



On Relation between the Joint Essential Spectrum and the Joint Essential Numerical Range of Aluthge Transform

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Author's contribution

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Abstract

Associated with every commuting m -tuples of operators on a complex Hilbert space X is its Aluthge transform. In this paper we show that every commuting m -tuples of operators on a complex Hilbert space X and its Aluthge transform have the same joint essential spectrum. Further, it is shown that the joint essential spectrum of Aluthge transform is contained in the joint essential numerical range of Aluthge transform.

Keywords: Aluthge transform; joint essential spectrum; joint essential numerical range.

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1 Introduction

Denote by $B(X)$ the algebra of (bounded) linear operators acting on complex Hilbert space X with inner product $\langle \cdot, \cdot \rangle$. For each operator $T \in B(X)$, its numerical range denoted by $W(T)$ is a subset of complex plane \mathbb{C} defined by $W(T) = \{ \langle Tx, x \rangle : x \in X, \langle x, x \rangle = 1 \}$. This implies that $W(T)$ is the image of the unit sphere $\{x \in X : \|x\| = 1\}$ of X under the (bounded) quadratic

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form $x \rightarrow \langle Tx, x \rangle$. This concept of numerical range, also known as the classical field of values on a Hilbert space, was introduced in 1918 by Toeplitz [1] for matrices. Since its conception, there has been an extensive research on this concept and its generalization. For instance, it is known that the spectrum of T denoted by $\sigma(T)$ is contained in the closure of numerical range, $\overline{W(T)}$. Here, the spectrum of an operator T is defined as $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$. The spectrum of an $n \times n$ matrix is vital in giving many properties of the matrix. For instance, it is clear that a matrix A is invertible if and only if $0 \notin \sigma(A)$. The spectrum has various components, among them the approximate point spectrum which we define as the complex number $\lambda \in \mathbb{C}$ such that for a sequence $\{x_m\}$ of unit vectors in X we have $\|(T - \lambda I)x_m\| \rightarrow 0$. Dekker [2] extended the notion of numerical range to joint numerical range in 1969. Let $T = (T_1, \dots, T_m) \in B(X)$ be m -tuples of operators on a complex Hilbert space X , the joint numerical range is denoted and defined as $W_m(T) = \{(\langle T_1 x, x \rangle, \dots, \langle T_m x, x \rangle) : x \in X, \langle x, x \rangle = 1\}$.

Also related the study of numerical range is the notion of essential numerical range. The essential numerical range for a single operator was introduced and studied in [3] by Stampfli and Williams in 1968. It is denoted and defined as $W_e(T) = \{\lambda \in \mathbb{C} : \langle Tx_m, x_m \rangle \rightarrow \lambda, x_m \rightarrow 0 \text{ weekly}\}$. It is worth noting that much has been done on the essential numerical range. For instance, it was shown in [4] and [5] that the essential numerical range of an operator $T \in B(X)$ is related to the usual numerical range by the formula $W_e(T) = \bigcap \{W(T + K) : K \in \mathcal{K}(X)\}$, where $\mathcal{K}(X)$ is the ideal of all compact operators in $B(X)$. It is clear that the essential numerical of an operator T is contained in the closure of the numerical range and that $W_e(T)$ contains the essential spectrum of T , $\sigma_e(T)$ [4]. The essential spectrum $\sigma_e(T)$ is defined as $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}$. Recall that an operator $T \in B(X)$ is Fredholm if it has a closed range with infinite dimensional null space and its range of finite co-dimension.

