Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 9881237, 9 pages
https://doi.org/10.1155/2022/9881237

Research Article

Some Geometric Characterizations of f-Curves Associated with a Plane Curve via Vector Fields

Azeb Alghanemi and Abeer AlGhawazi

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Azeb Alghanemi; aalghanemi@kau.edu.sa

Received 16 February 2022; Accepted 28 March 2022; Published 27 April 2022

Academic Editor: Meraj Ali Khan

The differential geometry of plane curves has many applications in physics especially in mechanics. The curvature of a plane curve plays a role in the centripetal acceleration and the centripetal force of a particle traversing a curved path in a plane. In this paper, we introduce the concept of the f-curves associated with a plane curve which are more general than the well-known curves such as involute, evolute, parallel, symmetry set, and midlocus. In fact, we introduce the f-curves associated with a plane curve via its normal and tangent for both the cases, a Frenet curve and a Legendre curve. Moreover, the curvature of an f-curve has been obtained in several approaches.

1. Introduction

The differential geometry of a plane curve is an attractive area of research for geometers and physicists, owing to its applications in several areas such as mechanics, computer graphics, computer vision, and medical imaging. In mechanics, for example, the differential geometry of plane curves is used to study the motion of a particle in a plane. Moreover, the curvature of a curved path is used for computing the centripetal acceleration and the centripetal force of a particle moving along that path (cf. [1]).

In this paper, we define the f-curve associated with a given plane curve, for both the cases, a Frenet curve and a Legendre curve. Note that Legendre curves are more general than Frenet curves. Recently, the geometry of Legendre plane curves has been quite extensively studied, and in particular, their evolutes and involutes have been investigated (cf. [2–5]). An important achievement of this paper is that what we find a neat expression for the curvature of an f-curve.

This paper consists of five main sections. The first section is introductory, giving a general idea about the paper. The second section contains basic concepts of the differential geometry of Frenet plane curves and Legendre plane curves which will be used in the rest of this paper. In the third section, we introduce the concept of the f-curves associated with Frenet and Legendre curves via their normals and we study their curvatures in several cases. In the fourth section, we introduce and study the f-curves associated with a Frenet curve and a Legendre curve in a plane via their tangents. Moreover, we give formulae for the curvature of the f-curve associated with a plane curve in both the cases, a Frenet curve and a Legendre curve. In the fifth section, we give nontrivial examples of the f-curve associated with a regular curve via its normal and we draw these curves using the Maple.

2. Preliminaries

In this section, we are going to review basic concepts of the differential geometry of plane curves. For more detail about plane curves and their properties, we refer the reader to [6, 7]. A smooth plane curve γ is a map $\gamma : I \longrightarrow \mathbb{R}^2$ given by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are smooth functions on I, where I is an open interval of $\mathbb{R}$. If γ is a regular parametrized curve (i.e., $\gamma'(t)$=0 for all $t\in I$), then we define the unit tangent vector by $T_\gamma(t)=\gamma'(t)/\|\gamma'(t)\|$ and the unit normal vector by $N_\gamma(t)=J(T_\gamma(t))$, where J is the counterclockwise rotation by $\pi/2$ and $\|\gamma'(t)\|=\sqrt{\gamma'(t)\cdot\gamma'(t)}$. Now, the Frenet formula is given by

$$\begin{pmatrix} T'_\gamma(t) \\ N'_\gamma(t) \end{pmatrix}=\begin{pmatrix} 0 & \|\gamma'(t)\|\kappa_\gamma(t) \\ -\|\gamma'(t)\|\kappa_\gamma(t) & 0 \end{pmatrix}\begin{pmatrix} T_\gamma(t) \\ N_\gamma(t) \end{pmatrix}, \tag{1}$$

where prime is the derivative with respect to the parameter t and $\kappa_\gamma(t)$ is the curvature of γ which is given by $\kappa_\gamma(t)=(T'_\gamma(t)\cdot N_\gamma(t))/\|\gamma'(t)\|$. The pair $\{T_\gamma(t),N_\gamma(t)\}$ is called the moving frame of the regular curve γ. If γ is parametrized by its arc-length s, then the Frenet formula is given by

$$\begin{pmatrix} T'_\gamma(s) \\ N'_\gamma(s) \end{pmatrix}=\begin{pmatrix} 0 & \kappa_\gamma(s) \\ -\kappa_\gamma(s) & 0 \end{pmatrix}\begin{pmatrix} T_\gamma(s) \\ N_\gamma(s) \end{pmatrix}, \tag{2}$$

where $T_\gamma(s)=\gamma'(s)$ is the unit tangent vector, $N_\gamma(s)=J(T_\gamma(s))$ is the unit normal vector, and $\kappa_\gamma(s)=\pm\|T'_\gamma(s)\|$. Also, the curvature is defined by $\kappa_\gamma(s)=d\chi(s)/ds$, where χ is the angle function between the horizontal lines and the tangent of γ and s is the arc length of γ.

