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Research Article

The Existence of Least Energy Sign-Changing Solution for Kirchhoff-Type Problem with Potential Vanishing at Infinity

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In this paper, we study the Kirchhoff-type equation: $-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=Q(x)f(u)$, in $\mathbb{R}^3$, where a, $b>0$, $f\in C^1(\mathbb{R}^3,\mathbb{R})$, and $V,Q\in C^1(\mathbb{R}^3,\mathbb{R}^+)$. $V(x)$ and $Q(x)$ are vanishing at infinity. With the aid of the quantitative deformation lemma and constraint variational method, we prove the existence of a sign-changing solution u to the above equation. Moreover, we obtain that the sign-changing solution u has exactly two nodal domains. Our results can be seen as an improvement of the previous literature.

1. Introduction

Consider the Kirchhoff-type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=Q(x)f(u),\quad x\in\mathbb{R}^3, \tag{1}$$

where $a,b>0$, $V,Q:\mathbb{R}^3\to\mathbb{R}^+$, and $f:\mathbb{R}^3\to\mathbb{R}$ are continuous functions. Moreover, V and Q are vanishing at infinity.

Similar to [1], we say $(V,Q)\in\mathcal{Q}$, if $Q(x)$ and $V(x)$ satisfy the following conditions:

(VQ1) $V(x)$, $Q(x)>0$ for any $x\in\mathbb{R}^3$ and $Q\in L^\infty(\mathbb{R}^3)$

(VQ2) If $\{A_n\}_n\subset\mathbb{R}^3$ is a sequence of Borel sets such that their Lebesgue measures $|A_n|\le R$ for all $n\in\mathbb{N}$ and some $R>0$, then

$$\lim_{r\to\infty}\int_{A_n\cap B_r^c(0)}Q(x)=0 \text{ uniformly in } n\in\mathbb{N}. \tag{2}$$

And one of the below conditions holds:

(VQ3) $Q/V\in L^\infty(\mathbb{R}^3)$ or

(VQ4) there exists $p\in(2,6)$ such that

$$\frac{Q(x)}{V(x)^{(6-p)/4}}\to 0 \text{ as } |x|\to+\infty. \tag{3}$$

Problem (1) is related to the following Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^3. \tag{4}$$

Problem (4) is nonlocal due to the existence of $\int_{\mathbb{R}^3}|\nabla u|^2dx$ which causes some mathematical analysis difficulties and makes problem (4) more interesting. The nonlocal operator $\int_{\mathbb{R}^3}|\nabla u|^2dx$ comes from the Dirichlet problem

$$\begin{cases}-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{5}$$

where a, $b>0$, $\Omega=\mathbb{R}^N$ or $\Omega\subset\mathbb{R}^N$ is a bounded domain. Equation (5) is related to the following stationary analogue of the Kirchhoff-type equation:

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x, u), \tag{6}$$

which was introduced by Kirchhoff [2] as generalization of the famous D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x, u) \tag{7}$$

for free vibration of elastic strings.

After Lion [3] investigated problem (5) involving an abstract framework, Kirchhoff-type equations have been extensively researched by many scholars. Hence, numerous interesting works to (5) or similar problems are obtained in the last decades. Please see [4–7] and the references therein.

In particular, many scholars dedicated to searching sign-changing solutions to (4) and similar problems. Indeed, they obtained a lot of interesting results. For example, Li et al. [8] investigated the sign-changing solution to the following problem by using the constraint variational method

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^3. \tag{8}$$

They supposed that $f(x, t)$ satisfies the following conditions:

(f1) $f(x, t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^3$

(f2) $f(x, t) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $|f(x, t)| \le c(|t| + |t|^{q-1})$ for some $q \in [2, 2^*)$, where $c > 0$ and $2^* = 2N/(N-2)$

(f3) $f(x, t)/t^4 \to \infty$ as $t \to \infty$ uniformly in $x \in \mathbb{R}^3$

(f4) $f(x, t)/t^3$ is an increasing function of $t \in \mathbb{R} \setminus \{0\}$.

In [9], by using a similar main method and theorem, Wang, Zhang, and Cheng proved that if V and f satisfy (f1), (f3)-(f4) and

(V1) $\exists a > 0$ such that $V(x) \ge a > 0$ for any $x \in \mathbb{R}^3$

(V2) $\lim_{|x|\to+\infty} V(x) = V_\infty := \sup_{x\in\mathbb{R}^3} V(x) < +\infty$

(V3) there exist $R_0 > 0$ and $\rho : (R_0,\infty) \to (0,\infty)$ is a non-increasing function such that

$$\lim_{r\to\infty} \rho(r)e^{\delta r} = \infty, \tag{9}$$

(f5) $\lim_{t\to\infty} f(t)/t^5 = 0$, then (4) has a sign-changing solution with two nodal domains. It is worth pointing out that they considered the associated "limiting problem"

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V_\infty u = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{10}$$

with the corresponding energy functional

$$\begin{aligned} J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V_\infty u^2\right)dx \\ + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} F(u)dx. \end{aligned} \tag{11}$$

Moreover, they obtained $c_0 \le c + c_\infty$, which is crucial to show that $\mathscr{M}_b \ne \emptyset$. Where

$$\begin{aligned} c_0 &\triangleq \inf_{\mathscr{M}_b} J(u), \\ c_\infty &\triangleq \inf_{\mathscr{M}_b^\infty} J_\infty(u), \\ \mathscr{N}_\infty &\triangleq \left\{u \in H^1(\mathbb{R}^3) \setminus \{0\}: \left\langle J_\infty'(u), u\right\rangle = 0\right\}, \\ \mathscr{M}_b &\triangleq \left\{u \in H^1(\mathbb{R}^3)\{0\}: \left\langle J'(u), u^+\right\rangle \right. \\ &= \left.\left\langle J'(u), u^-\right\rangle = 0\right\}, \end{aligned}$$

$$\begin{aligned} J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)u^2\right)dx \\ + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} F(u)dx. \end{aligned} \tag{12}$$

For more interesting results, see [10–14].