In the sequel, a bounded linear operator $T \in B(X)$ is said to be an isometry if $\|Tx\| = \|x\| \forall x \in X$. We say that T is a partial isometry if it is an isometry on the orthogonal complement of its kernel, that is, for every $x \in \ker(T)^\perp$, $\|Tx\| = \|x\|$ where $\ker(T)$ denotes the kernel of a bounded linear operator T . In this paper, we recall that Aluthge transform \tilde{T} of a bounded linear operator T on a complex Hilbert space X is the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T = U|T|$ is any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. Here, an operator T^* denotes the adjoint of $T \in B(X)$. Recall that the adjoint of $T \in B(X)$ is a linear operator $T^* \in B(X)$ defined by the relation $\langle Tx, y \rangle = \langle x, T^*y \rangle \forall y, x \in X$. The adjoint $T^* \in B(X)$ of an operator should not be construed as the adjoint of a matrix A denoted by $Adj(A)$ which is the transpose of the cofactor matrix. Aluthge transform has been extensively studied due to its usefulness in the study of p -hyponormal and semi-hyponormal operators.

The notion of numerical range $W(T)$ of an operator and its generalisation was later extended to the study of numerical range $W(\tilde{T})$ of Aluthge transform \tilde{T} . For instance, Yuan Wu [6] proved that the closure of the numerical range of \tilde{T} is contained in that of T . Let $*$ -Aluthge transform \tilde{T}^* be defined as $|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$. Both \tilde{T} and \tilde{T}^* are independent of the choice of the partial isometry U in the polar decomposition of T . In 2007, Guoxing, Liu and Li [7] studied the essential numerical range and the maximal numerical range of Aluthge transform. They proved that the essential numerical range of \tilde{T} is contained in that of T and is the same as that of \tilde{T}^* .

Various connections between an operator T and its associated Aluthge transform \tilde{T} were studied by Jung, Ko, and Pearcy, [8] in particular with regard to the invariant subspace problem.

Proposition 1.1. *If $T \in B(X)$ and \tilde{T} has a nontrivial invariant subspace, then so does T .*

See [8] for the proof and more.

Yuan [6] studied the relation between numerical range of T and its associated Aluthge transform \tilde{T} and proved that the containment $W(\tilde{T}) \subseteq \overline{W(T)}$ holds for any operator $T \in B(X)$.

Theorem 1.1. $\overline{W(\tilde{T})} \subseteq \overline{W(T)}$ for any operator $T \in B(X)$.

See [6] for the proof.

In this report, firstly, we show a property of Aluthge transform on the joint essential spectrum, that the joint essential spectrum of T coincides that of \tilde{T} .

Secondly, we show that the joint essential spectrum of \tilde{T} is contained in the joint essential numerical range of \tilde{T} .

2 Joint Essential Spectrum of Aluthge Transform

In this section, we give certain properties of the joint essential spectrum of commuting m -tuples of complex Hilbert space operators. Further, we show that the joint essential spectrum of T coincides with the joint essential spectrum of \tilde{T} . Let $T = (T_1, \dots, T_m) \in B(X)$ be commuting m -tuples of operators on a complex Hilbert space X and $T_j = U_j|T_j|$ for $1 \leq j \leq m$ be any joint polar decomposition of $T = (T_1, \dots, T_m) \in B(X)$ with U_j a joint partial isometry and $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$. The joint Aluthge transform is defined by $\tilde{T} = \left(|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}} \right)$. Dash [9] defined the joint essential spectrum of commuting m -tuples of operators $T = (T_1, \dots, T_m) \in B(X)$ as $\sigma_{em}(T) = \sigma_{em}^l(T) \cup \sigma_{em}^r(T)$. Dash [9] also showed that $\sigma_{em}(T) \subset \sigma_m(T)$ where $\sigma_m(T)$ is the joint spectrum of commuting m -tuples of operators $T = (T_1, \dots, T_m) \in B(X)$. Motivated by this definition, the set $\sigma_{em}(\tilde{T})$ of Aluthge transform is equivalently defined as $\sigma_{em}(\tilde{T}) = \sigma_{em}^l(\tilde{T}) \cup \sigma_{em}^r(\tilde{T})$ where the left (right) joint essential spectrum $\sigma_{em}^l(\tilde{T})$ ($\sigma_{em}^r(\tilde{T})$) of Aluthge transform are defined as

$$\sigma_{em}^l(\tilde{T}) = \left\{ (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m : B_1(\tilde{T}_1 - \lambda_1 I) + \dots + B_m(\tilde{T}_m - \lambda_m I) \text{ is not a Fredholm operator for all operators } B = (B_1, \dots, B_m) \text{ on } X \right\} \text{ and}$$

$$\sigma_{em}^r(\tilde{T}) = \left\{ (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m : (\tilde{T}_1 - \lambda_1 I)B_1 + \dots + (\tilde{T}_m - \lambda_m I)B_m \text{ is not a Fredholm operator for all operators } B = (B_1, \dots, B_m) \text{ on } X \right\}.$$

