Research Article

Christ–Lee Model: (Anti-)chiral Supervariable Approach to BRST Formalism

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We derive the off-shell nilpotent of order two and absolutely anticommuting Becchi-Rouet-Stora-Tyutin (BRST), anti-BRST, and (anti-)co-BRST symmetry transformations for the Christ–Lee (CL) model in one $(0 + 1)$-dimension (1D) of spacetime by exploiting the (anti-)chiral supervariable approach (ACSA) to BRST formalism where the *quantum* symmetry (i.e., (anti-)BRST along with (anti-)co-BRST) invariant quantities play a crucial role. We prove the nilpotency and absolute anticommutativity properties of the (anti-)BRST along with (anti-)co-BRST conserved charges within the scope of ACSA to BRST formalism where we take only *one* Grassmannian variable into account. We also show the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian within the scope of ACSA.

1. Introduction

Gauge theories describe three (i.e., strong, weak, and electromagnetic) out of four fundamental interactions of nature which are characterized by first-class constraints in the context of Dirac's prescription for the classification scheme of constraints [1, 2]. The existence of the first-class constraints, in a given system, is the key signature of a gauge theory. Many interesting theories, in the domain of physics, are expressed by the suitable Lagrangians that are invariant under the gauge symmetry transformations. These symmetries are generated by the first-class constraints in a given gauge theory. For the covariant canonical quantization of the gauge theory, the Becchi-Rouet-Stora-Tyutin (BRST) quantization procedure plays a decisive role where we replace the infinitesimal local gauge parameter by ghost and antighost fields [3–6]. Thus, in this formalism, we have two fermionic-type global BRST (s_b) and anti-BRST (s_{ab}) transformations at the *quantum* level (for a given local gauge symmetry transformation at the classical level). These symmetry transformations are endowed with two important properties: (i) nilpotency of order two (i.e., $s_b^2 = 0, s_{ab}^2 = 0$) and (ii) absolute anticommutativity (i.e., $s_b s_{ab} + s_{ab} s_b = 0$).

The first property signifies that these quantum BRST and anti-BRST symmetry transformations are fermionic in nature whereas the second property shows that both symmetry transformations are linearly independent of each other. Besides the (anti-)BRST symmetry transformations, we have two more fermionic and linearly independent symmetry transformations which are christened as the co-BRST (s_d) and anti-co-BRST (s_{ad}) symmetry transformations. The latter fermionic-type symmetry transformations are valid for any D-dimensional p-form $(p = 1, 2, 3, \cdots)$ gauge theories in $D = 2p$ dimensions of spacetime. We point out that there are some specific systems such as rigid rotor and Christ–Lee (CL) model in one dimension (1D) that respect the (anti-)co-BRST transformations along with the (anti-)BRST transformations [7–10].

The geometrical interpretation and the emergence of nilpotent (anti-)BRST symmetry transformations have been shown within the ambit of Bonora-Tonin (BT) superfield formalism [11–13] where the Grassmannian variables $(\vartheta, \bar{\vartheta})$ and their corresponding derivatives $(\partial_\vartheta, \partial_{\bar{\vartheta}})$ (with properties $\vartheta^2 = \bar{\vartheta}^2 = 0, \partial_\vartheta^2 = \partial_{\bar{\vartheta}}^2 = 0$ and $\vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = 0, \partial_\vartheta \partial_{\bar{\vartheta}} + \partial_{\bar{\vartheta}} \partial_\vartheta = 0$) play a very important role. In BT-superfield approach, we see the connections between the (anti-)BRST symmetry

transformations ($s_{(a)b}$) (with properties $s_b^2 = s_{ab}^2 = 0$ and $s_b s_{ab} + s_{ab} s_b = 0$) and Grassmannian translational generators ($\partial_\vartheta, \partial_{\bar\vartheta}$) because of the fact that both have the same algebraic structure. In this formalism, any D-dimensional Minkowskian manifold is generalized onto the $(D, 2)$-dimensional supermanifold. This suitably *chosen* supermanifold is denoted by the superspace coordinates $(x^\mu, \vartheta, \bar\vartheta)$ where x^μ ($\mu = 0, 1, 2, \cdots, D - 1$) are the spacetime coordinates and ($\vartheta, \bar\vartheta$) are a pair of Grassmannian variables.

The CL model is one of the simplest examples of gauge-invariant system which is described by a singular Lagrangian [14]. Physically, the CL model represents the motion of a point particle moving in a plane under the influence of a central potential. The CL model has been studied at the classical and quantum levels in different perspectives [14–20]. This model is endowed with the first-class constraints in the Dirac's terminology for the classification scheme of constraints [1, 2]. This model is also quantized by using the Faddeev-Jackiw quantization where all the primary and derived constraints are treated on equal footing without any type of further classification [20]. Within the framework of BRST formalism, the CL model respects *six* independent continuous symmetries (i.e., BRST, anti-BRST, co-BRST, anti-co-BRST, ghost-scale, and bosonic symmetries) (see, e.g., [9] for detail). The BT-superfield formalism has been applied to obtain the absolutely anticommuting and off-shell nilpotent (anti-)BRST as well as (anti-)co-BRST symmetry transformations where the techniques of celebrated horizontality condition (HC) and dual horizontality condition have been used [10], respectively.

In our recent set of papers [21–25], we have used a *newly* proposed formalism which has been called by us as the (anti-)chiral superfield/supervariable approach (ACSA) to BRST formalism. In this approach, we take into account only one Grassmannian variable in the expression for the superfield/supervariable. Thus, the resulting superfield/supervariable turns into (anti-)chiral version of the superfield/supervariable. In other words, in this formalism, any D-dimensional Minkowskian manifold is generalized onto $(D, 1)$-dimensional super-submanifolds of the most *general* $(D, 2)$-dimensional supermanifold. The proof of the absolute anticommutativity property of Noether's conserved charges is obvious in the case of BT-superfield formalism where the full super expansions of superfields/supervariables are taken into account. In the case of ACSA, we have also been able to show the nilpotency and absolute anticommutativity properties of conserved charges despite the fact that we have taken only one Grassmannian variable into account. In our present endeavor, we derive the (anti-)BRST together with (anti-)co-BRST symmetry transformations where some specific sets of (anti-)BRST and (anti-)co-BRST invariant restrictions play a very important role. We also show the absolute anticommutativity as well as nilpotency properties of (anti-)BRST and (anti-)co-BRST conserved charges within the realm of ACSA to BRST formalism.