Definition 1. Let $\gamma: I\longrightarrow\mathbb{R}^2$ be a regular parametrized curve and $\tilde{\gamma}(t)=\gamma(t)+\lambda N_\gamma(t)$, where λ is a constant. Then, $\tilde{\gamma}$ and γ are parallel curves.

Definition 2. Let $\gamma: I\longrightarrow\mathbb{R}^2$ be a regular parametrized curve with nonvanishing curvature. Then, the evolute of γ is given by

$$E_\gamma(t)=\gamma(t)+\frac{1}{\kappa_\gamma(t)}N_\gamma(t). \tag{3}$$

It is a well-known result that E_γ is regular at $t_0\in I$ if and only if $\kappa'_\gamma(t_0)\neq 0$.

In the rest of this section, we review some basic concepts of Legendre plane curves, and for more information, we refer the reader to [2–5, 8].

Definition 3. The map (γ,ω): $I\longrightarrow\mathbb{R}^2\times S^1$ is called a Legendre curve if $\gamma'(t)\cdot\omega(t)=0$ for all $t\in I$, where S^1 is the unit circle and $\omega: I\longrightarrow S^1$ is a smooth unit vector field. The map (γ,ω) is a Legendre immersion if it is an immersion.

We call $\gamma: I\longrightarrow\mathbb{R}^2$ a frontal if there exists a smooth function $\omega: I\longrightarrow S^1$ such that (γ,ω) is a Legendre curve, and we call $\gamma: I\longrightarrow\mathbb{R}^2$ a wavefront if there exists a smooth function $\omega: I\longrightarrow S^1$ such that (γ,ω) is a Legendre immersion.

For a Frenet curve γ, if γ has a singular point, then the moving frame is not well-defined. For a Legendre curve, (γ,ω): $I\longrightarrow\mathbb{R}^2\times S^1$, an alternative frame is well-defined at any point. This frame is given by $\{\omega,\mu\}$, where $\mu(t)=J(\omega(t))$. Also, we have the following formula:

$$\begin{pmatrix} \omega'(t) \\ \mu'(t) \end{pmatrix}=\begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix}\begin{pmatrix} \omega(t) \\ \mu(t) \end{pmatrix}, \tag{4}$$

where $\ell(t)=\omega'(t)\cdot\mu(t)$. We call the pair $\{\omega(t),\mu(t)\}$ a moving frame of a Legendre curve γ. In addition, there exists a smooth function $\beta(t)$ such that $\beta(t)=\gamma'(t)\cdot\mu(t)$. The curvature of the Legendre curve is $(\ell(t),\beta(t))$.

Definition 4 (see [4]). Suppose that γ is a frontal with the curvature (ℓ,β). If there exists a unique smooth function $\rho: I\longrightarrow\mathbb{R}$ such that $\beta(t)=\rho(t)\ell(t)$ for all $t\in I$, then the evolute $E_\gamma: I\longrightarrow\mathbb{R}^2$ of γ is given by $E_\gamma(t)=\gamma(t)-\rho(t)\omega(t)$. Moreover, E_γ is a frontal with the curvature $(\ell(t),\rho'(t))$. This means that $(E_\gamma,J(\omega))$: $I\longrightarrow\mathbb{R}^2\times S^1$ is a Legendre curve, where J is the counterclockwise rotation by $\pi/2$ on $\mathbb{R}^2$.

Definition 5 (see [4]). Suppose that γ is a frontal with the curvature (ℓ,β). The involute of frontal γ at $t_0\in I$ is given by $I_\gamma(t_0)=\gamma(t)-\int_{t0}^{t}\beta(u)du$, and $(I_\gamma,J^{-1}(\omega))$: $I\longrightarrow\mathbb{R}^2\times S^1$ is a Legendre curve with curvature $(\ell(t),(\int_{t_0}^{t}\beta(u)du)\ell)$, where J^{-1} is the clockwise rotation by $\pi/2$ on $\mathbb{R}^2$.

3. f-Curve via the Normal Vector Associated with a Plane Curve

3.1. Regular Curve. In this section, we study the f-curve via the normal vector associated with a regular plane curve. Also, the curvature of this curve will be obtained in two different ways.

Definition 6. Let $\gamma: I\longrightarrow\mathbb{R}^2$ be a regular parametrized curve. Then, the f-curve via the normal vector associated with γ is defined by $\alpha(t)=\gamma(t)+f(t)N_\gamma(t)$, where $f: I\longrightarrow\mathbb{R}$ is a smooth function.

In the following lemma, we give the necessary and sufficient condition for the curve α to be a regular curve.

Lemma 7. *Let $\gamma: I\longrightarrow\mathbb{R}^2$ be a unit speed curve. Then, the curve α is regular if and only if $f\kappa_\gamma\neq 1$ or $f'\neq 0$.*

Proof. The proof of this lemma is obvious. □

Remark 8. From Lemma 7, it can be easily obtained that all singular points of the f-curve via the normal vector associated with γ lie on the evolute of γ.