Remark 2.1. If $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{em}^l((\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1) + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m))$ then $0 \in \sigma_{em}((\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1) + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m))$.

If $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{em}^r((\tilde{T}_1 - \lambda_1)(\tilde{T}_1 - \lambda_1)^* + \dots + (\tilde{T}_m - \lambda_m)(\tilde{T}_m - \lambda_m)^*)$ then $0 \in \sigma_{em}((\tilde{T}_1 - \lambda_1)(\tilde{T}_1 - \lambda_1)^* + \dots + (\tilde{T}_m - \lambda_m)(\tilde{T}_m - \lambda_m)^*)$.

The converse of a) and b) are true.

Theorem 2.1 characterizes the sets $\sigma_{em}^l(\tilde{T})$ and $\sigma_{em}^r(\tilde{T})$.

Theorem 2.1. Let $T = (T_1, \dots, T_m)$ be commuting m -tuples operators on X and let $T = (U_1|T|, \dots, U_m|T|)$ be the associated joint polar decomposition. Then: $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{em}^l(\tilde{T})$ if and only if there exists a sequence $\{x_m\}$ of unit vectors in X with $x_m \rightarrow 0$ weakly such that

$\|(\tilde{T}_1 - \lambda_1)x_m \dots (\tilde{T}_m - \lambda_m)x_m\| \rightarrow 0$ as $m \rightarrow \infty$. $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{e_m}^r(\tilde{T})$ if and only if there exists a sequence $\{x_m\}$ of unit vectors in X with $x_m \rightarrow 0$ weakly such that $\|(\tilde{T}_1^* - \lambda_1^*)x_m \dots (\tilde{T}_m^* - \lambda_m^*)x_m\| \rightarrow 0$ as $m \rightarrow \infty$.

Moreover, the sequence $\{x_m\}$ can be chosen orthonormal.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{e_m}^l(\tilde{T})$. Then $B_1(\tilde{T}_1 - \lambda_1) + \dots + B_m(\tilde{T}_m - \lambda_m) \neq 1$ for all operators $B = (B_1, \dots, B_m)$ on X . Thus, $(\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1) + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m)$ lacks left inverse in X . Therefore, there exists a sequence $\{x_m\}$ of unit vectors with $x_m \rightarrow 0$ weakly such that $(\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1) + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m) \rightarrow 0$ as $m \rightarrow \infty$.

Now,

$$\begin{aligned} \|(\tilde{T}_1 - \lambda_1)x_m\|^2 + \dots + \|(\tilde{T}_m - \lambda_m)x_m\|^2 &= \left\langle (\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1)x_m, x_m \right\rangle + \dots \\ &\quad + \left\langle (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m)x_m, x_m \right\rangle \\ &= \|(\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1)x_m + \dots \\ &\quad + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m)x_m\| \rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Therefore, $\|(\tilde{T}_1 - \lambda_1)x_m \dots (\tilde{T}_m - \lambda_m)x_m\| \rightarrow 0$ as $m \rightarrow \infty$.

