\,\Gamma(\alpha(j+1)-k)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda k}$$

$$\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda(k+1)-1} = \sum_{i=0}^{\infty}\frac{(-1)^l\Gamma(\lambda(k+1))}{l!\,\Gamma(\lambda(k+1)-l)}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1}$$

$$f(x) = 2\alpha\theta\frac{\beta\lambda}{x^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} \tag{8}$$

4 Statistical Properties

In this section, some of the statistical properties of extended generalized inverse exponential distribution will be obtained as follows:

4.1 Moments

Moments is used to study many important properties of distribution such as dispersion, tendency, skewness and kurtosis. The r^{th} moments of the extended generalized inverse exponential distribution is obtained as follow:

$$E(X^r) = \int_0^\infty x^r f(x)dx \tag{9}$$

Using (6), we have,

$$E(X^r) = \int_0^\infty x^r\, 2\alpha\theta \frac{\beta\lambda}{x^2} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} dx \tag{10}$$

$$E(X^r) = 2\alpha\theta\beta\lambda \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \int_0^\infty x^{r-2} \left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} \tag{11}$$

$$E(X^r) = 2\alpha\theta\beta^r\lambda \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))(l+1)^{r-1}\Gamma(1-\mathrm{r})}{i!j!k!l!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \tag{12}$$

4.2 Moment generating function

The moment generating function (mgf) of X can be obtained using the equation

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \tag{13}$$

$$e^{tx} = \sum_{m=0}^\infty \frac{t^m x^m}{m!} \tag{14}$$

$$M_x(t) = 2\alpha\theta\beta^m\lambda \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))t^m(l+1)^{m-1}\Gamma(1-\mathrm{m})}{i!j!k!l!m!\Gamma(\theta-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \tag{15}$$

4.3 Quantile function

The Quantile function is given by;

$$Q(u) = F^{-1}(u) \tag{16}$$

Therefore, the corresponding quantile function for the extended generalized inverse exponential model is given by;

$$x = Q(u) = \frac{\beta}{-log\left\{1-\left[1-\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{\lambda}}\right\}} \tag{17}$$

where U has the uniform $U(0,1)$ distribution.

4.4 Median

The median of the extended generalized inverse exponential distribution is obtained by setting $U = 0.5$ in (17) to obtain,

$$x_m = Q(0.5) = \frac{\beta}{-log\left\{1-\left[1-\left(1-\left(1-0.5^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{\lambda}}\right\}} \tag{18}$$

4.5 Survival function

The survival function, which is the probability of an item not failing prior to some time, can be defined as

$$S(x) = 1 - F(x) \tag{19}$$

$$S(x) = 1 - \left\{1 - \left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta} \tag{20}$$

4.6 Hazard rate function

The hazard rate function is given as

$$\tau(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \tag{21}$$

$$\tau(x) = \frac{2\alpha\theta\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}}{1-\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta}} \tag{22}$$

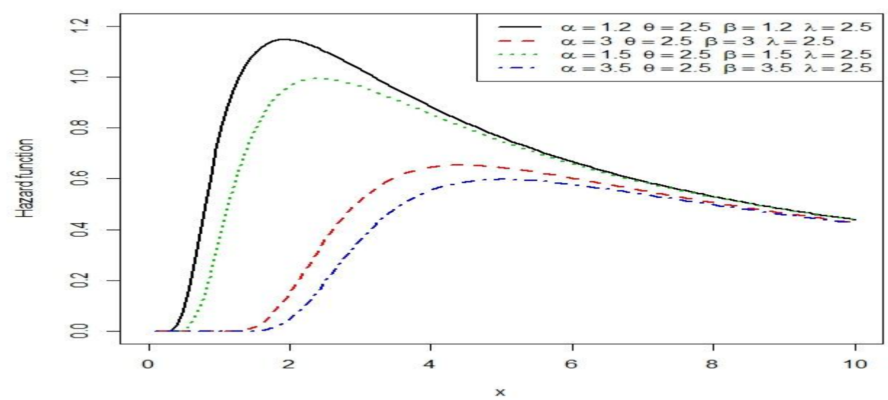

Fig. 5. Plots of hrf of extended generalized inverse exponential distribution at different parameter values