The following theorem provides a useful formula for the curvature of the f-curve via the normal vector associated with γ in the case of its regularity.

Theorem 9. *Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve and α be its associated f-curve via the normal vector.*

(1) If $f\kappa_\gamma \neq 1$, then $\kappa_\alpha = ((\kappa_\gamma \pm \theta')\cos\theta)/(1 - f\kappa_\gamma)$, where θ is the angle between the tangent vector of α and the unit tangent vector of γ

(2) If $f' \neq 0$, then $\kappa_\alpha = ((\kappa_\gamma \mp \phi')\cos\phi)/f'$, where ϕ is the angle between the tangent vector of α and the unit normal vector of γ

Proof. Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve and α be its associated f-curve via the normal vector. Then, we have

$$\alpha' = \left(1 - f\kappa_\gamma\right)T_\gamma + f' N_\gamma. \tag{5}$$

Case 1. If $f\kappa_\gamma$=1, then from Lemma 7, α is a regular curve, and we have

$$\alpha' \cdot T_\gamma = 1 - f\kappa_\gamma, \tag{6}$$

which gives

$$\|\alpha'\|\|T_\gamma\| \cos\theta = 1 - f\kappa_\gamma. \tag{7}$$

So, we have

$$\|\alpha'\| \cos\theta = 1 - f\kappa_\gamma. \tag{8}$$

Equation (8) can be rewritten as

$$\|\alpha'\| = \left(1 - f\kappa_\gamma\right)\sec\theta. \tag{9}$$

From equation (5), we have

$$\|\alpha'\|^2 = \left(1 - f\kappa_\gamma\right)^2 + \left(f'\right)^2. \tag{10}$$

Now, using equation (8) in equation (10), we get

$$\|\alpha'\|^2 = \|\alpha'\|^2 \cos^2\theta + \left(f'\right)^2. \tag{11}$$

So,

$$f' = \pm\|\alpha'\| \sin\theta. \tag{12}$$

Substituting (8) and (12) in (5), we have

$$\alpha' = \left(\|\alpha'\| \cos\theta\right)T_\gamma \pm \left(\|\alpha'\| \sin\theta\right)N_\gamma. \tag{13}$$

The unit tangent vector of α, T_α, can be written as

$$T_\alpha = (\cos\theta)T_\gamma \pm (\sin\theta)N_\gamma, \tag{14}$$

and the unit normal vector of α, N_α, can be written as

$$N_\alpha = (\mp\sin\theta)T_\gamma + (\cos\theta)N_\gamma. \tag{15}$$

Now,

$$T'_\alpha \cdot N_\alpha = \kappa_\gamma \pm \theta'. \tag{16}$$

Hence,

$$\kappa_\alpha = \frac{T'_\alpha \cdot N_\alpha}{\|\alpha'\|} = \frac{\left(\kappa_\gamma \pm \theta'\right)\cos\theta}{1 - f\kappa_\gamma}. \tag{17}$$

Case 2. If f'=0, then from Lemma 7, α is a regular curve, and we have

$$\alpha' \cdot N_\gamma = f', \tag{18}$$

which gives

$$\|\alpha\|\|N_\gamma\| \cos\phi = f'. \tag{19}$$

So, we have

$$\|\alpha'\| \cos\phi = f'. \tag{20}$$

Equation (20) can be rewritten as

$$\|\alpha'\| = f' \sec\phi. \tag{21}$$

Now, using equation (20) in equation (10), we get

$$\|\alpha'\|^2 = \left(1 - f\kappa_\gamma\right)^2 + \|\alpha'\|^2 \cos^2\phi. \tag{22}$$

Now,

$$1 - f\kappa_\gamma = \pm\|\alpha'\| \sin\phi. \tag{23}$$

Substituting (20) and (23) in (5), we get

$$\alpha' = \left(\pm\|\alpha'\| \sin\phi\right)T_\gamma + \left(\|\alpha'\| \cos\phi\right)N_\gamma. \tag{24}$$

So,

$$T_\alpha = (\pm\sin\phi)T_\gamma + (\cos\phi)N_\gamma, \tag{25}$$

$$N_\alpha = -(\cos\phi)T_\gamma \pm (\sin\phi)N_\gamma. \tag{26}$$

Now,

$$T'_\alpha \cdot N_\alpha = \kappa_\gamma \mp \phi'. \tag{27}$$

Hence,

$$\kappa_\alpha = \frac{\left(\kappa_\gamma \mp \phi'\right)\cos\phi}{f'}. \tag{28}$$

□

As an application of Theorem 9, the curvatures of parallel curves and evolute become special cases of this theorem. Precisely, we have the following corollary.