Against the background of the above paragraph, it has been found that the nilpotency of the $\mathcal{N} = 2$ super charges is true for any $\mathcal{N} = 2$ supersymmetric (SUSY) quantum

mechanical models (see, e.g., [26–28]) within the ambit of (anti-)chiral supervariable approach to BRST formalism. However, the application of ACSA, in the realm of $\mathcal{N} = 2$ SUSY quantum mechanical model, does not lead to the absolute anticommutativity of the $\mathcal{N} = 2$ super conserved charges. Rather, it has been found that the anticommutator of the above $\mathcal{N} = 2$ SUSY conserved charges leads to the time translation of the variable on which it acts. Thus, it is crystal clear that ACSA to BRST formalism does not lead to the derivation of absolute anticommutativity of the charges for all $\mathcal{N} = 2$ SUSY theories.

The different sections of our present paper are arranged as follows. In Section 2, we discuss the (anti-)BRST and (anti-)co-BRST symmetry transformations for the CL model and derive the conserved charges. Our Section 3 deals with the ACSA to BRST formalism where we derive the (anti-)BRST symmetry transformations. Section 4 is devoted to the derivation of (anti-)co-BRST symmetry transformations by using the ACSA to BRST formalism where the super expansions of (anti-)chiral supervariables are utilized in a fruitful manner. In Section 5, we express the conserved (anti-)BRST and (anti-)co-BRST charges on the $(1, 1)$-dimensional super-submanifolds (of the most *general* $(1, 2)$-dimensional supermanifold) on which our theory is generalized and provide the proof of nilpotency and absolute anticommutativity properties of the (anti-)BRST along with (anti-)co-BRST charges within the ambit of ACSA to BRST formalism. In Section 6, we discuss the (anti-)BRST and (anti-)co-BRST invariances of the Lagrangian within the scope of ACSA. Finally, we point out our key results and discovery in Section 7 and mention a few future scopes for further investigation.

2. Preliminaries: Symmetries and Their Corresponding Generator for the Christ–Lee Model

The first-order and gauge-invariant Lagrangian of the Christ–Lee (CL) model [the other equivalent second-order Lagrangian (L_s) [14] associated with the Christ–Lee model is $L_s = (1/2)\,\dot{r}^2 + (1/2)\,\dot{r}^2\,(\dot\varphi - z)^2 - V(r)$; but we choose only the first-order Lagrangian L_f in our present work because it respects maximum symmetry transformations] in $(0 + 1)$-dimension (1D) of spacetime in polar coordinates system is given by [14, 16, 19]

$$L_f = \dot{r} p_r + \dot\varphi p_\varphi - \frac{1}{2} p_r^2 - \frac{1}{2r^2} p_\varphi^2 - z p_\varphi - V(r), \qquad (1)$$

where $\dot{r}$ and $\dot\varphi$ are the generalized velocities, p_r and p_φ are their corresponding canonical momenta, respectively, and z is a Lagrange multiplier which enforces a constraint $p_\varphi \approx 0$. This Lagrangian explains that a two-dimensional particle moves under the influence of the central potential $V(r)$ bounded from below.

The above system has a primary constraint as follows:

$$\Phi_1 = \frac{\partial L_f}{\partial \dot{z}} = p_z \approx 0. \tag{2}$$

The time derivative of the primary constraint Φ_1 leads to the following secondary constraint:

$$\frac{d\Phi_1}{d\tau} = \frac{d}{d\tau}\left(\frac{\partial L_f}{\partial \dot{z}}\right) \approx 0 \Rightarrow \Phi_2 = p_\varphi \approx 0. \tag{3}$$

It is clear that both Φ_1 and Φ_2 are first-class constraints. The gauge symmetry transformation generator can be written in terms of first-class constraints as

$$G = \dot{\chi}(\tau)\Phi_1 + \chi(\tau)\Phi_2, \tag{4}$$

where $\chi(\tau)$ is an infinitesimal and time-dependent local gauge parameter and $\dot{\chi}(\tau) = d\chi/d\tau$. Using the definition of a generator

$$\delta\phi(\tau) = -i[\phi(\tau), G], \quad \phi = r, p_r, \varphi, p_\varphi, z, \tag{5}$$

where ϕ is the generic variable that is present in the first-order Lagrangian L_f. We deduce the following local gauge transformations by exploiting Equation (5), namely,

$$\delta z(\tau) = \dot{\chi}(\tau),$$
$$\delta \varphi(\tau) = \chi(\tau), \tag{6}$$
$$\delta\left[r(\tau), p_r(\tau), p_\varphi(\tau), V(r)\right] = 0.$$

It is elementary to check that, under the above local gauge symmetry transformations, the Lagrangian under consideration remains invariant (i.e., $\delta L_f = 0$).

The (anti-)BRST invariant Lagrangian for the $(0 + 1)$-dimensional CL model containing the gauge-fixing and Faddeev–Popov ghost terms is given by [15]

$$L = \dot{r}p_r + \dot{\varphi}p_\varphi - \frac{1}{2}p_r^2 - \frac{1}{2r^2}p_\varphi^2 - zp_\varphi - V(r)$$
$$+ \frac{1}{2}\mathscr{B}^2 + \mathscr{B}(\dot{z} + \varphi) + i\dot{\bar{C}}C - i\dot{\bar{C}}\dot{C}, \tag{7}$$

where the Nakanishi–Lautrup-type *auxiliary* variable $\mathscr{B}$ is used to linearize the gauge-fixing term and the Faddeev–Popov (anti-)ghost variables $(\bar{C})C$ are used to make the Lagrangian BRST invariant. These fermionic variables $(\bar{C})$ C (with $C^2 = \bar{C}^2 = 0$, $C\bar{C} + \bar{C}C = 0$) have ghost numbers (-1) $+1$, respectively. The above Lagrangian respects the following off-shell nilpotent (i.e., $s_{(a)b}^2 = s_{(a)d}^2 = 0$) and absolutely anticommuting (i.e., $s_b s_{ab} + s_{ab} s_b = 0$ and $s_d s_{ad} + s_{ad} s_d = 0$)