On the other hand, let $\{x_m\}$ be a sequence of unit vectors with $x_m \rightarrow 0$ weakly such that

$$\|(\tilde{T}_1 - \lambda_1)x_m \dots (\tilde{T}_m - \lambda_m)x_m\| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Then

$$\|(\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1)x_m + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m)x_m\| \leq \|(\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1)x_m\| + \dots + \|(\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m)x_m\| \rightarrow 0.$$

This implies that

$0 \in \sigma_{e_m}^l((\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1) + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m)) = \sigma_{e_m}((\tilde{T}_1 - \lambda_1)^*(\tilde{T}_1 - \lambda_1) + \dots + (\tilde{T}_m - \lambda_m)^*(\tilde{T}_m - \lambda_m))$ implying that $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{e_m}^l(\tilde{T})$ by Remark 2.1. Proof of part 2.1 follows as that of part 2.1 by taking adjoint. This completes the proof. \square

Throughout the remaining part of this section, let $A = (A_1, \dots, A_m)$ and $B = (B_1, \dots, B_m)$ be two m -tuples of operators on X . We define $AB = (A_1B_1, \dots, A_mB_m)$ and $BA = (B_1A_1, \dots, B_mA_m)$. If $A_iB_jA_k = A_kB_jA_i$ and $B_iA_jB_k = B_kA_jB_i$ for all $i, j, k = 1, \dots, m$ then A and B commute. If there is no danger of confusion, we write A instead of $A = (A_1, \dots, A_m)$ and B instead of $B = (B_1, \dots, B_m)$.

Note that if A and B commute and AB is commuting then BA is also commuting.

Lemma 2.2. *Suppose that $A = (A_1, \dots, A_m)$ and $B = (B_1, \dots, B_m)$ are two m -tuples of operators on X and $A = A^*$. Then AB is invertible if and only if BA is invertible.*

Proof. Let $A = A^*$ and AB be invertible. Then there exists an operator Y such that $ABY = I$. Thus, $(BY)^*A = I$ which implies that A is invertible. Therefore, both A and AB are invertible meaning that B is invertible and so is BA .

Conversely, assume BA is invertible. Then there is an operator Z such that $BAZ = I$. Then $(AZ)^*B = I$ so that B is invertible. Since BA and B are invertible, A is invertible. Thus, AB is invertible. \square

The following proposition by Halmos [10] will be used in the sequel.

Proposition 2.1. *Let $A, B \in B(X)$. Then $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$*

See [11] for the proof.

Theorem 2.3. Let $T = (T_1, \dots, T_m)$ or $\tilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ be left invertible.

Then $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is invertible.

Proof. Assume $T = (T_1, \dots, T_m)$ is left invertible. Then $T_1^*T_1 + \dots + T_m^*T_m$ is left invertible implying that $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is invertible.

Now let $\tilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ be left invertible. Then it is bounded below. This means that there is a constant $k > 0$ such that

$$\tilde{T}x = \left\| |T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}}x \right\|^2 + \dots + \left\| |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}}x \right\|^2 \geq k^2 \|x\|^2.$$

Since (U_1, \dots, U_m) is a joint partial isometry, it follows that

$$\left\| |T|^{\frac{1}{2}}x \right\|^2 + \dots + \left\| |T|^{\frac{1}{2}}x \right\|^2 \geq \frac{k^2}{\| |T|^{\frac{1}{2}} \|} \|x\|^2.$$

Therefore, $|T|^{\frac{1}{2}}$ is bounded below and hence $|T|$ is invertible. □

As a consequence, we state the following remark.

Remark 2.2. If the operator $T = (T_1, \dots, T_m) \in B(X)$ is invertible, then $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is invertible and $U = (U_1, \dots, U_m)$ is unitary.

Theorem 2.4. The operator $\tilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ is invertible if and only if $T = (T_1, \dots, T_m) \in B(X)$ is invertible.

Proof. From definition of \tilde{T} , if $T = (T_1, \dots, T_m) \in B(X)$ is invertible then

$$\tilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$$
 is invertible.

It now remains to show that if $\tilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ is invertible then

$|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is invertible. Assume the contrary that $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is not invertible. Then $|T|^{\frac{1}{2}}$ is not invertible meaning $|T|^{\frac{1}{2}}$ is not bounded below. Therefore, there is a sequence $\{x_n\}$ of unit vectors such that $\| |T|^{\frac{1}{2}}x_n \| \rightarrow 0$.