When $a = 1$, $b = 0$, (4) reduces to

$$-\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^3. \tag{13}$$

Recently, (13) and similar problems have also been received far-reaching research. For example, in [15], when $V(x) \equiv 0$, $f(x, u) = f(u)$, Chen considered (13) on a bounded domain $\Omega \subset \mathbb{R}^N$ and obtained the sign-changing solution to (13) under some sufficient hypothesis. Precisely, they assumed that $f(x, t)$ satisfies the following conditions:

(A1) $f \in C(\Omega \times \mathbb{R})$, $F(x, t) \triangleq \int_0^t f(x, s)ds \ge 0$ and $f(x, t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \Omega$

(A2) $f(x, t) = V_\infty(x)t + f_1(x, t)$, $V_\infty \in C(\Omega)$ and $f_1(x, t) = o(|t|)$ as $|t| \to +\infty$ uniformly in $x \in \Omega$

(A3) $t \mapsto f(x, t)/|t|$ is strictly increasing on $(-\infty,0) \cup (0,\infty)$ for every $x \in \Omega$

(A4) $\tilde{F}(x, t) := (1/2)f(x, t)t - F(x, t) \to +\infty$ as $t \to +\infty$ uniformly in $x \in \Omega$

(A5) $\inf_{x\in\Omega} V_\infty(x) > \mu \triangleq \inf_{u\in\Pi} \max\{|\nabla u^+|_2^2, |\nabla u^-|_2^2\}$, where

$$\Pi := \left\{u \in H_0^1(\Omega): u^\pm \ne 0, \int_\Omega |u^\pm|^2 \mathrm{d}x = 1\right\}. \tag{14}$$

Besides, letting λ_1 be the first eigenvalue of $-\Delta$. Then by (A5) and the definition of μ, they concluded $\lambda \le \mu < \infty$. By using variational methods, Liu and Wang [16] obtained the least energy nodal solution to (13) in $\mathbb{R}^N$, when $V(x) = \lambda V(x)$ and $N \ge 3$, where λ is a parameter. They discussed two situations, the most important of which is that a nodal solution to (13) can be found for any $\lambda \in \mathbb{R}^+$ under certain conditions, thanks to assumptions.

(S1) $\Omega \triangleq \text{int}\,(V^{-1}(0)) \neq \emptyset$ and $\exists \Omega_1, \Omega_2 \subset \Omega$ such that

$$\bar{\Omega}_1 \cap \bar{\Omega}_2 = \emptyset, \tag{15}$$

where Ω_1 and Ω_2 are bounded smooth domains.

(S2) $\lim_{|t|\to\infty} f(t)/t = L \in [\mu, +\infty)$,where $\mu = \max\{\mu_1, \mu_2\}$ and $\mu_i = \inf\{\int_{\Omega_i} |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_{\Omega_i} u^2 dx = 1\}$, $i = 1, 2$.

It is worth noting that Liu and Wang [16] used the first eigenfunction of $(-\Delta, H_0^1(\Omega_i))$, $i = 1, 2$, when they showed that the constraint set is nonempty. For researching on problem (4) under conditions similar to [16], see [13]. For more relevant results, see [17–21] and the references therein.

In fact, not only the sign-changing solutions of Kirchhoff-type equation has received widespread attention but also other kinds of solutions. Particularly, many scholars have investigated positive solutions, ground state solutions, multiple solutions, etc., for the Kirchhoff equation. Please see the literature [22–25]. Moreover, in recent years, there are also a lot of works on the following Kirchhoff-Schrö dinger-Poisson systems

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u + \phi u = f(x, u), & x \in \mathbb{R}^N, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^N. \end{cases} \tag{16}$$

Readers can refer to [26–32]. These results are also helpful for us to study (1), since (1) and (16) have a great correlation.

Motivated by above-mentioned results, we will establish the existence of sign-changing solution to (1). Throughout this paper, we will make the following hypothesis about f:

(F1) $\lim_{t\to 0} f(t)/t^3 = 0$ if (VQ3) holds and $\lim_{t\to 0} f(t)/|t|^{p-1} = 0$ for some $p \in (4, 6)$ if (VQ4) holds

(F2) $\lim_{t\to\infty} f(t)/t^6 = 0$

(F3) $\lim_{t\to\infty} F(t)/t^4 = \infty$, where $F(t) = \int_0^t f(s)ds$

(F4) $f(t)/t^3$ is nondecreasing for $|t| > 0$.

Obviously, (F2)-(F3) are weaker than (f2)-(f3). In [9], the authors assume that f satisfies (F1)-(F4), but they have more other assumptions on V than this paper. This also makes the problem challenging and more interesting. Moreover, our hypothesis does not involve eigenvalue problems. Hence, our results can be regarded as the improvement and complementary of [15, 16]. In the paper, we use a direct method to obtain the sign-changing solution.

In this paper, we consider our problem on the following space:

$$E \triangleq \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty\right\} \tag{17}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)u^2\right)dx. \tag{18}$$

Since $(V, Q) \in \mathcal{Q}$, the embedding $E \hookrightarrow L_Q^q(\mathbb{R}^3)$ is compact for some $q \in (2, 6)$ (see Lemma 1), where

$$L_Q^q(\mathbb{R}^3) = \left\{u : u \text{ismeasurableon} \mathbb{R}^3 \text{and} \int_{\mathbb{R}^3} Q(x)|u|^q dx < +\infty\right\} \tag{19}$$

endowed with the norm

$$\|u\|_Q = \left(\int_{\mathbb{R}^3} Q(x)|u|^q dx\right)^{1/q}. \tag{20}$$

It is clear that $(E, \|\bullet\|)$ is a Banach space. Let $|\cdot|_p$ be the usual norm in $L^p(\mathbb{R}^3)$. Hence, the embedding $E \hookrightarrow L^6(\mathbb{R}^3)$ is continuous. Set

$$S_6 = \inf_{u \in E, u \neq 0} \frac{\|u\|^2}{\left(\int_{\mathbb{R}^3} u^6 dx\right)^{1/3}}. \tag{21}$$

Similar to [1], we can prove the following lemmas.