(anti-)BRST along with (anti-)co-BRST symmetry transformations:

$$s_{ab}z = \dot{\bar{C}},$$
$$s_{ab}z = \dot{\bar{C}},$$
$$s_{ab}\varphi = \bar{C},$$
$$s_{ab}C = -i\mathscr{B},$$
$$s_{ab}\left[r, p_r, p_\varphi, \mathscr{B}, \bar{C}\right] = 0,$$
$$s_b z = \dot{C}, \tag{8}$$
$$s_b \varphi = C,$$
$$s_b \bar{C} = i\mathscr{B},$$
$$s_b\left[r, p_r, p_\varphi, \mathscr{B}, C\right] = 0,$$

$$s_{ad} z = C,$$
$$s_{ad} \varphi = -\dot{C},$$
$$s_{ad}\bar{C} = -ip_\varphi,$$
$$s_{ad}\left[r, p_r, p_\varphi, \mathscr{B}, C\right] = 0,$$
$$s_d z = \bar{C}, \tag{9}$$
$$s_d \varphi = -\dot{\bar{C}},$$
$$s_d C = ip_\varphi,$$
$$s_d\left[r, p_r, p_\varphi, \mathscr{B}, \bar{C}\right] = 0.$$

It can be clearly checked that under the above (anti-)BRST (Equation (8)) and (anti-)co-BRST (Equation (9)) symmetry transformations the Lagrangian (Equation (7)) remains quasi-invariant (i.e., modulo a total time derivative):

$$s_b L = \frac{d}{d\tau}\left(\mathscr{B}\dot{C}\right),$$
$$s_{ab}L = \frac{d}{d\tau}\left(\mathscr{B}\dot{\bar{C}}\right),$$
$$s_d L = -\frac{d}{d\tau}\left(p_\varphi \dot{\bar{C}}\right), \tag{10}$$
$$s_{ad}L = -\frac{d}{d\tau}\left(p_\varphi \dot{C}\right).$$

As a result, the action integral $S = \int d\tau L$ remains invariant under the (anti-)BRST as well as (anti-)co-BRST symmetry transformations (i.e., $s_{(a)b} S = 0$, $s_{(a)d}S = 0$). According to Noether's theorem, the invariance of the above Lagrangian under the nilpotent (anti-)BRST together with (anti-)co-BRST symmetry transformations leads to the following

(anti-)BRST charges $[\mathcal{Q}_{(a)b}]$ and (anti-)co-BRST charges $[\mathcal{Q}_{(a)d}]$, namely,

$$\mathcal{Q}_{ab} = \mathcal{B}\dot{\bar{C}} + p_\varphi\bar{C} \equiv \mathcal{B}\dot{\bar{C}} - \dot{\mathcal{B}}\bar{C},$$
$$\mathcal{Q}_b = \mathcal{B}\dot{C} + p_\varphi C \equiv \mathcal{B}\dot{C} - \dot{\mathcal{B}}C,$$
(11)

$$\mathcal{Q}_{ad} = \mathcal{B}C - p_\varphi\dot{C} \equiv \mathcal{B}C + \dot{\mathcal{B}}\dot{C},$$
$$\mathcal{Q}_d = \mathcal{B}\bar{C} - p_\varphi\dot{\bar{C}} \equiv \mathcal{B}\bar{C} + \dot{\mathcal{B}}\dot{\bar{C}},$$
(12)

where the equivalent forms of the above charges are written with the help of the equation of motion: $p_\varphi = -\mathcal{B}$. The above charges are nilpotent of order two (i.e., $\mathcal{Q}_{(a)b}^2 = \mathcal{Q}_{(a)d}^2 = 0$) and anticommuting in nature (i.e., $\mathcal{Q}_b\mathcal{Q}_{ab} + \mathcal{Q}_{ab}\mathcal{Q}_b = 0$ and $\mathcal{Q}_d\mathcal{Q}_{ad} + \mathcal{Q}_{ad}\mathcal{Q}_d = 0$). The conservation law for these charges (i.e., $(d/d\tau)\mathcal{Q}_{(a)b} = 0$ and $(d/d\tau)\mathcal{Q}_{(a)d} = 0$) can be easily proven by using the following interesting Euler-Lagrange equations of motion [besides these EOMs, we use an equation $\ddot{\mathcal{B}} + \mathcal{B} = 0$ derived from the EOMs (13) to prove the conservation law for the (anti-)BRST together with (anti-)co-BRST conserved charges (Equations (11) and (12))] (EOMs) derived from Lagrangian L of our theory (Equation (7)), namely,

$$\mathcal{B} + p_\varphi = 0,$$
$$\mathcal{B} = \dot{p}_\varphi,$$
$$\mathcal{B} = -(\dot{z} + \varphi),$$
$$\dot{p}_r - \frac{p_\varphi^2}{r^3} + V'(r) = 0,$$
$$\dot{r} = p_r,$$
(13)
$$\dot{\varphi} - z - \frac{p_\varphi}{r^2} = 0,$$
$$\bar{C} + \ddot{\bar{C}} = 0,$$
$$C + \ddot{C} = 0.$$

The (anti-)co-BRST and (anti-)BRST conserved charges are the generators of the (anti-)co-BRST and (anti-)BRST symmetry transformations, respectively. As one can easily check that the following relationships are true:

$$s_d\xi = -i[\xi, Q_d]_\pm,$$
$$s_{ad}\xi = -i[\xi, \mathcal{Q}_{ad}]_\pm,$$
$$s_b\xi = -i[\xi, Q_b]_\pm,$$
$$s_{ab}\xi = -i[\xi, \mathcal{Q}_{ab}]_\pm,$$
(14)

where ξ denotes any generic variable present in the Lagrangian L of our theory. The subscript $(\pm)$ on the square brackets denotes the (anti)commutator which depends on the nature of generic variables ξ being (fermionic) bosonic in nature.