Since $\tilde{T}x = \left\| |T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}}x \right\|^2 + \dots + \left\| |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}}x \right\|^2$ we have

$$\| \tilde{T}x \| \leq \left\| |T|^{\frac{1}{2}}U_1 \right\| \left\| |T|^{\frac{1}{2}}x \right\| + \dots + \left\| |T|^{\frac{1}{2}}U_m \right\| \left\| |T|^{\frac{1}{2}}x \right\|.$$

This implies that $\| \tilde{T}x \| \rightarrow 0$ so that \tilde{T} is not bounded below and is thus not invertible. This contradicts and thus $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is invertible and so is $\tilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$. Since $|T| = (T_1^*T_1 + \dots + T_m^*T_m)^{\frac{1}{2}}$ is invertible and $T = |T|^{-12}\tilde{T}|T|^{12}$ it implies T is invertible. □

The following theorem shows that T and \tilde{T} have the same joint essential spectrum.

Theorem 2.5. Let $T = (T_1, \dots, T_m) \in B(X)$ be m -tuples arbitrary operator such that $T = (U_1|T|, \dots, U_m|T|)$ with joint partial isometry U_1, \dots, U_m . Then $\sigma_{em}(T) = \sigma_{em}(\tilde{T})$.

Proof. Clearly, $\sigma_{em}(T) \setminus \{0\} = \sigma_{em}(U_1|T|, \dots, U_m|T|) \setminus \{0\}$ and

$$\sigma_{em}(\tilde{T}) \setminus \{0\} = \sigma_{em}(|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}}) \setminus \{0\}$$
 by applying Proposition 2.1.

$$\text{Thus } \sigma_{em}(|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}}) \setminus \{0\} = \sigma_{em}(U_1|T|^{\frac{1}{2}}|T|^{\frac{1}{2}} \dots U_m|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}) \setminus \{0\}.$$

Since $0 \in \sigma_{em}(T)$ if and only if $0 \in \sigma_{em}(\tilde{T})$, by Lemma 2.2 we have that $\sigma_{em}(T) = \sigma_{em}(\tilde{T})$. □

We proceed by studying the finer components of the joint spectra of T and \tilde{T} . We write $\sigma_{\pi_m}(T)$ for the joint approximate point spectrum of $T = (T_1, \dots, T_m) \in B(X)$ and define it as the complex numbers $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ such that for a sequence $\{x_m\}$ of unit vectors in X we have $\| (T_1 - \lambda_1 I)x_m, \dots, (T_m - \lambda_m I)x_m \| \rightarrow 0$. A joint eigenvalue $\sigma_{p_m}(T)$ of $T = (T_1, \dots, T_m) \in B(X)$ is

the complex numbers $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ such that for a nonzero joint eigenvector x there is $(T_1 - \lambda_1)x \dots (T_m - \lambda_m)x = 0$. In general, we have $\sigma_{\pi_m}(T) \subset \sigma_{p_m}(T)$. Note that $\sigma_{\pi_m}(T) = \sigma_{p_m}(T)$ holds for semi-hyponormal operators [12].

Theorem 2.6. For every $T = (U_1|T|, \dots, U_m|T|)$ in $B(X)$ with joint partial isometry U_1, \dots, U_m , $\sigma_{\pi_m}(T) = \sigma_{\pi_m}(\tilde{T})$.

Proof. If $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{\pi_m}(T)$ there is a sequence $\{x_m\}$ of unit vectors in X such that $\|(T_1 - \lambda_1)x_m, \dots, (T_m - \lambda_m)x_m\| \rightarrow 0$. Thus,

$$\|(U_1|T| - \lambda_1)x_m, \dots, (U_m|T| - \lambda_m)x_m\| \rightarrow 0. \tag{2.1}$$

Let $\lambda = (\lambda_1, \dots, \lambda_m) = 0$, then $\| |T|x_m \| \rightarrow 0$. This implies that $\| |T|^{12}x_m \| \rightarrow 0$ and $\| \tilde{T}x_m \| \rightarrow 0$ meaning that $0 \in \sigma_{\pi_m}(\tilde{T})$.