Lemma 1. *Let $(V, Q) \in \mathcal{Q}$. If (VQ3) occurs, then the embedding $E^\circ L_Q^q(\mathbb{R}^3)$ is continuous and compact. If (VQ4) satisfies, then the embedding $E^\circ L_Q^p(\mathbb{R}^3)$ is continuous and compact.*

Lemma 2. *Assume that $(V, Q) \in \mathcal{Q}$. If $\{u_n\} \subset E$ and $u_n \rightharpoonup u$ in E , then*

$$\begin{aligned} \int_{\mathbb{R}^3} Q(x)F(u_n)dx &\to \int_{\mathbb{R}^3} Q(x)F(u)dx, \\ \int_{\mathbb{R}^3} Q(x)f(u_n)u_n dx &\to \int_{\mathbb{R}^3} Q(x)f(u)u dx. \end{aligned} \tag{22}$$

Our main result is as follows.

Theorem 3. *If $(V, Q) \in \mathcal{Q}$ and (F1)-(F4) hold, then problem (4) has at least one least-energy sign-changing solution u , which has precisely two nodal domains.*

Throughout this paper, $C_1, C_2, \cdots$ denote positive constants possibly different in different places.

The rest of this paper is organized as follows: in Section 2, some frameworks are demonstrated. In Section 3, the proof of the main result is given.

2. Preliminaries

It is no doubt that the weak solution for (1) corresponds to the critical point of the energy functional

$$I(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2-\int_{\mathbb{R}^3}Q(x)F(u)dx. \tag{23}$$

Meanwhile, $I\in C^1(E,\mathbb{R})$ and we define

$$\begin{aligned}\left\langle I'(u),v\right\rangle &=\int_{\mathbb{R}^3}(a\nabla u\nabla v+V(x)uv)dx\\&+b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}\nabla u\nabla vdx\\&-\int_{\mathbb{R}^3}Q(x)f(u)vdx.\end{aligned} \tag{24}$$

And if u is a solution to (1) with $u^\pm\neq 0$, then u is called the sign-changing solution of (1), where

$$\begin{aligned}u^+(x)&=\max\{u(x),0\},\\u^-(x)&=\min\{u(x),0\}.\end{aligned} \tag{25}$$

Through a straightforward calculation, we obtain

$$\begin{aligned}I(u)&=I(u^+)+I(u^-)+\frac{b}{2}\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx,\\\left\langle I'(u),u^+\right\rangle&=\left\langle I'(u^+),u^+\right\rangle+b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx,\\\left\langle I'(u),u^-\right\rangle&=\left\langle I'(u^-),u^-\right\rangle+b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx.\end{aligned} \tag{26}$$

The only thing needed to obtain a sigh-changing solution to (1) is to find a minimizer of the energy of I over the following constraint:

$$\mathscr{M}=\left\{u\in H^1(\mathbb{R}^3):\ u^\pm\neq 0,\left\langle I'(u),u^+\right\rangle=\left\langle I'(u),u^-\right\rangle=0\right\}. \tag{27}$$

For proving our result, we will adopt a regular process. First, we need to obtain the existence of a unique pair (s_u, t_u) such that $s_uu^++t_uu^-\in\mathscr{M}$ and (s_u,t_u) is the unique maximum point of $I(su^++tu^-)$. That is to prove the following lemma.

Lemma 4. *Let $(V,Q)\in\mathcal{Q}$. Suppose that (F1)-(F4) hold, if $u\in E$ with $u^\pm\neq 0$, then there exists a unique pair (s_u,t_u) of positive numbers such that $s_uu^++t_uu^-\in\mathscr{M}$ and*

$$I(s_uu^++t_uu^-)=\max_{s,t\geq 0}\{I(su^++tu^-)\}. \tag{28}$$

Proof. Set $u\in E$ with $u^\pm\neq 0$. Letting

$$\begin{aligned}g(s,t)&=\left\langle I'(su^++tu^-),su^+\right\rangle\\&=\int_{\mathbb{R}^3}[a\nabla(su^++tu^-)\nabla(su^+)+V(x)(su^++tu^-)su^+]dx\\&+b\int_{\mathbb{R}^3}\left|\nabla(su^++tu^-)\right|^2dx\int_{\mathbb{R}^3}\nabla(su^++tu^-)\nabla(su^+)dx\\&+b\int_{\mathbb{R}^3}\left|\nabla(su^++tu^-)\right|^2dx\int_{\mathbb{R}^3}\nabla(su^++tu^-)\nabla(su^+)dx\\&-\int_{\mathbb{R}^3}Q(x)f(su^++tu^-)su^+dx=s^2\|u^+\|^2\\&+bs^4\left(\int_{\mathbb{R}^3}|\nabla u^+|^2dx\right)^2+bs^2t^2\int_{\mathbb{R}^3}|\nabla u^+|^2dx\\&\cdot\int_{\mathbb{R}^3}\left|\nabla u^-\right|^2dx-\int_{\mathbb{R}^3}Q(x)f(su^+)su^+dx,\\h(s,t)&=\left\langle I'(su^++tu^-),tu^-\right\rangle\\&=\int_{\mathbb{R}^3}[a\nabla(su^++tu^-)\nabla(tu^-)+V(x)(su^++tu^-)tu^-]dx\\&+b\int_{\mathbb{R}^3}|\nabla(su^++tu^-)|^2dx\int_{\mathbb{R}^3}\nabla(su^++tu^-)\nabla(tu^-)dx\\&-\int_{\mathbb{R}^3}Q(x)f(su^++tu^-)tu^-dx=t^2\|u^-\|^2\\&+bt^4\left(\int_{\mathbb{R}^3}|\nabla u^-|^2dx\right)^2+bs^2t^2\int_{\mathbb{R}^3}|\nabla u^+|^2dx\\&\cdot\int_{\mathbb{R}^3}|\nabla u^-|^2dx-Q(x)\int_{\mathbb{R}^3}f(tu^-)tu^-dx.\end{aligned} \tag{29}$$