3. Nilpotent Quantum (Anti-)BRST Symmetry Transformations: (Anti-)chiral Supervariable Approach

In this section, we determine the nilpotent (anti-)BRST symmetry transformations (cf. Equation (8)) by using (anti-)chiral supervariable approach (ACSA) to BRST formalism where we shall use the (anti-)chiral super expansions of supervariables. Towards this goal, first of all, we generalize the ordinary variables of the Lagrangian (7) onto $(1,1)$-dimensional *antichiral* super-submanifold (of the most *common* $(1,2)$-dimensional supermanifold) as follows:

$$z(\tau) \longrightarrow \mathcal{Z}(\tau, \bar{\vartheta}) = z(\tau) + \bar{\vartheta}f_1(\tau),$$
(15)

$$\varphi(\tau) \longrightarrow \Theta(\tau, \bar{\vartheta}) = \varphi(\tau) + \bar{\vartheta}f_2(\tau),$$
(16)

$$C(\tau) \longrightarrow F(\tau, \bar{\vartheta}) = C(\tau) + i\bar{\vartheta}b_1(\tau),$$
(17)

$$\bar{C}(\tau) \longrightarrow \bar{F}(\tau, \bar{\vartheta}) = \bar{C}(\tau) + i\bar{\vartheta}b_2(\tau),$$
(18)

$$r(\tau) \longrightarrow R(\tau, \bar{\vartheta}) = r(\tau) + \bar{\vartheta}f_3(\tau),$$
(19)

$$p_r(\tau) \longrightarrow P_r(\tau, \bar{\vartheta}) = p_r(\tau) + \bar{\vartheta}f_4(\tau),$$
(20)

$$p_\varphi(\tau) \longrightarrow P_\varphi(\tau, \bar{\vartheta}) = p_\varphi(\tau) + \bar{\vartheta}f_5(\tau),$$
(21)

$$\mathcal{B}(\tau) \longrightarrow \tilde{\mathcal{B}}(\tau, \bar{\vartheta}) = \mathcal{B}(\tau) + \bar{\vartheta}f_6(\tau),$$
(22)

where b_1, b_2 are the bosonic derived variables and $f_1, f_2, f_3, f_4, f_5, f_6$ are the fermionic derived variables due to fermionic nature of $\bar{\vartheta}$. We determine the precise value of these derived variables in terms of the *auxiliary* and *basic* variables present in the BRST invariant Lagrangian (7) by using the BRST invariant quantities/restrictions.

According to the basic principles of ACSA, the BRST invariant quantities *must* remain independent of the Grassmannian variable $(\bar{\vartheta})$ when they are generalized onto the $(1,1)$-dimensional *antichiral* super-submanifold. The BRST invariant quantities are the specific combinations of the variables present in Lagrangian (7). These are given as follows:

$$s_b\left(r, p_r, p_\varphi, \mathcal{B}, C\right) = 0,$$
$$s_b\left(z\dot{C}\right) = 0,$$
$$s_b(\varphi C) = 0,$$
$$s_b\left(\mathcal{B}z + i\dot{C}\dot{C}\right) = 0,$$
(23)
$$s_b(\dot{\varphi} - z) = 0,$$
$$s_b\left(\mathcal{B}\varphi + i\bar{C}C\right) = 0.$$

We generalize the above BRST invariant restrictions onto the $(1,1)$-dimensional *antichiral* super-submanifolds (of the suitably chosen most *common* $(1,2)$-dimensional supermanifold)

$$R(\tau, \bar{\vartheta}) = r(\tau), P_r(\tau, \bar{\vartheta}) = p_r(\tau), P_\varphi(\tau, \bar{\vartheta}) = p_\varphi(\tau),$$
$$\tilde{\mathscr{B}}(\tau, \bar{\vartheta}) = \mathscr{B}(\tau), F(\tau, \bar{\vartheta}) = C(\tau), \tag{24}$$

$$\mathscr{Z}(\tau, \bar{\vartheta}) \dot{F}(\tau, \bar{\vartheta}) = z(\tau)\dot{C}(\tau), \Theta(\tau, \bar{\vartheta}) F(\tau, \bar{\vartheta}) = \varphi(\tau)C(\tau), \tag{25}$$

$$\dot{\tilde{\mathscr{B}}}(\tau, \bar{\vartheta}) \mathscr{Z}(\tau, \bar{\vartheta}) + i\dot{F}(\tau, \bar{\vartheta})\dot{C}(\tau, \bar{\vartheta}) = \dot{\mathscr{B}}(\tau)z(\tau) + i\dot{C}(\tau)\dot{C}(\tau), \tag{26}$$

$$\dot{\Theta}(\tau, \bar{\vartheta}) - \mathscr{Z}(\tau, \bar{\vartheta}) = \dot{\varphi}(\tau) - z(\tau), \tag{27}$$

$$\tilde{\mathscr{B}}(\tau, \bar{\vartheta})\Theta(\tau, \bar{\vartheta}) + i\dot{F}(\tau, \bar{\vartheta})F(\tau, \bar{\vartheta}) = \mathscr{B}(\tau)\varphi(\tau) + i\dot{C}(\tau)C(\tau). \tag{28}$$

The above restrictions lead to the derivation of the derived variables in terms of the *basic* and *auxiliary* variables. To determine the value of these variables, we perform the step-by-step explicit calculations. For this purpose, first of all, we use the generalization of the *trivial* BRST invariant restrictions given in the first line of Equation (24) as

$$P_\varphi(\tau, \bar{\vartheta}) = p_\varphi(\tau) \Rightarrow f_5 = 0,$$
$$\tilde{\mathscr{B}}(\tau, \bar{\vartheta}) = \mathscr{B}(\tau) \Rightarrow f_6 = 0,$$
$$R(\tau, \bar{\vartheta}) = r(\tau) \Rightarrow f_3 = 0, \tag{29}$$
$$F(\tau, \bar{\vartheta}) = C(\tau) \Rightarrow b_1 = 0,$$
$$P_r(\tau, \bar{\vartheta}) = p_r(\tau) \Rightarrow f_4 = 0.$$