Now let $\lambda = (\lambda_1, \dots, \lambda_m) \neq 0$. Then, from (2.1), $\| |T|^{12}x_m \| \not\rightarrow 0$. Applying $|T|^{12}$ to (2.1) we obtain

$$\| (\tilde{T}|T|^{12} - \lambda_1|T|^{12})x_m, \dots, (\tilde{T}|T|^{12} - \lambda_m|T|^{12})x_m \| \rightarrow 0.$$

Thus $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{\pi_m}(\tilde{T})$ and $\sigma_{\pi_m}(T) \subset \sigma_{\pi_m}(\tilde{T})$. It remains to show that $\sigma_{\pi_m}(T) \supset \sigma_{\pi_m}(\tilde{T})$.

Now let $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{\pi_m}(\tilde{T})$. There is a sequence $\{x_m\}$ of unit vectors in X such that

$$\| (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} - \lambda_1)x_m, \dots, (|T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}} - \lambda_m)x_m \| \rightarrow 0. \tag{2.2}$$

If $\lambda = (\lambda_1, \dots, \lambda_m) = 0$, then either $\| |T|^{12}x_m \| \rightarrow 0$ so that $\| Tx_m \| \rightarrow 0$ or $\| |T|^{12}x_m \| \not\rightarrow 0$ meaning that

$\| U_1|T|^{12}x_m, \dots, U_m|T|^{12}x_m \| \not\rightarrow 0$. But $|T|^{\frac{1}{2}}$ (and therefore T) maps $\| U_1|T|^{12}x_m, \dots, U_m|T|^{12}x_m \| \not\rightarrow 0$ to a null sequence. Thus $0 \in \sigma_{\pi_m}(T)$.

Now let $\lambda = (\lambda_1, \dots, \lambda_m) \neq 0$. Then, from (2.2) $\| |T|^{12}x_m \| \not\rightarrow 0$ meaning that $\| U_1|T|^{12}x_m, \dots, U_m|T|^{12}x_m \| \not\rightarrow 0$. Apply $\| U_1|T|^{12}, \dots, U_m|T|^{12} \|$ to (2.2) to get $\| (TU_1|T|^{\frac{1}{2}} - \lambda_1U_1|T|^{\frac{1}{2}})x_m, \dots, (TU_m|T|^{\frac{1}{2}} - \lambda_mU_m|T|^{\frac{1}{2}})x_m \| \rightarrow 0$.

Thus $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{\pi_m}(T)$ and $\sigma_{\pi_m}(T) \supset \sigma_{\pi_m}(\tilde{T})$. Thus $\sigma_{\pi_m}(T) = \sigma_{\pi_m}(\tilde{T})$. □

Theorem 2.7. Let $T = (T_1, \dots, T_m) \in B(X)$ be m -tuples arbitrary operator with joint polar decomposition $T = (U_1|T|, \dots, U_m|T|)$. Then $\sigma_{p_m}(T) = \sigma_{p_m}(\tilde{T})$.

The proof is omitted since it runs through as that of Theorem 2.6 if we replace $\{x_m\}$ with x .

3 Joint Essential Numerical Range of Aluthge Transform

The notion of the joint essential numerical range of m -tuples of operators $T = (T_1, \dots, T_m) \in B(X)$ has been studied by various authors. It is related to the joint numerical range by the formula $W_{e_m}(T) = \bigcap \{ \overline{W_m(T_1 + K_1, \dots, T_m + K_m)} : K = (K_1, \dots, K_m) \in \mathcal{K}(X) \}$. Clearly,

$W_{e_m}(T) \subset \overline{W_m(T)}$. We denote by $W_{e_m}(T^*) = [W_{e_m}(T)]^*$ the complex conjugate of $W_{e_m}(T)$. Cyprian, Masibayi and Okelo, together studied the convexity of the joint essential numerical ranges in [13]. Later, Cyprian [14] generalised this notion to the study of the joint essential numerical range $W_{e_m}(\tilde{T})$ of Aluthge transform and proved various interesting results. In this section, we examine some of the properties of the set $W_{e_m}(\tilde{T})$ and show that $\sigma_{e_m}(\tilde{T}) \subset W_{e_m}(\tilde{T})$. We begin with the following theorem.