First, we will show that there exists $0<r<R$ such that

$$g(r,t)>0,\quad g(R,t)<0,\quad \forall t\in[r,R] \tag{30}$$

$$h(s,r)>0,\quad h(s,R)<0,\quad \forall s\in[r,R]. \tag{31}$$

Case 1. If (VQ2) holds, from (F1)-(F2), for each $\varepsilon>0$, there exists a $C_\varepsilon>0$ such that

$$f(t)t\leq\varepsilon t^2+C_\varepsilon|t|^6,\quad \forall t\in\mathbb{R}. \tag{32}$$

Then by Sobolev imbedding theorem, one has

$$\begin{aligned}h(s,t)&\geq t^2\|u^-\|^2+bt^4\left(\int_{\mathbb{R}^3}|\nabla u^-|^2dx\right)^2+bs^2t^2\\&\cdot\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx-\varepsilon\int_{\mathbb{R}^3}Q(x)|tu^-|^2dx\\&-C_\varepsilon\int_{\mathbb{R}^3}Q(x)|tu^-|^6dx\geq t^2\|u^-\|^2-C_1\varepsilon Q(x)t^2\|u^-\|^2\\&-C_\varepsilon t^6\int_{\mathbb{R}^3}Q(x)|u^-|^6dx\geq\left(1-C_1\varepsilon\|QV^{-1}\|_\infty\right)t^2\|u^-\|^2\\&-C_2S_6^{-3}t^6\|Q(x)\|_\infty\|u\|^6.\end{aligned} \tag{33}$$

Case 2. If (VQ4) holds, refer to the argument in [1], there exists R, $C_p > 0$ such that

$$\int_{B_R^c(0)} Q(x)|u|^p dx \le \varepsilon \int_{B_R^c(0)} \left(V(x)u^2 + u^6\right)dx, \quad \forall u \in E, \varepsilon \in \left(0, C_p\right). \tag{34}$$

Then by (F1)-(F2), we deduce that

$$\begin{aligned} h(s,t) \ge{}& t^2\|u^-\|^2 - C_3 t^p \int_{\mathbb{R}^3} Q(x)|u^-|^p dx \\ &- C_4 t^6 \int_{\mathbb{R}^3} Q(x)\left(u^-\right)^6 dx. \end{aligned} \tag{35}$$

According to (34), (VQ1) and Hölder inequality, we obtain

$$\begin{aligned} h(s,t) \ge{}& t^2\|u^-\|^2 - C_4\|Q\|_\infty t^6 \int_{\mathbb{R}^3} \left(u^-\right)^6 dx - C_3 \varepsilon t^p \\ &\cdot \int_{B_R^c(0)} \left(V(x)\left(u^-\right)^2 + \left(u^-\right)^6\right)dx - C_3 t^p \|Q\|_{6/6-p\left(B_{R(0)}\right)} \\ &\cdot \left(\int_{B_R(0)} \left(u^-\right)^6 dx\right)^{p/6} \ge t^2\|u^-\|^2 - C_4 t^6 \|Q\|_\infty S_6^{-3}\|u^-\|^6 \\ &- C_3 t^p \varepsilon \|u^-\|^2 + C_3 S_6^{-3} t^p \varepsilon \|u^-\|^6 + C_3 S_6^{-p/2}\|Q\|_{(6/(6-p))\left(B_{R(0)}\right)}\|u^-\|^p t^p. \end{aligned} \tag{36}$$

From the above discussion and $p > 2$, we can conclude that there is $t_1 > 0$ enough small such that

$$h(s,t) > 0 \text{ for } t \in \left(0, t_1\right). \tag{37}$$

Due to $u^- \neq 0$, then $\exists K > 0$ such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^3, u^- > K\right\} > 0. \tag{38}$$

Besides, it follows from (F3) that for all $L > 0$, there exists $M > 0$ such that

$$\frac{f(y)}{y^3} \ge L, \text{ for any } y \in (M,\infty). \tag{39}$$

Hence, as t satisfies $tK > M$, we get

$$\begin{aligned} \int_{\mathbb{R}^3} Q(x) f\left(tu^-\right) tu^- dx &\ge \int_{\{u^-(x)>K\}} Q(x)\frac{f\left(tu^-\right)}{\left(tu^-\right)^3}\left(tu^-\right)^4 \\ &\ge Lt^4 \int_{\{u^-(x)>K\}} Q(x)\left(u^-\right)^4 dx. \end{aligned} \tag{40}$$

Assume $s \le t$. We have

$$\begin{aligned} h(s,t) \le{}& t^2\|u^-\|^2 + bs^2t^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx + bt^4 \\ &\cdot \left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2 - Lt^4 \int_{\{u^-(x)>K\}} Q(x)\left(u^-\right)^4 dx \\ \le{}& t^2\|u^-\|^2 + \frac{b}{2}s^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 + \frac{b}{2}t^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2 \\ &+ bt^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2 - Lt^4 \int_{\{u^-(x)>K\}} Q(x)\left(u^-\right)^4 dx \\ \le{}& t^2\|u^-\|^2 + \frac{3b}{2}t^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2 + \frac{b}{2}t^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 \\ &- Lt^4 \int_{\{u^-(x)>K\}} Q(x)\left(u^-\right)^4 dx. \end{aligned} \tag{41}$$