After substituting the above value of derived variables from (29) to (15), we get the following expressions for the *antichiral* supervariables, namely,

$$C(\tau) \longrightarrow F^{(b)}(\tau, \bar{\vartheta}) = C(\tau) + \bar{\vartheta}(0) \equiv C(\tau) + \bar{\vartheta}[s_b C(\tau)],$$
$$r(\tau) \longrightarrow R^{(b)}(\tau, \bar{\vartheta}) = r(\tau) + \bar{\vartheta}(0) \equiv r(\tau) + \bar{\vartheta}[s_b r(\tau)],$$
$$p_r(\tau) \longrightarrow P_r^{(b)}(\tau, \bar{\vartheta}) = p_r(\tau) + \bar{\vartheta}(0) \equiv p_r(\tau) + \bar{\vartheta}[s_b p_r(\tau)],$$
$$p_\varphi(\tau) \longrightarrow P_\varphi^{(b)}(\tau, \bar{\vartheta}) = p_\varphi(\tau) + \bar{\vartheta}(0) \equiv p_\varphi(\tau) + \bar{\vartheta}\left[s_b p_\varphi(\tau)\right],$$
$$\mathscr{B}(\tau) \longrightarrow \tilde{\mathscr{B}}^{(b)}(\tau, \bar{\vartheta}) = \mathscr{B}(\tau) + \bar{\vartheta}(0) \equiv \mathscr{B}(\tau) + \bar{\vartheta}[s_b \mathscr{B}(\tau)], \tag{30}$$

where the superscript (b) on the *antichiral* supervariables denotes that these supervariables have been obtained after the use of BRST invariant quantities. It is clear that the coefficients of Grassmannian variable $\bar{\vartheta}$ are simply quantum BRST symmetries (8). Now, in the case of *nontrivial* BRST invariant restrictions: $s_b(z\dot{C}) = 0$ and $s_b(\varphi C) = 0$, the following generalizations onto $(1, 1)$-dimensional super-submanifold, namely,

$$\mathscr{Z}(\tau, \bar{\vartheta}) \dot{F}^{(b)}(\tau, \bar{\vartheta}) = z(\tau) \dot{C}(\tau), \Theta(\tau, \bar{\vartheta}) F^{(b)}(\tau, \bar{\vartheta}) = \varphi(\tau) C(\tau), \tag{31}$$

lead to the following interesting results:

$$f_1(\tau) \dot{C}(\tau) = 0 \Rightarrow f_1(\tau) \propto \dot{C}(\tau), \Rightarrow f_1(\tau) = \kappa_1 \dot{C}(\tau),$$
$$f_2(\tau)C(\tau) = 0 \Rightarrow f_2(\tau) \propto C(\tau), \Rightarrow f_2(\tau) = \kappa_2 C(\tau), \tag{32}$$

where κ_1 and κ_2 are the proportionality constants. To determine the value of these constants, we use the generalization of BRST invariant restriction $s_b(\dot{\varphi} - z) = 0$ as

$$\dot{\Theta}(\tau, \bar{\vartheta}) - \mathscr{Z}(\tau, \bar{\vartheta}) = \dot{\varphi}(\tau) - z(\tau) \Rightarrow \kappa_1 = \kappa_2. \tag{33}$$

Finally, to determine the value of constants, we generalize the BRST invariant restrictions $s_b(\dot{\mathscr{B}}z + i\dot{C}\dot{C}) = 0$ and $s_b(\dot{\mathscr{B}}\varphi + i\,\bar{C}C) = 0$ onto $(1, 1)$-dimensional super-submanifold as

$$\dot{\tilde{\mathscr{B}}}^{(b)}(\tau, \bar{\vartheta}) \mathscr{Z}(\tau, \bar{\vartheta}) + i\dot{F}(\tau, \bar{\vartheta}) \dot{F}^{(b)}(\tau, \bar{\vartheta})$$
$$= \dot{\mathscr{B}}(\tau) z(\tau) + i\dot{C}(\tau)\dot{C}(\tau) \Rightarrow b_2(\tau) = \kappa_1 \dot{\mathscr{B}}(\tau), \tag{34}$$

$$\tilde{\mathscr{B}}^{(b)}(\tau, \bar{\vartheta})\Theta(\tau, \bar{\vartheta}) + i\bar{F}(\tau, \bar{\vartheta}) F^{(b)}(\tau, \bar{\vartheta})$$
$$= \mathscr{B}(\tau)\varphi(\tau) + i\bar{C}(\tau)C(\tau) \Rightarrow b_2(\tau) = \kappa_2 \mathscr{B}(\tau). \tag{35}$$

Using the results obtained in Equations (33) and (34), it is clear that $\kappa_1 = \kappa_2 = 1$. Therefore, we obtain the value of derived variables as $f_1(\tau) = \dot{C}(\tau), f_2(\tau) = C(\tau), b_2(\tau) = \mathscr{B}(\tau)$; thus, we get the following expansions for antichiral supervariables:

$$z(\tau) \longrightarrow \mathscr{Z}^{(b)}(\tau, \bar{\vartheta}) = z(\tau) + \bar{\vartheta}\left[\dot{C}(\tau)\right] \equiv z(\tau) + \bar{\vartheta}[s_b z(\tau)], \tag{36}$$

$$\varphi(\tau) \longrightarrow \Theta^{(b)}(\tau, \bar{\vartheta}) = \varphi(\tau) + \bar{\vartheta}[C(\tau)] \equiv \varphi(\tau) + \bar{\vartheta}[s_b \varphi(\tau)], \tag{37}$$

$$\bar{C}(\tau) \longrightarrow \bar{F}^{(b)}(\tau, \bar{\vartheta}) = \bar{C}(\tau) + \bar{\vartheta}[i\mathscr{B}(\tau)] \equiv \bar{C}(\tau) + \bar{\vartheta}[s_b \bar{C}(\tau)]. \tag{38}$$

Thus, in view of Equation (36), we have a connection between the BRST symmetry transformation (s_b) and partial derivative $(\partial_{\bar{\vartheta}})$ on the *antichiral* super-submanifold defined by the mapping: $s_b \leftrightarrow \partial_{\bar{\vartheta}}$ (see, e.g., [11–13] for details). To be more clear, the BRST transformation of any generic variable $\psi(\tau)$ is equal to the translation of the corresponding *antichiral* supervariable $\Psi^{(b)}(\tau, \bar{\vartheta})$ along the $\bar{\vartheta}$-direction. Mathematically, it can be represented as $s_b \psi(\tau) = (\partial/\partial\bar{\vartheta})\Psi^{(b)}(\tau, \bar{\vartheta}) = \partial_{\bar{\vartheta}} \Psi^{(b)}(\tau, \bar{\vartheta})$. In other words, one can say that the coefficient of $\bar{\vartheta}$ in the expansion of an *antichiral* supervariable is simply the quantum BRST symmetry transformation of the corresponding variable.