Theorem 3.1. Suppose X is an infinite-dimensional complex Hilbert space and $T = (T_1, \dots, T_m) \in B(X)$. Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ and $k = 1, \dots, m$. Let P be an infinite-dimensional projection such that $P(T_k - \lambda_k I)P \in \mathcal{K}(X)$ then $\lambda \in W_{e_m}(T) = \bigcap \{ \overline{W_m(T + K)} : K = (K_1, \dots, K_m) \in \mathcal{K}(X) \}$.

See [15] for the proof.

Lemma 3.2. Let $T = (T_1, \dots, T_m) \in B(X)$ be m -tuples arbitrary operator such that $T = (U_1|T|, \dots, U_m|T|)$ with joint partial isometry U_1, \dots, U_m . Then $(\widetilde{T + K_1 - \widetilde{T}}, \dots, \widetilde{T + K_m - \widetilde{T}}) \in \mathcal{K}(X)$ for all $K = K_1, \dots, K_m \in \mathcal{K}(X)$.

We omit the proof since it runs as [7, Lemma 1] which was done for a single operator $T \in B(X)$. We only highlight the following which are key to the proof. If $U_1|T|, \dots, U_m|T|$ is the joint polar decomposition of T with U_1, \dots, U_m a joint partial isometry, we write $T = (U_1|T|, \dots, U_m|T|)$. Similarly, $V_1|T + K_1|, \dots, V_m|T + K_m|$ is the joint polar decomposition of $T + K_1, \dots, T + K_m$ with V_1, \dots, V_m a joint partial isometry. In this case we write $T + K_1, \dots, T + K_m = (V_1|T + K_1|, \dots, V_m|T + K_m|)$. It is easy to see that $K_1, \dots, K_m = (V_1|T + K_1|, \dots, V_m|T + K_m|) - T$ or $K_1, \dots, K_m = (V_1|T + K_1|, \dots, V_m|T + K_m|) - (U_1|T|, \dots, U_m|T|) \in \mathcal{K}(X)$. Note also that $\widetilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ and $\widetilde{T + K_1}, \dots, \widetilde{T + K_m} = (|T + K_1|^{\frac{1}{2}}V_1|T + K_1|^{\frac{1}{2}} \dots |T + K_m|^{\frac{1}{2}}V_m|T + K_m|^{\frac{1}{2}})$.

Therefore, $(\widetilde{T + K_1 - \widetilde{T}}, \dots, \widetilde{T + K_m - \widetilde{T}}) = (|T + K_1|^{\frac{1}{2}}V_1|T + K_1|^{\frac{1}{2}} - |T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T + K_m|^{\frac{1}{2}}V_m|T + K_m|^{\frac{1}{2}} - |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ We leave the rest of the proof to the reader.

Theorem 3.3. Let X be an infinite-dimensional complex Hilbert space and $\widetilde{T} = (\widetilde{T}_1, \dots, \widetilde{T}_m) \in B(X)$.

Then $\sigma_{e_m}(\widetilde{T}) \subseteq W_{e_m}(\widetilde{T})$.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{e_m}(\widetilde{T})$. It should be shown that $\lambda = (\lambda_1, \dots, \lambda_m) \in W_{e_m}(\widetilde{T})$.

To do this, since $\sigma_{e_m}(\widetilde{T}) = \sigma_{e_m}^l(\widetilde{T}) \cup \sigma_{e_m}^r(\widetilde{T})$, it is enough to show that both $\sigma_{e_m}^l(\widetilde{T})$ and $\sigma_{e_m}^r(\widetilde{T})$ are contained in $W_{e_m}(\widetilde{T})$. Now suppose $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{e_m}^l(\widetilde{T})$. Then there is a sequence $\{x_m\}$ of unit vectors in X such that $\|(\widetilde{T}_1 - \lambda_1 I)x_m \dots (\widetilde{T}_m - \lambda_m I)x_m\| \rightarrow 0$ as $x_m \rightarrow 0$ weakly.