Corollary 10. *Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve with nonvanishing curvature and α be its associated f-curve via the normal vector.*

(1) If f is a constant and $f \neq 1/\kappa_\gamma$, then (γ, α) are parallel curves and $\kappa_\alpha = \kappa_\gamma/|1 - f\kappa_\gamma|$

(2) If $f' \neq 0$ and $f = 1/\kappa_\gamma$, then α is the evolute of γ and $\kappa_\alpha = \mp\kappa_\gamma^3/\kappa_\gamma'$

3.2. Legendre Curve. In this section, we consider the case when γ is a Legendre curve.

Definition 11. Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve. Then, the f-curve via the vector field ω associated with γ is given by $\tilde{\alpha}(t) = \gamma(t) + f(t)\omega(t)$, where $f : I \longrightarrow \mathbb{R}$ is a smooth function.

Lemma 12. *Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve. Then, the f-curve via the vector field ω associated with γ is regular if and only if $f \neq (-\beta)/\ell$ or $f' \neq 0$.*

Proof. Let $\tilde{\alpha} = \gamma + f\omega$ be associated with f-curve via the vector field ω, then we have

$$\tilde{\alpha}' = \gamma' + f'\omega + f\omega'. \tag{29}$$

Since $\gamma' = \beta\mu$ and $\omega' = \ell\mu$, we have

$$\tilde{\alpha}' = f'\omega + (\beta + f\ell)\mu. \tag{30}$$

Now, $\tilde{\alpha}' \neq \vec{0}$ if and only if $\beta + f\ell \neq 0$ or $f' \neq 0$. □

Theorem 13. *Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve and $\tilde{\alpha}$ be its associated f-curve via the vector field ω.*

(1) If $f \neq -\beta/\ell$, then $\kappa_{\tilde{\alpha}} = ((\ell \mp \psi')\cos\psi)/(\beta + \ell f)$, where ψ is the angle between the tangent vector of $\tilde{\alpha}$ and the unit vector field μ

(2) If $f' \neq 0$, then $\kappa_{\tilde{\alpha}} = ((\ell \pm \xi')\cos\xi)/f'$, where ξ is the angle between the tangent vector of $\tilde{\alpha}$ and the unit vector field ω

Proof. Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve and $\tilde{\alpha}$ be its associated f-curve via the vector field ω. Then, we have

$$\tilde{\alpha}' = f'\omega + (\beta + f\ell)\mu. \tag{31}$$

Case 1. If $f = -\beta/\ell$, then from Lemma 12, $\tilde{\alpha}$ is a regular curve, and we have

$$\tilde{\alpha}' \cdot \mu = \beta + f\ell, \tag{32}$$

which gives

$$\|\tilde{\alpha}'\|\|\mu\| \cos\psi = \beta + f\ell. \tag{33}$$

So, we have

$$\|\tilde{\alpha}'\| \cos\psi = \beta + f\ell. \tag{34}$$

Equation (34) can be rewritten as

$$\|\tilde{\alpha}'\| = (\beta + f\ell)\sec\psi. \tag{35}$$

From equation (31), we have

$$\|\tilde{\alpha}'\|^2 = (\beta + f\ell)^2 + \left(f'\right)^2. \tag{36}$$

Now, using equation (34) in (36), we get

$$\|\tilde{\alpha}'\|^2 = \|\tilde{\alpha}'\|^2 \cos^2\psi + \left(f'\right)^2. \tag{37}$$

So,

$$f' = \pm\|\tilde{\alpha}'\| \sin\psi. \tag{38}$$

From (34) and (38), equation (31) becomes

$$\tilde{\alpha}' = \left(\pm\|\tilde{\alpha}'\| \sin\psi\right)\omega + \left(\|\tilde{\alpha}'\| \cos\psi\right)\mu. \tag{39}$$

Now,

$$\begin{aligned} T_{\tilde{\alpha}} &= (\pm\sin\psi)\omega + (\cos\psi)\mu, \\ N_{\tilde{\alpha}} &= (-\cos\psi)\omega \pm (\sin\psi)\mu. \end{aligned} \tag{40}$$

Thus,

$$T_{\tilde{\alpha}}' \cdot N_{\tilde{\alpha}} = \ell \mp \psi'. \tag{41}$$

Hence,

$$\kappa_{\tilde{\alpha}} = \frac{\left(\ell \mp \psi'\right)\cos\psi}{\beta + \ell f}. \tag{42}$$

Case 2. If f'=0, then from Lemma 12, $\tilde{\alpha}$ is a regular curve, and we have

$$\tilde{\alpha}' \cdot \omega = f', \tag{43}$$

which gives

$$\|\tilde{\alpha}'\| \|\omega\| \cos \xi = f'. \tag{44}$$

So, we have

$$\|\tilde{\alpha}'\| \cos \xi = f'. \tag{45}$$

Equation (45) can be rewritten as

$$\|\tilde{\alpha}'\| = \left(f'\right) \sec \xi. \tag{46}$$

Now, using equation (45) in (36), we have

$$\|\tilde{\alpha}'\|^2 = (\beta + \ell f)^2 + \|\tilde{\alpha}'\|^2 \cos^2\xi. \tag{47}$$