We are now in the stage to derive the quantum anti-BRST symmetry transformations using *chiral* supervariable approach. In this context, we use the *chiral* super expansions of the *chiral* supervariables where we generalize $(0 + 1)$-dimensional variables onto the $(1, 1)$-dimensional super-submanifold of the suitably chosen most *common* $(1, 2)$

-dimensional supermanifold. The *chiral* super expansions of the ordinary variables are as follows:

$$z(\tau) \longrightarrow \mathscr{Z}(\tau, \vartheta) = z(\tau) + \vartheta \bar{f}_1(\tau), \tag{39}$$

$$\varphi(\tau) \longrightarrow \Theta(\tau, \vartheta) = \varphi(\tau) + \vartheta \bar{f}_2(\tau), \tag{40}$$

$$C(\tau) \longrightarrow F(\tau, \vartheta) = C(\tau) + i\,\vartheta \bar{b}_1(\tau), \tag{41}$$

$$\bar{C}(\tau) \longrightarrow \bar{F}(\tau, \vartheta) = \bar{C}(\tau) + i\,\vartheta \bar{b}_2(\tau), \tag{42}$$

$$r(\tau) \longrightarrow R(\tau, \vartheta) = r(\tau) + \vartheta \bar{f}_3(\tau), \tag{43}$$

$$p_r(\tau) \longrightarrow P_r(\tau, \vartheta) = p_r(\tau) + \vartheta \bar{f}_4(\tau), \tag{44}$$

$$p_\varphi(\tau) \longrightarrow P_\varphi(\tau, \vartheta) = p_\varphi(\tau) + \vartheta \bar{f}_5(\tau), \tag{45}$$

$$\mathscr{B}(\tau) \longrightarrow \tilde{\mathscr{B}}(\tau, \vartheta) = \mathscr{B}(\tau) + \vartheta \bar{f}_6(\tau), \tag{46}$$

where derived variables $\bar{b}_1, \bar{b}_2$ are the bosonic and derived variables $\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5, \bar{f}_6$ are fermionic in nature. The anti-BRST invariant restrictions also *must* remain independent of the Grassmannian variable (ϑ) when they are generalized onto the $(1, 1)$-dimensional *chiral* super-submanifold. The anti-BRST invariant restrictions are given as

$$s_{ab}\left(r, p_r, p_\varphi, \mathscr{B}, \bar{C}\right) = 0,$$
$$s_{ab}\left(z\dot{\bar{C}}\right) = 0,$$
$$s_{ab}\left(\varphi\bar{C}\right) = 0,$$
$$s_{ab}\left(\dot{\mathscr{B}}z + i\dot{C}\dot{C}\right) = 0, \tag{47}$$
$$s_{ab}\left(\dot{\varphi} - z\right) = 0,$$
$$s_{ab}\left(\mathscr{B}\varphi + i\bar{C}C\right) = 0.$$

As the physical quantities remain independent of the Grassmannian variable ϑ which imply that the anti-BRST invariant restrictions can be generalized onto the $(1, 1)$-dimensional super-submanifold of the most *common* $(1, 2)$-dimensional supermanifold as follows:

$$R(\tau, \vartheta) = r(\tau), P_r(\tau, \vartheta) = p_r(\tau), P_\varphi(\tau, \vartheta) = p_\varphi(\tau),$$
$$\tilde{\mathscr{B}}(\tau, \vartheta) = \mathscr{B}(\tau), \bar{F}(\tau, \vartheta) = \bar{C}(\tau),$$

$$\mathscr{Z}(\tau, \vartheta)\dot{\bar{F}}(\tau, \vartheta) = z(\tau)\dot{\bar{C}}(\tau), \Theta(\tau, \vartheta)\bar{F}(\tau, \vartheta) = \varphi(\tau)\bar{C}(\tau),$$

$$\dot{\tilde{\mathscr{B}}}(\tau, \vartheta)Z(\tau, \vartheta) + i\,\dot{\bar{F}}(\tau, \vartheta)\dot{F}(\tau, \vartheta) = \dot{\mathscr{B}}(\tau)\,z(\tau) + i\dot{\bar{C}}(\tau)\dot{C}(\tau),$$

$$\dot{\Theta}(\tau, \vartheta) - \mathscr{Z}(\tau, \vartheta) = \dot{\varphi}(\tau) - z(\tau),$$

$$\tilde{\mathscr{B}}(\tau, \vartheta)\Theta(\tau, \vartheta) + i\bar{F}(\tau, \vartheta)F(\tau, \vartheta) = \mathscr{B}(\tau)\varphi(\tau) + i\bar{C}(\tau)C(\tau). \tag{48}$$

The above generalizations of the anti-BRST invariant restrictions (Equation (47)) lead to the derivation of the *chiral* derived variables in terms of the *auxiliary* and *basic* variables present in the Lagrangian L, namely,

$$\bar{b}_2 = 0,$$

$$\bar{f}_3 = 0,$$

$$\bar{f}_4 = 0,$$

$$\bar{f}_5 = 0,$$

$$\bar{f}_6 = 0,$$

$$\bar{f}_1 = \dot{C},$$

$$\bar{f}_2 = \bar{C},$$

$$\bar{b}_1 = -\mathscr{B}. \tag{49}$$

The above values of *chiral* derived variables are obtained in a similar fashion as the *antichiral* derived variables are obtained. After the substitution of the above derived variables into the *chiral* super expansions (39), we get the following expressions for the *chiral* supervariables onto $(1, 1)$-dimensional super-submanifold as

$$z(\tau) \longrightarrow \mathscr{Z}^{(ab)}(\tau, \vartheta) = z(\tau) + \vartheta\left(\dot{C}\right) \equiv z(\tau) + \vartheta[s_{ab}\,z(\tau)],$$

$$\varphi(\tau) \longrightarrow \Theta^{(ab)}(\tau, \vartheta) = \varphi(\tau) + \vartheta\left(\bar{C}\right) \equiv \varphi(\tau) + \vartheta[s_{ab}\varphi(\tau)],$$

$$C(\tau) \longrightarrow F^{(ab)}(\tau, \vartheta) = C(\tau) + \vartheta(-i\mathscr{B}) \equiv C(\tau) + \vartheta[s_{ab}C(\tau)],$$

$$\bar{C}(\tau) \longrightarrow \bar{F}^{(ab)}(\tau, \vartheta) = \bar{C}(\tau) + \vartheta\,(0) \equiv \bar{C}(\tau) + \vartheta\left[s_{ab}\bar{C}(\tau)\right],$$