Now $|\langle (\widetilde{T}_1 - \lambda_1 I)x_m, x_m \rangle \dots \langle (\widetilde{T}_m - \lambda_m I)x_m, x_m \rangle| \leq \|(\widetilde{T}_i - \lambda_i I)x_m\| \rightarrow 0$

Therefore, $(\langle \widetilde{T}_1 x_m, x_m \rangle \dots \langle \widetilde{T}_m x_m, x_m \rangle) \rightarrow \lambda_i \forall i = (1, \dots, m)$. Thus $\lambda = (\lambda_1, \dots, \lambda_m) \in W_{e_m}(\widetilde{T})$.

Likewise, let $\lambda = (\lambda_1, \dots, \lambda_m) \in \sigma_{e_m}^r(\widetilde{T})$ then $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*) \in \sigma_{e_m}^l(\widetilde{T})^*$.

This gives $\lambda = (\lambda_1, \dots, \lambda_m) \in W_{e_m}(\widetilde{T})^* = [W_{e_m}(\widetilde{T})]^*$ (the complex conjugate of $W_{e_m}(\widetilde{T})$) implying that $\lambda = (\lambda_1, \dots, \lambda_m) \in W_{e_m}(\widetilde{T})$ which completes the proof. \square

Theorem 3.4. Let $\widetilde{T} = (|T|^{\frac{1}{2}}U_1|T|^{\frac{1}{2}} \dots |T|^{\frac{1}{2}}U_m|T|^{\frac{1}{2}})$ such that $T = (U_1|T|, \dots, U_m|T|)$ with joint partial isometry U_1, \dots, U_m . Then $W_{e_m}(\widetilde{T}) \subseteq W_{e_m}(T)$.

Proof. Since $W_{e_m}(T) = \bigcap \{ \overline{W_m(T_1 + K_1, \dots, T_m + K_m)} : K = (K_1, \dots, K_m) \in \mathcal{K}(X) \}$, we have

$W_{e_m}(T) = \overline{W_m(T_1 + K_1, \dots, T_m + K_m)}$ for all $K = (K_1, \dots, K_m) \in \mathcal{K}(X)$.

From Theorem 1.1, it is immediate that $\overline{W_m(\widetilde{T})} \subseteq \overline{W_m(T)}$ for any operator $T = (T_1, \dots, T_m) \in B(X)$. Using this together with Lemma 3.2 we get,

$$\begin{aligned}
W_{e_m}(\tilde{T}) &= \overline{W_{e_m}(T_1 + K_1, \dots, T_m + K_m)} \\
&\subseteq \overline{W_m(T_1 + K_1, \dots, T_m + K_m)} \\
&\subseteq \overline{W_m(T_1 + K_1, \dots, T_m + K_m)}.
\end{aligned}$$

This implies that $W_{e_m}(\tilde{T}) \subseteq \bigcap \{ \overline{W_m(T_1 + K_1, \dots, T_m + K_m)} : K = (K_1, \dots, K_m) \in \mathcal{K}(X) \} = W_{e_m}(T)$ which completes the proof. \square

4 Conclusions

Section 1 was a survey of what is known about the notions of numerical range, essential numerical range and essential spectrum of T and its associated \tilde{T} . In section 2, we gave certain properties of the joint essential spectrum of Aluthge transform. Further, we showed that $\sigma_{e_m}(T) = \sigma_{e_m}(\tilde{T})$. Section 3 delved into the notion of the joint essential numerical range of Aluthge transform associated with m -tuples of operator $T = (T_1, \dots, T_m) \in B(X)$ and among other results, showed that $\sigma_{e_m}(\tilde{T}) \subseteq W_{e_m}(\tilde{T})$ and $W_{e_m}(\tilde{T}) \subseteq W_{e_m}(T)$.

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Competing Interests

Author has declared that no competing interests exist

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