So,

$$\beta + \ell f = \pm\|\tilde{\alpha}'\| \sin \xi. \tag{48}$$

Substituting (45) and (48) in (31), we get

$$\tilde{\alpha}' = \left(\|\tilde{\alpha}'\| \cos \xi\right)\omega \pm \left(\|\tilde{\alpha}'\| \sin \xi\right)\mu. \tag{49}$$

Now,

$$\begin{aligned} T_{\tilde{\alpha}} &= (\cos \xi)\omega \pm (\sin \xi)\mu, \\ N_{\tilde{\alpha}} &= \mp(\sin \xi)\omega + (\cos \xi)\mu. \end{aligned} \tag{50}$$

Thus,

$$T'_{\tilde{\alpha}} \cdot N_{\tilde{\alpha}} = \ell \pm \xi'. \tag{51}$$

Hence,

$$\kappa_{\tilde{\alpha}} = \frac{\left(\ell \pm \xi'\right) \cos \xi}{f'}. \tag{52}$$

□

Now, we have the following corollary of Theorem 13.

Corollary 14. *Let γ be a frontal and $\tilde{\alpha}$ be its associated f-curve via the vector field ω.*

(1) If f is a unique smooth function such that $f = -\beta/\ell$ and f'=0, then $\tilde{\alpha}$ is the evolute of γ and it is a regular curve with $\kappa_{\tilde{\alpha}} = \pm\ell^3/(\ell'\beta - \ell\beta')$

(2) If f is a constant and $f \neq -\beta/\ell$, then $\kappa_{\tilde{\alpha}} = \pm\ell/(\beta + f\ell)$

4. f-Curve via the Tangent Vector Associated with a Plane Curve

4.1. Regular Curve

Definition 15. Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a regular parametrized curve. Then, the f-curve via the tangent vector associated with γ is given by $\Omega(t) = \gamma(t) + f(t)T_\gamma(t)$, where $f : I \longrightarrow \mathbb{R}$ is a smooth function.

Now, we state the following lemma.

Lemma 16. *Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve. Then, the curve Ω associated with γ is regular if and only if $f' \neq -1$ or $f\kappa_\gamma \neq 0$.*

Remark 17. From Lemma 16, if Ω has a singularity at t_0, then γ has an inflexion point at t_0 or $\Omega(t_0) = \gamma(t_0)$.

Theorem 18. *Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve and Ω be its associated f-curve via the tangent vector.*

(1) If $f' \neq -1$, then $\kappa_\Omega = ((\kappa_\gamma \pm \zeta') \cos \zeta)/(1 + f')$, where ζ is the angle between the tangent vector of Ω and the unit tangent vector of γ

(2) If $f\kappa_\gamma \neq 0$, then $\kappa_\Omega = ((\kappa_\gamma \mp \Psi') \cos \Psi)/f\kappa_\gamma$, where Ψ is the angle between the tangent vector of Ω and the unit normal vector of γ

Proof. Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve and Ω be its associated f-curve via the tangent vector. Then, we have

$$\Omega' = \left(1 + f'\right) T_\gamma + f\kappa_\gamma N_\gamma. \tag{53}$$

Case 1. If $f' \neq -1$, then from Lemma 16, Ω is a regular curve, and we have

$$\Omega' \cdot T_\gamma = 1 + f', \tag{54}$$

which gives

$$\|\Omega'\| \cos \zeta = 1 + f'. \tag{55}$$

From equation (53), we have

$$\|\Omega'\|^2 = \left(1 + f'\right)^2 + \left(f\kappa_\gamma\right)^2. \tag{56}$$

Substituting equation (55) in equation (56), we obtain that

$$f\kappa_\gamma = \pm\|\Omega'\| \sin \zeta. \tag{57}$$

From equation (55) and equation (57), equation (53) becomes

$$\Omega' = \left(\|\Omega'\| \cos \zeta\right) T_\gamma \pm \left(\|\Omega'\| \sin \zeta\right) N_\gamma. \tag{58}$$

Therefore,

$$\begin{aligned} T_\Omega &= (\cos \zeta) T_\gamma \pm (\sin \zeta) N_\gamma, \\ N_\Omega &= \mp(\sin \zeta) T_\gamma + (\cos \zeta) N_\gamma. \end{aligned} \tag{59}$$

Hence,

$$\kappa_\Omega = \frac{T'_\Omega \cdot N_\Omega}{\|\Omega'\|} = \frac{\left(\kappa_\gamma \pm \zeta'\right) \cos \zeta}{1 + f'}. \tag{60}$$

Case 2. If $f\kappa_\gamma$=0, then from Lemma 16, Ω is a regular curve, and we have

$$\Omega' \cdot N_\gamma = f\kappa_\gamma, \tag{61}$$

which gives

$$\|\Omega'\| \cos \Psi = f\kappa_\gamma. \tag{62}$$

Substituting (62) in (56), we obtain that

$$1 + f' = \pm\|\Omega\| \sin \Psi. \tag{63}$$

From equation (62) and equation (63), equation (53) becomes

$$\Omega' = \left(\pm\|\Omega'\| \sin \Psi\right) T_\gamma + \left(\|\Omega'\| \cos \Psi\right) N_\gamma. \tag{64}$$

Therefore,

$$\begin{aligned} T_\Omega &= (\pm\sin \Psi) T_\gamma + (\cos \Psi) N_\gamma, \\ N_\Omega &= -(\cos \Psi) T_\gamma \pm (\sin \Psi) N_\gamma. \end{aligned} \tag{65}$$

Hence,

$$\kappa_\Omega = \frac{T'_\Omega \cdot N_\Omega}{\|\Omega'\|} = \frac{\left(\kappa_\gamma \mp \Psi'\right) \cos \Psi}{f\kappa_\gamma}. \tag{66}$$

□

The following corollary can be easily obtained from Theorem 18.