And once again, if $t \le s$, we have

$$g(s,t) \ge \|u^+\|^2 - C_3 t^p \int_{\mathbb{R}^3} Q(x)|u^+|^p dx - C_4 t^6 \int_{\mathbb{R}^3} Q(x)\left(u^+\right)^6 dx, \tag{42}$$

or

$$\begin{aligned} g(s,t) \ge{}& s^2\|u^+\|^2 - C_4 s^6 \|Q\|_\infty S_6^{-3}\|u^+\|^6 - C_3 \\ &\cdot \left(\varepsilon\|u^+\|^2 + S_6^{-3}\varepsilon\|u^+\|^6 + S_6^{-p/2}\|Q\|_{(6/(6-p))\left(B_{R(0)}\right)}\|u^+\|^p\right)s^p. \end{aligned} \tag{43}$$

And

$$\begin{aligned} g(s,t) \le{}& s^2\|u^+\|^2 + \frac{3b}{2}s^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 \\ &+ \frac{b}{2}s^4\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2 \\ &- Ls^4 \int_{\{u^+(x)>K\}} Q(x)\left(u^+\right)^4 dx. \end{aligned} \tag{44}$$

Obviously, we can take L large enough so that

$$\begin{aligned} L\int_{\{u^\pm(x)>K\}} Q(x)\left(u^\pm\right)^4 dx \ge{}& 3b\left(\int_{\mathbb{R}^3} |\nabla u^\pm|^2 dx\right)^2 \\ &+ b\left(\int_{\mathbb{R}^3} |\nabla u^\mp|^2 dx\right)^2. \end{aligned} \tag{45}$$

By (33), (34), and (41)–(45), we can deduce that (30) and (31) hold. Then from the point of view of Maranda's theorem [2], there exists a pair $\left(s_u, t_u\right)$ of positive numbers such that

$$g\left(s_u, t_u\right) = 0, h\left(s_u, t_u\right) = 0, \text{ i.e.,} s_u u^+ + t_u u^- \in \mathcal{M}. \tag{46}$$

Next, we will prove the uniqueness of $\left(s_u, t_u\right)$.

Without loss of generality, the only thing we need to do is to show that s_u which makes $s_u u^+ + t_u u^- \in \mathcal{M}$ is unique.

Assume that there exists another pair (s_{u_1}, t_u) such that $s_{u_1} u^+ + t_u u^- \in \mathcal{M}$. Let $s_{u_1} < s_u$, by the definition of $\mathcal{M}$, we have

$$s_u^2\|u^+\|^2 + bs_u^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 + bs_u^2 t_u^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 dx - \int_{\mathbb{R}^3} Q(x) f(s_u u^+) s_u u^+ dx = 0, \tag{47}$$

$$bs_{u_1}^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 + bs_{u_1}^2 t_u^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx + s_{u_1}^2\|u^+\|^2 - \int_{\mathbb{R}^3} Q(x) f\left(s_{u_1} u^+\right) s_{u_1} u^+ dx = 0. \tag{48}$$

Combining (47) and (48), we have

$$\left(\frac{1}{s_u^2} - \frac{1}{s_{u_1}^2}\right)\left[\|u^+\|^2 + bt_u^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right] = \int_{\mathbb{R}^3} Q(x)\left(\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f\left(s_{u_1} u^+\right)}{\left(s_{u_1} u^+\right)^3}\right)(u^+)^4 dx. \tag{49}$$

Thanks to (F4) and $s_{u_1} < s_u$, the right side of the above equality is positive, which is a contradiction since the left is negative. So s_{u_1} must be equal to s_u, that is, s_u is unique. Similarly, we can show the uniqueness of t_u. In short, the pair (s_u, t_u) is unique.

Next, we prove that $I(s_u u^+ + t_u u^-) = \max_{s,t\geq 0}\{I(su^+ + t u^-)\}$.

Letting $W(s,t) = I(su^+ + tu^-)$. It is obvious that W attains the unique critical value at (s_u, t_u) in $\mathbb{R}^+ \times \mathbb{R}^+$. From (F3), we obtain

$$W(s,t) \to -\infty \text{ uniformly as } |(s,t)| \to \infty, \tag{50}$$

which implies that a maximum point of W cannot be established on the boundary of $\mathbb{R}^+ \times \mathbb{R}^+$. If we may assume that $(\tilde{s}, 0)$ is a maximum of W, for t small enough, we can easily obtain that

$$\begin{aligned} W_t'(\tilde{s}, t) &= (I(\tilde{s}u^+ + tu^-))_t' \\ &= t\|u^-\|^2 + t^3\left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx\right)^2 \\ &\quad + \tilde{s}^2 t \int_{\mathbb{R}^3} \left|\nabla u^+\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u^-\right|^2 dx \\ &\quad - \int_{\mathbb{R}^3} Q(x) f(tu^-) u^- dx > 0. \end{aligned} \tag{51}$$

Then for t small enough, $W(\tilde{s}, t)$ is an increasing function with respect to t. Hence, according to our discussion, the pair $(\tilde{s}, 0)$ is not a maximum point of W in $\mathbb{R}^+ \times \mathbb{R}^+$.

Remark 5. For proving the uniqueness of the pair (s_u, t_u), we can also discuss it in two situations. Case 1. $u \in \mathcal{M}$. We need to show that $s_u = t_u = 1$. The process of proving in Case 1 shall be similar to the proof in Lemma 4. Case 2. $u \notin \mathcal{M}$. By Lemma 4, there exists a pair of positive numbers (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathcal{M}$ when $u \notin \mathcal{M}$. Assume that there exists another pair (s_u', t_u') such that $s_u' u^+ + t_u' u^- \in \mathcal{M}$. Set $v \triangleq s_u u^+ + t_u u^-$ and $v' \triangleq s_u' u^+ + t_u' u^-$. Then, we have

$$\frac{s_u'}{s_u} v^+ + \frac{t_u'}{t_u} v^- = s_u' u^+ + t_u' u^- = v' \in \mathcal{M}, \tag{52}$$

which implies $s_u = s_u'$ and $t_u = t_u'$. Since $v' = v \in \mathcal{M}$ and by Case 1, we consider the minimization problem

$$m \triangleq \inf\{I(u): u \in \mathcal{M}\}. \tag{53}$$

Lemma 6. *Assume that (F1)-(F4) hold and $(V,Q) \in \mathcal{Q}$, then m is achieved.*

Proof. Step 1. $m > 0$.