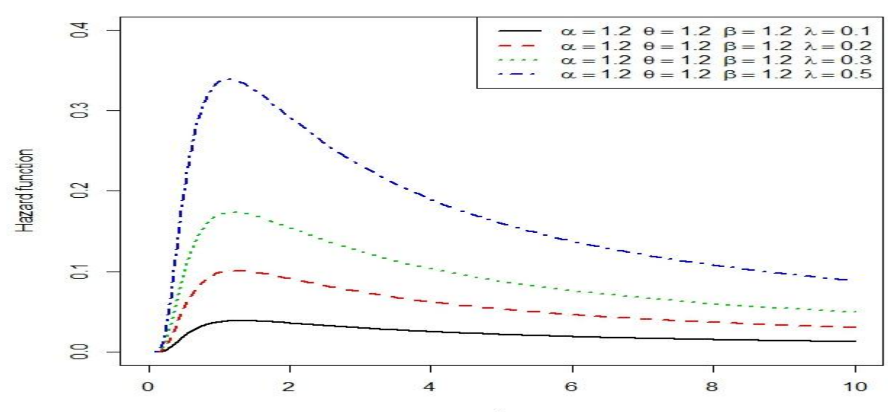

Fig. 6. Plots of hrf of extended generalized inverse exponential distribution with values of λ varies and other parameters fixed at 1.2

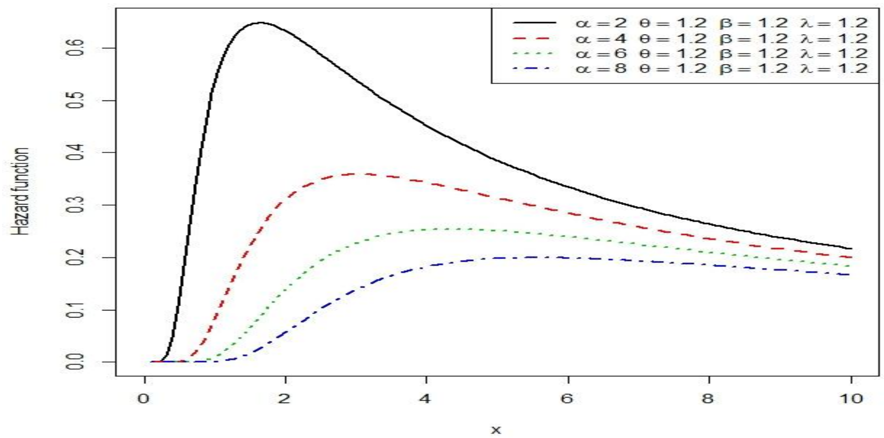

Fig. 7. Plots of hrf of extended generalized inverse exponential distribution with values of α varies and other parameters fixed at 1.2

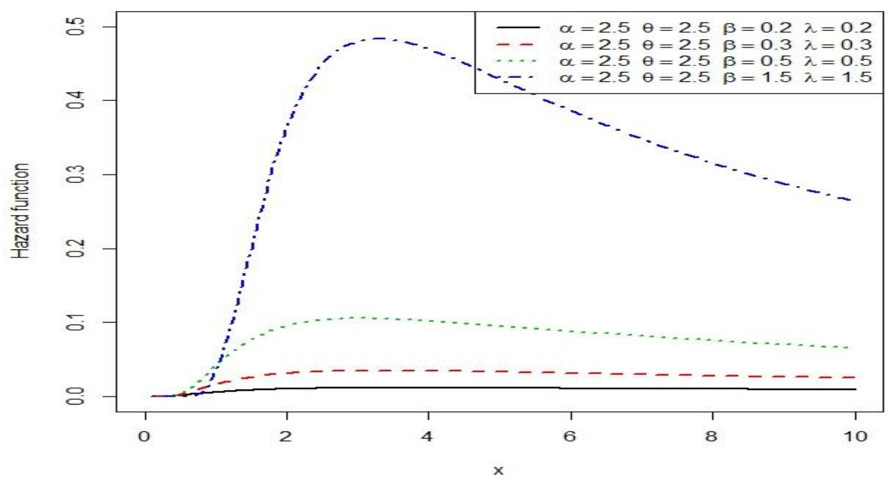

Fig. 8. Plots of hrf of extended generalized inverse exponential distribution with values of α and θ fixed at 2.5 and β and λ varies