Corollary 19. *Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve and Ω be its associated f-curve via the tangent vector. If $f' = -1$ (that is, $f(s) = C - s$ for some constant C) and $f \neq 0$, then Ω is the involute of γ and $\kappa_\Omega = sing(\kappa_\gamma)/|f|$.*

From Theorem 18, it can be easily obtained a simple and neat formula for the curvature of the regular part of the midlocus associated with a regular part of the symmetry set of a plane curve. The symmetry set of a plane curve is the closure of the locus of centers of the bitangent circles, and the associated midlocus is the set of all midpoints of the chords joining the tangency points. For more detail in the symmetry set of a plane curve and the associated midlocus, we refer the reader to [9–12].

In [9], the first author of this paper obtained the curvature of the midlocus associated with the regular part of the symmetry set of a plane curve. This curvature is given by the following formula:

$$\kappa_\Omega = \frac{\cos \Theta\left(\kappa_\gamma + \Theta'\right)}{\sin \varphi\left(\sin \varphi + r\varphi'\right)}, \tag{67}$$

where φ is the angle between the normal of a given curve γ and the tangent of its symmetry set and Θ is the angle between the tangent of the symmetry set and the tangent of midlocus and prime denotes the derivative with respect to the arc length of the symmetry set. This formula contains more factors than the following formula in the next corollary which gives a simple formula of the curvature of the midlocus. In the following corollary, γ is the symmetry set of a given curve, r is the radius function of the bitangent circles, and the associated f-curve via the tangent of γ is the associated midlocus.

Corollary 20. *Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed curve and Ω be its associated f-curve via the tangent vector such that $f = -rr'$*
.

(1) If $f' \neq -1$, then $\kappa_\Omega = ((\kappa_\gamma \pm \zeta') \cos \zeta)/(1 - r'2 - rr'')$

(2) If $f\kappa_\gamma \neq 0$, then $\kappa_\Omega = (-(\kappa_\gamma \mp \Psi') \cos \Psi)/rr'\kappa_\gamma$