For each $u \in \mathcal{M}$, we have $\langle I'(u), u\rangle = 0$. From (33) and (36), we derive that

$$\|u\|^2 \leq \varepsilon\|QV^{-1}\|_\infty \|u\|^2 + S_6^{-3} C_\varepsilon \|Q\|_\infty \|u\|^6 \tag{54}$$

or

$$\|u\|^2 \leq C_3\left(\varepsilon\|u\|^2 + S_6^{-3}\varepsilon\|u\|^6 + S_6^{-p/2}\|Q\|_{(6/(6-p))\left(B_{R(0)}\right)}\|u\|^p\right) + C_4\|Q\|_\infty S_6^{-3}\|u\|^6. \tag{55}$$

Choosing $\varepsilon = \min\{1/2C_3, 1/2\|QV^{-1}\|_\infty\}$. It is easy to verify that there exists $\delta > 0$ such that $\|u\|^2 > \delta$. According to (F4), we have

$$G(x,t) \triangleq \int_{\mathbb{R}^3} Q(x)[f(t) - 4F(t)] dx \geq 0, \quad \text{for all } x \in \mathbb{R}^3, t \in \mathbb{R}. \tag{56}$$

Then,

$$I(u) = I(u) - \frac{1}{4}\left\langle I'(u), u\right\rangle \geq \frac{1}{4}\|u\|^2 + \frac{1}{4} G(x,u) > 0, \tag{57}$$

i.e., $m \geq (1/4)\delta > 0$.

Step 2. m is achieved.

Let $\{u_n\} \subset \mathcal{M}$ such that $I(u_n) \to m$. Then, $\{u_n\}$ is bounded in E. Consequently, there exist u such that $u_n \rightharpoonup u$ in E. Due to $\{u_n\} \in \mathcal{M}$, then $\langle I'(u_n), u_n^\pm\rangle = 0$. That is

$$\|u_n^\pm\|^2 + b\left(\int_{\mathbb{R}^3} \left|\nabla u_n^\pm\right|^2 dx\right)^2 - \int_{\mathbb{R}^3} Q(x) f\left(u_n^\pm\right) u_n^\pm dx = 0. \tag{58}$$

Similar to (57), we can find a $\gamma > 0$ such that $\|u_n^\pm\| > \gamma > 0$ for any $n \in \mathbb{N}^+$. Moveover, according to (F1)-(F2), for every

$\rho > 0$, there is c_ρ such that

$$f(t)t \leq \rho t^2 + \rho|t|^6 + C_\rho|t|^q, \quad \forall t \in \mathbb{R}. \tag{59}$$

Therefore, from (58), we conclude

$$\begin{aligned}\gamma &\leq \|u_n^\pm\|^2 \\ &< \int_{\mathbb{R}^3} Q(x) f\left(u_n^\pm\right) u_n^\pm \\ &\leq \rho \int_{\mathbb{R}^3} Q(x)\left|u_n^\pm\right|^2 dx + \rho \int_{\mathbb{R}^3} Q(x)\left|u_n^\pm\right|^6 dx \\ &\quad + C_\rho \int_{\mathbb{R}^3} Q(x)\left|u_n^\pm\right|^q dx.\end{aligned} \tag{60}$$

Based on the boundedness of $\{u_n\}$, there is C_7 such that

$$\gamma \leq \rho C_7 + C_\rho \int_{\mathbb{R}^3} Q(x)\left|u_n^\pm\right|^q dx. \tag{61}$$

Selecting $C_7 = \gamma/2\rho$, by (61), we have

$$\int_{\mathbb{R}^3} Q(x)\left|u_n^\pm\right|^q dx \geq \frac{\gamma}{2C_\rho}. \tag{62}$$

From (62) and the compactness of embedding $E^\circ L^q(\mathbb{R}^3)$ for $q \in (2, 6)$, then we have

$$\int_{\mathbb{R}^3} Q(x)\left|u^\pm\right|^q dx \geq \frac{\gamma}{2C_\rho}, \tag{63}$$

i.e., $u^\pm \neq 0$. According to the weak semicontinuity of the norm, we have

$$\begin{aligned}&\|u^\pm\|^2 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \left|\nabla u^\pm\right|^2 dx \\ &\quad \leq \lim_{n\to\infty} \inf \left(\|u_n^\pm\|^2 + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \left|\nabla u_n^\pm\right|^2 dx\right).\end{aligned} \tag{64}$$

By (58), (64), and Lemma 2, we have

$$\|u^\pm\|^2 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \left|\nabla u^\pm\right|^2 dx \leq \int_{\mathbb{R}^3} Q(x) f\left(u^\pm\right) u^\pm dx, \tag{65}$$

i.e., $\langle I'(u), u^\pm\rangle \leq \liminf_{n\to\infty}\langle I'(u_n), u_n^\pm\rangle = 0$. Then, there exists (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathscr{M}$. Suppose that $s_u \geq t_u > 0$, then

$$\begin{aligned}&s_u^2\|u^+\|^2 + bs_u^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 \\ &\quad + bs_u^4 \int_{\mathbb{R}^3} \left|\nabla u^+\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u^-\right|^2 dx \\ &\geq s_u^2\|u^+\|^2 + bs_u^4\left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx\right)^2 \\ &\quad + bs_u^2 t_u^2 \int_{\mathbb{R}^3} \left|\nabla u^+\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u^-\right|^2 dx \\ &= \int_{\mathbb{R}^3} Q(x) f(s_u u^+) s_u u^+ dx.\end{aligned} \tag{66}$$