4.7 Odds function

The odds function is obtained using the relation

$$Q(x) = \frac{F(x)}{S(x)} \tag{23}$$

$$Q(x) = \frac{\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta}}{1-\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta}} \tag{24}$$

4.8 Reversed hazard rate function

The reverse hazard rate function of the extended generalized inverse exponential distribution is given as

$$\emptyset(x)=\frac{f(x)}{F(x)} \tag{25}$$

$$\emptyset(x)=\frac{2\alpha\theta\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}}{\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta}} \tag{26}$$

5 Distribution of Order Statistic

Let $X_1, X_2, X_3, \dots, X_n$ be a random sample and its ordered values are denoted as $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$. The pdf of order statistic is obtained using the below function

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \tag{27}$$

5.1 Minimum order statistic

The minimum order statistic is obtained by setting $r=1$ in (27) as

$$f_{1:n}(x)=nf(x)[1-F(x)]^{n-1} \tag{28}$$

Then the minimum order statistic of the extended generalized inverse exponential distribution is given as

$$f_{1:n}(x)=2n\alpha\theta\frac{\beta\lambda}{x^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+k+l+m}\Gamma(n)\Gamma(\theta(i+1))\Gamma(2(j+1))\Gamma(\alpha(k+1))\Gamma(\lambda(l+1))\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{m+1}}{i!j!k!l!m!\Gamma(n-i)\Gamma(\theta(i+1)-j)\Gamma(2(j+1)-k)\Gamma(\alpha(k+1)-l)\Gamma(\lambda(l+1)-m)} \tag{29}$$

5.2 Maximum order statistic

The maximum order statistic is obtained by setting $r=n$ in (27) as

$$f_{n:n}(x)=nf(x)[F(x)]^{n-1} \tag{30}$$

Then the maximum order statistic of the extended generalized inverse exponential distribution is given as

$$f_{n:n}(x)=2n\alpha\theta\frac{\beta\lambda}{x^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma(\theta(n-1)+1)\Gamma(2(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta(n-1)+-i)\Gamma(2(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} \tag{31}$$

6 Maximum Likelihood Estimates

Since maximum likelihood estimators give the maximum information about the population parameters, therefore this section presents the maximum likelihood estimates (MLEs) of the parameters that are inherent within the extended generalized inverse exponential distribution function given by the following: Let

$X_1, X_2, X_3, \dots, X_n$ be random variables of the extended generalized inverse exponential distribution of size n. Then sample likelihood function of extended generalized inverse exponential distribution is obtained as

$$L(x) = (2\alpha\theta\beta\lambda)^n \prod_{i=1}^{n} \left\{ \frac{1}{x_i^2} e^{-\left(\frac{\beta}{x_i}\right)} \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda-1} \left[1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right]^{\alpha-1} \left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right] \left\{1 - \left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2\right\}^{\theta-1} \right\} \quad (32)$$

The Log-likelihood function is $logL(x)$ and is given as

$$logL(x) = nlog2 + nlog\alpha + nlog\theta + nlog\beta + nlog\lambda + \sum_{i=1}^{n} log\left(\frac{1}{x_i^2}\right) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right) + (\lambda - 1)\sum_{i=1}^{n} log\left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right) + (\alpha - 1)\sum_{i=1}^{n} log\left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right) + \sum_{i=1}^{n} log\left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right] + (\theta - 1)\sum_{i=1}^{n} log\left\{1 - \left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2\right\} \quad (33)$$