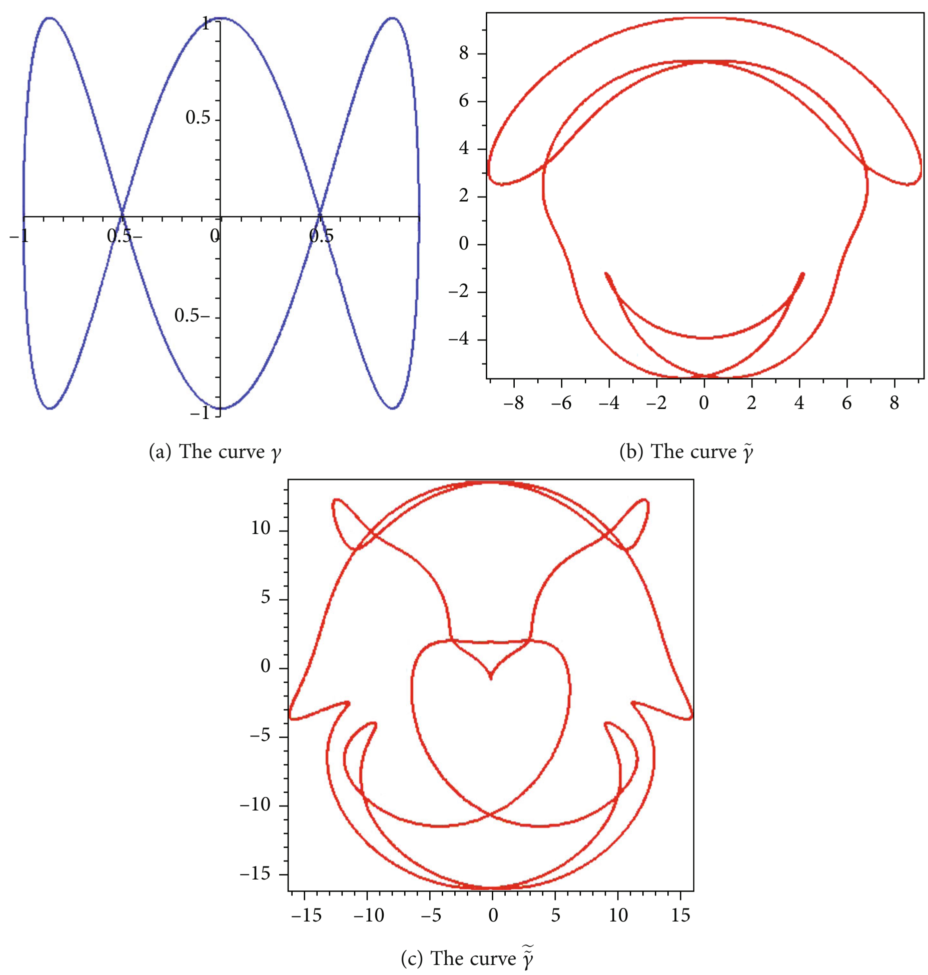
4.2. Legendre Curve

Definition 21. Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve. The f -curve via the vector filed μ associated with γ is given by $\tilde{\Omega}(t) = \gamma(t) + f(t)\mu(t)$, where $f : I \longrightarrow \mathbb{R}$ is a smooth function.

Lemma 22. *Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve. The f-curve via the vector field μ associated with γ is regular if and only if $\beta + f' \neq 0$ or ℓf=0.*

Theorem 23. *Let (γ, ω): $I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve and $\tilde{\Omega}$ be its associated f-curve via the vector field μ.*

(1) If $\beta + f' \neq 0$, then $\kappa_{\tilde{\Omega}} = ((\ell \mp \eta') \cos \eta)/(\beta + f')$, where η is the angle between the tangent vector of $\tilde{\Omega}$ and the unit vector field μ



(a) The curve γ (b) The curve $\tilde{\gamma}$

(c) The curve $\tilde{\tilde{\gamma}}$

Figure 1: The curve γ and its resulting f-curves.

(2) *If $\ell f \neq 0$, then $\kappa_{\tilde{\Omega}} = ((\ell \pm \varepsilon')\cos\varepsilon)/\ell f$, where ε is the angle between the tangent vector of $\tilde{\Omega}$ and the unit vector field ω*

Proof. Let $(\gamma, \omega)\colon I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve and $\tilde{\Omega}$ be its associated f-curve via the vector field μ. Then, we have

$$\tilde{\Omega}' = \left(\beta + f'\right)\mu - (\ell f)\omega. \tag{68}$$

Case 1. If $\beta + f' = 0$, then from Lemma 22, $\tilde{\Omega}$ is a regular curve, and we have

$$\tilde{\Omega}' \cdot \mu = \beta + f', \tag{69}$$

which gives

$$\left\|\tilde{\Omega}'\right\| \cos\eta = \beta + f'. \tag{70}$$

From equation (68), we have

$$\left\|\tilde{\Omega}'\right\|^2 = (\beta + f)^2 + (\ell f)^2. \tag{71}$$

Substituting (70) in (71), it can be shown that

$$\ell f = \pm\left\|\tilde{\Omega}'\right\| \sin\eta. \tag{72}$$

From (70) and (72), equation (68) becomes

$$\tilde{\Omega}' = \left(\left\|\tilde{\Omega}'\right\| \cos\eta\right)\mu \mp \left(\left\|\tilde{\Omega}'\right\| \sin\eta\right)\omega. \tag{73}$$

Now,

$$\begin{aligned} T_{\tilde{\Omega}} &= (\mp\sin\eta)\omega + (\cos\eta)\mu, \\ N_{\tilde{\Omega}} &= (-\cos\eta)\omega \mp (\sin\eta)\mu. \end{aligned} \tag{74}$$

Hence,

$$\kappa_{\tilde{\Omega}} = \frac{\left(\ell \pm \eta'\right)\cos\eta}{\beta + \ell f'}. \quad (75)$$

Case 2. If $\ell f \neq 0$, then from Lemma 22, $\tilde{\Omega}$ is a regular curve, and we have

$$\tilde{\Omega}' \cdot \omega = -\ell f, \quad (76)$$

which gives

$$\left\|\tilde{\Omega}'\right\| \cos\varepsilon = -\ell f. \quad (77)$$

Using equation (77) in (71), it can be shown that

$$\beta + f' = \pm\left\|\tilde{\Omega}'\right\| \sin\varepsilon. \quad (78)$$

From (77) and (78), equation (68) becomes

$$\tilde{\Omega}' = \left(\left\|\tilde{\Omega}'\right\| \cos\varepsilon\right)\omega \pm \left(\left\|\tilde{\Omega}'\right\| \sin\varepsilon\right)\mu. \quad (79)$$

Now,

$$\begin{aligned} T_{\tilde{\Omega}} &= (\cos\varepsilon)\omega \pm (\sin\varepsilon)\mu, \\ N_{\tilde{\Omega}} &= (\mp\sin\varepsilon)\omega + (\cos\varepsilon)\mu. \end{aligned} \quad (80)$$

Hence,

$$\kappa_{\tilde{\Omega}} = \frac{\left(\ell \pm \varepsilon'\right)\cos\varepsilon}{\ell f}. \quad (81)$$