From (65) and (66), we have

$$\left(\frac{1}{s_u^2} - 1\right)\|u^+\|^2 \geq \int_{\mathbb{R}^3} Q(x)\left(\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}\right)(u^+)^4 dx. \tag{67}$$

Similar to the discussion in Lemma 4, by (67), we obtain $s_u \leq 1$. Letting $\bar{u} = s_u u^+ + t_u u^-$. Then by $G(x, t) \geq 0$, we obtain

$$\begin{aligned}m &\leq I(\bar{u}) = I(\bar{u}) - \frac{1}{4}\left\langle I'(\bar{u}), \bar{u}\right\rangle \\ &= \frac{1}{4}\left[\|s_u u^+\|^2 + \|t_u u^-\|^2 + \int_{\mathbb{R}^3} Q(x)(f(s_u u^+) s_u u^+ \right. \\ &\quad \left. - 4F(s_u u^+)) dx\right] + \frac{1}{4} \\ &\quad \cdot \int_{\mathbb{R}^3} Q(x)(f(t_u u^-) t_u u^- - 4F(t_u u^-)) dx \\ &\leq \frac{1}{4}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)(f(u)u - 4F(u)) dx \\ &\leq \lim_{n\to\infty} \inf\left(I(u_n) - \frac{1}{4}\left\langle I'(u_n), u_n\right\rangle\right) = m.\end{aligned} \tag{68}$$

(68) implies that $s_u = t_u = 1$. Then, $\bar{u} = u$ and $I(u) = m$. The proof is completed.

Lemma 7. *Assume (F1)-(F4) hold, then u is a critical point of I.*

Proof. Due to $u \in \mathscr{M}$, then $I'(u)u^+ = I'(u)u^- = 0$. By Lemma 4, for any $(s, t) \in (\mathbb{R}^+, \mathbb{R}^+)$ and $(s, t) \neq (1, 1)$, we have

$$I(su^+ + tu^-) < I(u^+ + u^-) = m. \tag{69}$$

Assume $I'(u) \neq 0$, then there exists $\mu > 0$ and $\theta > 0$ such that $\|I'(u)\| \geq \mu$ for every $\|u - v\| \leq 3\theta$ and $u \in E$. Let $D = (1 - \sigma, 1 + \sigma) \times (1 - \sigma, 1 + \sigma)$ and $\varphi(s, t) \triangleq su^+ + tu^-$, $(s, t) \in D$, where $\sigma = \min\{1/2, \theta^2/\|u\|^2\}$. Then, from Lemma 4, we have

$$\bar{m} \triangleq \max_{\partial D} I \circ \varphi < m. \tag{70}$$

For $S_{2\theta} = \{u \in E : \|u - v\| \le 2\theta\}$, Lemma 6 in [2] shows that there is a deformation η such that

(i) $\eta(1, u) = u$ if $u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\theta}$

(ii) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$

(iii) $I(\eta(1, u)) \le I(u)$ for all $u \in E$.

We claim that

$$\max_{(s,t)\in \bar{D}} I(\eta(1, \varphi(s, t))) < m. \tag{71}$$

By Lemma 4 and (iii), we have $I(\varphi(s, t)) \le m < m + \varepsilon$, i.e., $\varphi(s, t) \in I^{m+\varepsilon}$. Moveover, from the definition of $\varphi(s, t)$, it follows that

$$\|\varphi(s, t) - u\|^2 \le (s - 1)^2\|u^+\|^2 + (t - 1)^2\|u^-\|^2 \le \sigma\|u\|^2 \le \theta^2, \tag{72}$$

which implies that $\varphi(s, t) \in S_\theta$ for all $(s, t) \in \bar{D}$. So $\varphi(s, t) \in I^{m+\varepsilon} \cap S_\theta$. Then by (ii), we can obtain that (71) holds.

Next, we will show that $\eta(1, \varphi(D)) \cap \mathcal{M} \ne \emptyset$.

Letting $\kappa(s, t) \triangleq \eta(1, \varphi(s, t))$,

$$\begin{aligned}\Psi_0(s, t) &\triangleq \left(\left\langle I'(su^+ + tu^-), u^+\right\rangle, \left\langle I'(su^+ + tu^-), u^-\right\rangle\right)\\ &\triangleq (\psi_1(s, t), \psi_2(s, t)),\end{aligned}$$

$$\Psi_1(s, t) \triangleq \left(\frac{1}{s}\left\langle I'(\kappa(s, t)), \kappa^+(s, t)\right\rangle, \frac{1}{t}\left\langle I'(\kappa(s, t)), \kappa^-(s, t)\right\rangle\right). \tag{73}$$

By direct calculation, we get

$$\begin{aligned}\left.\frac{\partial\psi_1(s, t)}{\partial s}\right|_{(1,1)} &= \|u^+\|^2 + 3b\left(\int_{\mathbb{R}^3}|\nabla u^+|^2dx\right)^2\\ &\quad + b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2\\ &\quad \cdot dx - \int_{\mathbb{R}^3}Q(x)f'(u^+)(u^+)^2dx,\\ \left.\frac{\partial\psi_2(s, t)}{\partial t}\right|_{(1,1)} &= \|u^-\|^2 + 3b\left(\int_{\mathbb{R}^3}|\nabla u^-|^2dx\right)^2\\ &\quad + b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2\\ &\quad \cdot dx - \int_{\mathbb{R}^3}Q(x)f'(u^-)(u^-)^2dx,\\ \left.\frac{\partial\psi_1(s, t)}{\partial t}\right|_{(1,1)} &= 2b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx = \frac{\partial\psi_2(s, t)}{\partial s}.\end{aligned} \tag{74}$$