Therefore, The MLE's of parameters $\alpha, \beta, \theta, \lambda$ which maximize the above log-likelihood function must satisfy the normal equations. We take the first derivative of the above log-likelihood equation with respect to each parameter and equate to zero respectively.

$$\frac{\partial logL(x)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right) + \sum_{i=1}^{n} \frac{\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha} log\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)}{1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}} + (\theta - 1)\sum_{i=1}^{n} \frac{2\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right] log\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]}{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2} \quad (34)$$

$$\frac{\partial logL(x)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + (\lambda - 1)\sum_{i=1}^{n} \frac{\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}}{\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)} + (\alpha - 1)\sum_{i=1}^{n} \frac{\lambda\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)}{1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}} + \sum_{i=1}^{n} \frac{\alpha\lambda\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda-1}\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}}{1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}} + (\theta - 1)\sum_{i=1}^{n} \frac{2\alpha\lambda\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda-1}\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]}{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2} \quad (35)$$

$$\frac{\partial logL(x)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log\left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right) + (\alpha - 1)\sum_{i=1}^{n} \frac{\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda} log\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)}{1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}} + \sum_{i=1}^{n} \frac{\alpha\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda} log\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}}{1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}} + (\theta - 1)\sum_{i=1}^{n} \frac{2\alpha\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda} log\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]}{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^2} \quad (36)$$

$$\frac{\partial logL(x)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log\left\{1 - \left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\} \tag{37}$$

Since the above derived equations (34) to (37) are in the complex form, therefore the exact solution of ML estimator for unknown parameters is not possible. So it is convenient to use nonlinear Newton Raphson algorithm for exact numerically solution to maximize the above likelihood function.

7 Applications

In this section, we applied two data sets to illustrate the usefulness of the proposed model and observe its flexibility over some existing models. The models considered are:

- ✓ Inverse exponential (IEx) distribution

$$f(x) = \frac{\beta}{x^2} e^{\left(-\frac{\beta}{x}\right)}$$

- ✓ Exponentiated generalized inverse exponential (ExGIEx) distribution

$$f(x) = \frac{\beta\alpha\theta}{x^2} e^{\left(-\frac{\beta}{x}\right)} \left[1 - e^{\left(-\frac{\beta}{x}\right)}\right]^{\alpha-1} \left[1 - \left(1 - e^{\left(-\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{\theta-1}$$

- ✓ Generalized Inverse exponential (GIEx) distributiion

$$f(x) = \frac{\beta\lambda}{x^2} e^{\left(-\frac{\beta}{x}\right)} \left[1 - e^{\left(-\frac{\beta}{x}\right)}\right]^{\lambda-1}$$

Data set 1:

The first data set represents the breaking stress of carbon fibers of 50 mm length (GPa) which was reported by Nicholas and Padgett [17]. This data was used by Yousof et al. [18]. The data set is:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 1. MLEs and selection criteria for data set 1

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC
EGIEx	1.18861	13.0965	0.3690	19.5331	93.9751	195.9502
IEx	-	2.2992	-	-	136.0285	274.0570
ExGIEx	39.7875	13.8792	0.4454	-	94.0114	196.0228
GIEx	-	13.2879	-	7.6019	99.6202	203.2403

Data set 2:

The second data set was given by Lee [19] and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. It has also been applied by Ramos et al. [20]. The data set is as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Table 2. MLEs and selection criteria for data set 2

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC
EGIEx	63.7813	0.0040	2.2264	0.5013	622.6595	1253.3190
IEx	-	10.3215	-	-	677.2791	1357.5580
ExGIEx	0.2142	0.0021	4.6702	-	743.3310	1492.6620
GIEx	-	0.5595	-	6.4034	664.1239	1332.2480