□

Corollary 24. *Let $(\gamma, \omega): I \longrightarrow \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β) and $\tilde{\Omega}$ be its associated f-curve via the vector field μ. If $f = -\int_{t_0}^{t}\beta(u)du$ and $\ell f \neq 0$, then $\tilde{\Omega}$ is the involute of the frontal and it is a regular curve with $\kappa_{\tilde{\Omega}} = \pm 1/\int_{t_0}^{t}\beta(u)du$.*

5. Examples

In this section, we give an example in the case of a regular curve and we repeat the process to the resulting curve but with different function f. We use the Maple for calculation and drawing pictures. First, let us consider the curve $\gamma(t) = (\cos(t), \sin(3t))$, $t \in (0, 2\pi)$. It is clear that γ is a closed curve (see Figure 1(a)), and we choose $f_1(t) = 3\sin(t) - 5e^{\sin(t)}$. The f_1-curve via the normal vector associated with γ is given by $\tilde{\gamma}(t) = \gamma(t) + f_1(t)N_\gamma(t) = (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))$, where

$$\begin{aligned} \tilde{\gamma}_1(t) &= \cos(t) + \frac{15\cos(3t)e^{\sin(t)} - 9\cos(3t)\sin(t)}{\sqrt{144\cos^6(t) - 216\cos^4(t) + 80\cos^2(t) + 1}}, \\ \tilde{\gamma}_2(t) &= \sin(t) + \frac{5\sin(t)e^{\sin(t)} + 3\cos^2(t) - 3}{\sqrt{144\cos^6(t) - 216\cos^4(t) + 80\cos^2(t) + 1}}. \end{aligned} \quad (82)$$

The graph of $\tilde{\gamma}$ is shown in Figure 1(b), and we may call this graph Clown's face.

In what follows, we consider the f-curve via the normal vector associated with $\tilde{\gamma}$ with $f(t) = f_2(t) = -5 - 2e^{\sin(9t)}$; this curve is given by $\tilde{\tilde{\gamma}}(t) = \tilde{\gamma}(t) + f_2(t)N_{\tilde{\gamma}}$. The calculation becomes very tedious, and the forms of $\tilde{\tilde{\gamma}}_1$ and $\tilde{\tilde{\gamma}}_2$ are too long and very complicated. We just use the Maple for drawing the graph of this curve which is shown in Figure 1(c). Again, the resulting graph is interesting as well as the graph of $\tilde{\gamma}$ and we may call this graph Tiger's face.

Remark 25. From this section, one can observe the importance of f-curves associated with a plane curve in the theory of computer graphics. We hope this method for creating curves from a given curve will find its applications to computer graphics and related topics soon.

6. Final Remark

Throughout this paper, we introduce the concept of the f-curves associated with a plane curve in the cases of Frenet and Legendre curves. The curvatures of these new curves have been obtained in several ways. This work has direct applications to the motion of particles in the plane. Also, it is useful for calculating the centripetal acceleration and the centripetal force of a particle traversing a curved path such as f-curve in the plane.

Data Availability

No external data has been used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] C. E. Mungan, "A refresher on curvature for application to centripetal acceleration," *Latin-American Journal of Physics Education*, vol. 4, pp. 27–30, 2010.

[2] T. Fukunaga and M. Takahashi, "Existence and uniqueness for Legendre curves," *Journal of Geometry*, vol. 104, no. 2, pp. 297–307, 2013.

[3] T. Fukunaga and M. Takahashi, "Evolutes of fronts in the Euclidean plane," *Journal of Singularities*, vol. 10, pp. 92–107, 2014.

[4] T. Fukunaga and M. Takahashi, “Evolutes and involutes of frontals in the Euclidean plane,” *Demonstratio Mathematica*, vol. 48, no. 2, pp. 147–166, 2015.

[5] S. Isumiya and N. Takahashi, “Evolutoids and pedaloids of plane curves,” *Note di Matematica*, vol. 39, pp. 13–24, 2019.

[6] A. Pressley, *Elementary Differential Geometry*, 2010, Springer London.

[7] M. Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey, 1976.

[8] T. Fukunaga and M. Takahashi, “Involutes of fronts in the Euclidean plane,” *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, vol. 57, no. 3, pp. 637–653, 2016.

[9] A. Alghanemi, “The curvature of the midlocus in the plane,” *International Journal of Scientific and Innovative Mathematical Research*, vol. 4, pp. 1–6, 2016.

[10] A. Alghanemi, *Geometry of skeletal structures and symmetry sets, [Ph. D. thesis]*, University of Leeds, 2012.

[11] J. W. Bruce, P. J. Giblin, and C. G. Gibson, “Symmetry sets,” *Proceedings of the Royal Society of Edinburgh*, vol. 101, no. 1-2, pp. 163–186, 1985.

[12] P. J. Giblin and S. A. Barassett, “Local symmetry of plane curves,” *The American Mathematical Monthly*, vol. 92, no. 10, pp. 689–707, 1985.