$$r(\tau) \longrightarrow R^{(ab)}(\tau, \vartheta) = r(\tau) + \vartheta\,(0) \equiv r(\tau) + \vartheta[s_{ab}r(\tau)],$$

$$p_r(\tau) \longrightarrow P_r^{(ab)}(\tau, \vartheta) = p_r(\tau) + \vartheta\,(0) \equiv p_r(\tau) + \vartheta[s_{ab}p_r(\tau)],$$

$$p_\varphi(\tau) \longrightarrow P_\varphi^{(ab)}(\tau, \vartheta) = p_\varphi(\tau) + \vartheta\,(0) \equiv p_\varphi(\tau) + \vartheta\left[s_{ab}p_\varphi(\tau)\right],$$

$$\mathscr{B}(\tau) \longrightarrow \tilde{\mathscr{B}}^{(ab)}(\tau, \vartheta) = \mathscr{B}(\tau) + \vartheta\,(0) \equiv \mathscr{B}(\tau) + \vartheta[s_{ab}\mathscr{B}(\tau)]. \tag{50}$$

Here, the coefficients of ϑ are simply the anti-BRST symmetry transformations (see, e.g., [11–13] for detail). In fact, the anti-BRST transformation of any *generic* variable $\psi(\tau)$ is simply the translation of the corresponding *chiral* supervariable $\Psi^{(ab)}(\tau, \vartheta)$ along the ϑ-direction. Mathematically, this statement can be corroborated as $s_{ab}\psi(\tau) = (\partial/\partial\vartheta)\Psi^{(ab)}(\tau, \vartheta) = \partial_\vartheta\Psi^{(ab)}(\tau, \vartheta)$. Thus, it is clear that there is a mapping between the quantum anti-BRST symmetry transformation (s_{ab}) and the Grassmannian partial derivative (∂_ϑ) defined on the *chiral* super-submanifold with the mapping: $s_{ab} \leftrightarrow \partial_\vartheta$.

4. Nilpotent (Anti-)co-BRST Symmetry Transformations: (Anti-)chiral Supervariable Approach

In this section, we derive the nilpotent (anti-)co-BRST symmetry transformations using the (anti-)chiral supervariable approach (ACSA) where we use the expansions of the (anti-)chiral supervariables and the (anti-)co-BRST invariant restrictions. Toward this goal in our mind, first of all, we determine the co-BRST symmetries by exploiting the *chiral* super expansions given in Equation (39) and the co-BRST invariant restrictions. The co-BRST invariant restrictions are given as

$$s_d\left(r, p_r, p_\varphi, \mathscr{B}, \bar{C}\right) = 0,$$
$$s_d\left(z\bar{C}\right) = 0,$$
$$s_d\left(\varphi\dot{\bar{C}}\right) = 0,$$
$$s_d\left(\varphi\dot{p}_\varphi + i\dot{\bar{C}}\dot{C}\right) = 0,$$
$$s_d\left(z\,p_\varphi - i\bar{C}C\right) = 0,$$
$$s_d(\varphi + \dot{z}) = 0.$$

$$(51)$$

According to the *basic* rules of ACSA, the above co-BRST invariant restrictions can be generalized onto the $(1, 1)$-dimensional super-submanifold (of the suitably chosen most *common* $(1, 2)$-dimensional supermanifold) as

$$R(\tau, \vartheta) = r(\tau), P_r(\tau, \vartheta) = p_r(\tau), P_\varphi(\tau, \vartheta) = p_\varphi(\tau),$$
$$\tilde{\mathscr{B}}(\tau, \vartheta) = \mathscr{B}(\tau), \bar{F}(\tau, \vartheta) = \bar{C}(\tau),$$

$$(52)$$

$$\mathscr{Z}(\tau, \vartheta)\,\bar{F}(\tau, \vartheta) = z(\tau)\bar{C}(\tau), \Theta(\tau, \vartheta)\,\dot{\bar{F}}(\tau, \vartheta) = \varphi(\tau)\dot{\bar{C}}(\tau),$$

$$(53)$$

$$\Theta(\tau, \vartheta)\,\dot{P}_\varphi(\tau, \vartheta) + i\bar{F}(\tau, \vartheta)\dot{F}(\tau, \vartheta) = \varphi(\tau)\,\dot{p}_\varphi(\tau) + i\dot{\bar{C}}(\tau)\dot{C}(\tau),$$

$$(54)$$

$$\mathscr{Z}(\tau, \vartheta)\,P_\varphi(\tau, \vartheta) - i\bar{F}(\tau, \vartheta)F(\tau, \vartheta) = z(\tau)p_\varphi(\tau) - i\bar{C}(\tau)C(\tau),$$

$$(55)$$

$$\Theta(\tau, \vartheta) + \dot{\mathscr{Z}}(\tau, \vartheta) = \varphi(\tau) + \dot{z}(\tau). \quad (56)$$

At this stage, we determine the value of derived variables of Equation (39) using the above generalizations of the co-BRST invariant restrictions. To derive the value of the derived variables, first of all, we use the first line entry of Equation (52) where the trivial co-BRST invariant quantities are generalized which implies the following relationships:

$$P_\varphi(\tau, \vartheta) = p_\varphi(\tau) \Rightarrow \bar{f}_5 = 0,$$
$$\tilde{\mathscr{B}}(\tau, \vartheta) = \mathscr{B}(\tau) \Rightarrow \bar{f}_6 = 0,$$
$$R(\tau, \vartheta) = r(\tau) \Rightarrow \bar{f}_3 = 0, \quad (57)$$
$$\bar{F}(\tau, \vartheta) = \bar{C}(\tau) \Rightarrow \bar{b}_2 = 0,$$
$$P_r(\tau, \vartheta) = p_r(\tau) \Rightarrow \bar{f}_4 = 0.$$