From (F4), we have

$$\begin{aligned}&f'(t)t^2 - 3f(t)t > 0, \quad \text{for any } t \in \mathbb{R}.\\ &\left.\frac{\partial\psi_1(s, t)}{\partial s}\right|_{(1,1)} < 2b\left(\int_{\mathbb{R}^3}|\nabla u^+|^2dx\right)^2 - 2\int_{\mathbb{R}^3}Q(x)f(u^+)u^+dx,\\ &\left.\frac{\partial\psi_2(s, t)}{\partial t}\right|_{(1,1)} < 2b\left(\int_{\mathbb{R}^3}|\nabla u^-|^2dx\right)^2 - 2\int_{\mathbb{R}^3}Q(x)f(u^-)u^-dx.\end{aligned} \tag{75}$$

Then,

$$\begin{aligned}\left.\frac{\partial\psi_1(s, t)}{\partial s}\right|_{(1,1)} &\le -2\|u^+\|^2 - 2b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx,\\ \left.\frac{\partial\psi_2(s, t)}{\partial t}\right|_{(1,1)} &\le -2\|u^-\|^2 - 2b\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx.\end{aligned} \tag{76}$$

Let

$$A \triangleq \begin{bmatrix}\dfrac{\partial\psi_1(s, t)}{\partial s} & \dfrac{\partial\psi_2(s, t)}{\partial s}\\ \dfrac{\partial\psi_1(s, t)}{\partial t} & \dfrac{\partial\psi_2(s, t)}{\partial t}\end{bmatrix}, \tag{77}$$

then $\det A > 0$. Therefore, $(1, 1)$ is the unique isolated zero point of Ψ_0. Then by degree theorem, we have that deg $(\Psi_0, D, 0) = 1$. From (70) and (i), we have $\kappa = \varphi$. Hence, deg $(\Psi_1, D, 0) =$ deg $(\Psi_0, D, 0) = 1$. Then, there is $(s_0, t_0) \in D$ such that $\Psi_1(s_0, t_0) = 0$. Hence, $\eta(1, \varphi(s_0, t_0)) = \Psi(s_0, t_0) \in \mathcal{M}$, which is a contradiction with (71).

3. Proof of Theorem 3

Proof of Theorem 3. From Lemmas 6 and 7, there is a $u \in \mathcal{M}$ such that $I(u) = m$ and $I'(u) = 0$. Next, we will prove that u has exactly two nodal domains.

By a contradiction, let $u = u_1 + u_2 + u_3$, $u_i \ne 0$ and $\langle I'(u), u_i\rangle = 0 (i = 1, 2, 3)$, where

$$\begin{aligned}&u_1 \ge 0, \quad u_2 \le 0, \quad \Omega_3 \cap \Omega_4 = \emptyset, \quad u_3|_{\Omega_3\cup\Omega_4} = 0,\\ &\Omega_3 \coloneqq \{x \in \Omega : u_1(x) > 0\}, \quad \Omega_4 \coloneqq \{x \in \Omega : u_2(x) < 0\}.\end{aligned} \tag{78}$$

Letting $v = u_1 + u_2$. Then $v^+ = u_1$, $v^- = u_2$, that is, $v^\pm \ne 0$. By Lemmas 4 and 6, there exists a unique pair $(s_v, t_v) \in (0, 1] \times (0, 1]$ such that $s_v v^+ + t_v v^- \in \mathcal{M}$. Then,

$$I(s_v v^+ + t_v v^-) \ge m. \tag{79}$$

Therefore, we have

$$\begin{aligned}0 &= \frac{1}{4}\left\langle I'(u), u_3\right\rangle \\ &= \frac{1}{4}\|u_3\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_3|^2 dx\right)^2 + \frac{b}{4}\int_{\mathbb{R}^3} |\nabla u_1|^2 dx \\ &\quad \cdot \int_{\mathbb{R}^3} \left|\nabla u_3\right|^2 dx + \frac{b}{4}\int_{\mathbb{R}^3} \left|\nabla u_2\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u_3\right|^2 dx \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3} Q(x) f(u_3) u_3 dx \\ &< I(u_3) + \frac{b}{4}\int_{\mathbb{R}^3} \left|\nabla u_1\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u_3\right|^2 dx \\ &\quad + \frac{b}{4}\int_{\mathbb{R}^3} \left|\nabla u_2\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u_3\right|^2 dx.\end{aligned} \tag{80}$$

On the other hand, we have

$$\begin{aligned}m &\le I(s_v v^+ + t_v v^-) \\ &= I(s_v v^+ + t_v v^-) - \frac{1}{4}\left\langle I'(s_v v^+ + t_v v^-), s_v v^+ + t_v v^-\right\rangle \\ &\le \frac{1}{4}s_v^2\|u_1\|^2 + \frac{1}{4}t_v^2\|u_2\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)(f(s_v u_1)s_v u_1 \\ &\quad - 4F(s_v u_1))dx + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)(f(t_v u_2)t_v u_2 - 4F(t_v u_2))dx \\ &\le \frac{1}{4}\|u_1\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)(f(u_1)u_1 - 4F(u_1))dx + \frac{1}{4}\|u_2\|^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} Q(x)(f(u_2)u_2 - 4F(u_2))dx = I(u_1 + u_2) \\ &\quad - \frac{1}{4}\left\langle I'(u_1 + u_2), u_1 + u_2\right\rangle \\ &\le I(u_1) + I(u_2) + I(u_3) + \frac{b}{2}\int_{\mathbb{R}^3} |\nabla u_1|^2 dx \int_{\mathbb{R}^3} |\nabla u_2|^2 dx \\ &\quad + \frac{b}{2}\int_{\mathbb{R}^3} \left|\nabla u_1\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u_3\right|^2 dx \\ &\quad + \frac{b}{2}\int_{\mathbb{R}^3} \left|\nabla u_2\right|^2 dx \int_{\mathbb{R}^3} \left|\nabla u_3\right|^2 dx = I(u) = m,\end{aligned} \tag{81}$$

which is impossible. Hence, u has two exactly two nodal domains. The proof is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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