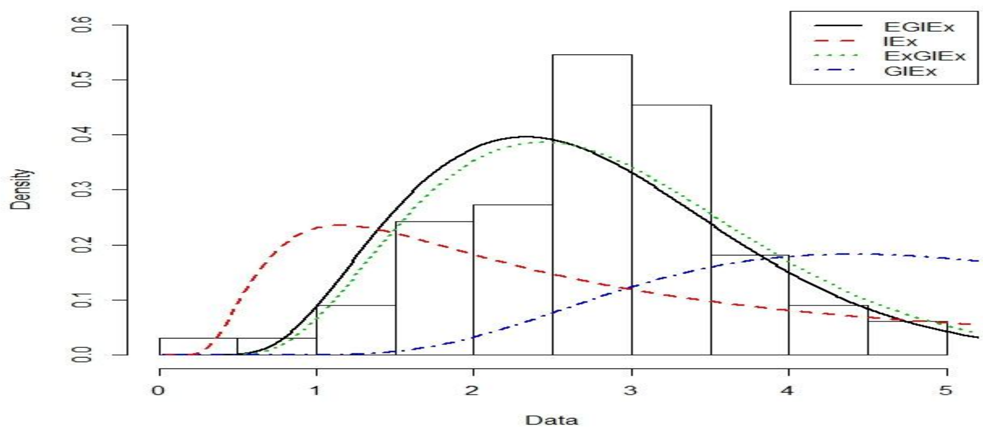

Fig. 9. Histogram and fitted models of data set 1

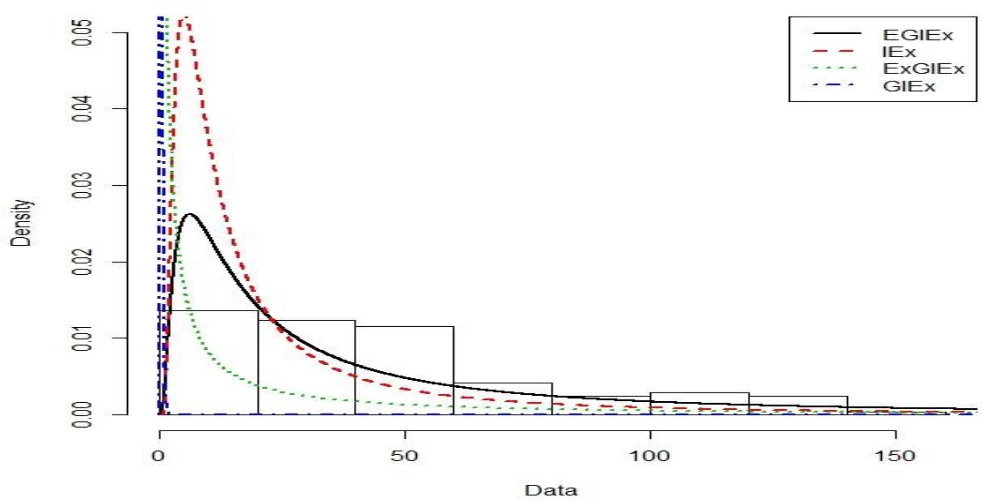

Fig. 10. Histogram and fitted models of data set 2

8 Conclusion

In this paper, a four parameter model called the extended generalized inverse exponential distribution which extends the generalized inverse exponential distribution has been studied. The pdf plots in Fig. 1 to Fig. 4 showed that this distribution can be used to model to skewed and symmetric data, also, in the hrt plots in Fig. 5 to Fig. 8 showed that the distribution can have increasing, decreasing and unimodal hrf. This new distribution is capable of modeling survival data. We derived explicit expressions for some of its statistical and mathematical properties including the moments, generating function, quantile function, survival function, hazard rate function, reversed hazard rate function and odd function. The minimum and maximum distributions of order statistics of the new model were derived. The model parameters were estimated by using maximum likelihood method based on complete sample. From the results showed in Table 1 and Table 2, we observed that the extended generalized inverse exponential distribution provides better fit than the distributions considered on the two real data sets based on the value of AIC and also from the histograms and the fitted plots of the pdfs in Fig. 9 and Fig. 10.

Competing Interests

Authors have declared that no competing interests exist.

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