After substituting the above value of derived variables into the expressions of the *chiral* super expansions (Equation (39)), we obtain the following *chiral* super expansions:

$$\bar{C}(\tau) \longrightarrow \bar{F}^{(d)}(\tau, \vartheta) = \bar{C}(\tau) + \vartheta(0) \equiv \bar{C}(\tau) + \vartheta\left[s_d\bar{C}(\tau)\right], \quad (58)$$

$$r(\tau) \longrightarrow R^{(d)}(\tau, \vartheta) = r(\tau) + \vartheta(0) \equiv r(\tau) + \vartheta\left[s_d r(\tau)\right], \quad (59)$$

$$p_r(\tau) \longrightarrow P_r^{(d)}(\tau, \vartheta) = p_r(\tau) + \vartheta(0) \equiv p_r(\tau) + \vartheta\left[s_d p_r(\tau)\right], \quad (60)$$

$$p_\varphi(\tau) \longrightarrow P_\varphi^{(d)}(\tau, \vartheta) = p_\varphi(\tau) + \vartheta(0) \equiv p_\varphi(\tau) + \vartheta\left[s_d p_\varphi(\tau)\right], \quad (61)$$

$$\mathscr{B}(\tau) \longrightarrow \tilde{\mathscr{B}}^{(d)}(\tau, \vartheta) = \mathscr{B}(\tau) + \vartheta(0) \equiv \mathscr{B}(\tau) + \vartheta\left[s_d\mathscr{B}(\tau)\right], \quad (62)$$

where superscript (d) on the *chiral* supervariables denotes the supervariables obtained after the application of the co-BRST (i.e., dual-BRST) invariant restrictions. For the nontrivial case, first of all, we generalize the co-BRST invariant restriction $s_d(z\,\bar{C}) = 0$ and $s_d(\varphi\dot{\bar{C}}) = 0$ onto $(1, 1)$-dimensional super-submanifold as

$$\mathscr{Z}(\tau, \vartheta)\,\bar{F}^{(d)}(\tau, \vartheta) = z(\tau)\,\bar{C}(\tau), \Theta(\tau, \vartheta)\,\dot{\bar{F}}^{(d)}(\tau, \vartheta) = \varphi(\tau)\,\dot{\bar{C}}(\tau),$$

$$(63)$$

which lead to the following relationships for the derived variables:

$$\bar{f}_1(\tau)\,\bar{C}(\tau) = 0 \Rightarrow \bar{f}_1(\tau) \propto \bar{C}(\tau), \Rightarrow \bar{f}_1(\tau) = -\bar{\kappa}_1\bar{C}(\tau),$$
$$\bar{f}_2(\tau)\,\dot{\bar{C}}(\tau) = 0 \Rightarrow \bar{f}_2(\tau) \propto \dot{\bar{C}}(\tau), \Rightarrow \bar{f}_2(\tau) = \bar{\kappa}_2\dot{\bar{C}}(\tau),$$

$$(64)$$

where $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are the proportionality constants. To determine the value of these constants, we further use the generalizations of the co-BRST invariant restrictions $s_d(\varphi + \dot{z}) = 0, s_d(\varphi\dot{p}_\varphi + i\dot{\bar{C}}\dot{C}) = 0$ and $s_d(z\,p_\varphi - i\bar{C}C) = 0$ as

$$\Theta(\tau, \vartheta) + \dot{\mathscr{Z}}(\tau, \vartheta) = \varphi(\tau) + \dot{z}(\tau) \Rightarrow \bar{\kappa}_1 = \bar{\kappa}_2, \quad (65)$$

$$\Theta(\tau, \vartheta)\dot{P}_\varphi^{(d)}(\tau, \vartheta) + i\,\dot{\bar{F}}^{(d)}(\tau, \vartheta)\dot{F}(\tau, \vartheta) = \varphi(\tau)\dot{p}_\varphi(\tau) + i\dot{\bar{C}}(\tau)\dot{C}(\tau) \Rightarrow \dot{\bar{b}}_1(\tau) = -\bar{\kappa}_2\,\dot{p}_\varphi(\tau),$$

$$(66)$$

$$\mathscr{Z}(\tau, \vartheta)P_\varphi^{(d)}(\tau, \vartheta) - i\,\bar{F}^{(d)}(\tau, \vartheta)F(\tau, \vartheta) = z(\tau)p_\varphi(\tau) - i\bar{C}(\tau)C(\tau) \Rightarrow \bar{b}_1(\tau) = -\bar{\kappa}_1\,p_\varphi(\tau).$$

$$(67)$$

The results of the above three relations in Equation (65) imply that proportionality constant is equal (i.e., $\bar{\kappa}_1$

$= \bar{\kappa}_2$) and their values are equal to minus one (i.e., $\bar{\kappa}_1 = \bar{\kappa}_2 = -1$). Therefore, we get the value of the derived variables as $\bar{f}_1 = \bar{C}, \bar{f}_2 = -\dot{\bar{C}}, \bar{b}_1 = p_\varphi$. As a result, we have the following *chiral* super expansions of the ordinary variables:

$$z(\tau) \longrightarrow \mathcal{Z}^{(d)}(\tau, \vartheta) = z(\tau) + \vartheta\left[\bar{C}(\tau)\right] \equiv z(\tau) + \vartheta[s_d\, z(\tau)],$$

$$\varphi(\tau) \longrightarrow \Theta^{(d)}(\tau, \vartheta) = \varphi(\tau) + \vartheta\left[-\dot{\bar{C}}(\tau)\right] \equiv \varphi(\tau) + \vartheta[s_d\, \varphi(\tau)],$$

$$C(\tau) \longrightarrow F^{(d)}(\tau, \vartheta) = C(\tau) + \vartheta\left[i\, p_\varphi(\tau)\right] \equiv C(\tau) + \vartheta[s_d\, C(\tau)],$$

$$(68)$$

where superscript (d) on the supervariables denotes the same meaning as in Equation (58). From the above equations, it is clear that the translation of any generic *chiral* supervariable $\Psi^{(d)}(\tau, \vartheta)$ along the ϑ-direction generates co-BRST symmetry transformation (s_d) of the corresponding variable $\psi(\tau)$. Mathematically, we can express these statements as $s_d\psi(\tau) = \partial_\vartheta\Psi^{(d)}(\tau, \vartheta)$. In other words, the coefficients of ϑ are simply the co-BRST symmetry transformations. Thus, the co-BRST symmetry transformation (s_d) is connected with the Grassmannian derivative ∂_ϑ (i.e., $s_d \leftrightarrow \partial_\vartheta$) [11–13].

Now, for the derivation of anti-co-BRST symmetry transformations, we use the *antichiral* super expansions of the supervariables (Equation (15)) and anti-co-BRST invariant restrictions. The anti-co-BRST invariant restrictions are listed as

$$s_{ad}\left(r, p_r, p_\varphi, \mathcal{B}, C\right) = 0,$$

$$s_{ad}(zC) = 0,$$

$$s_{ad}\left(\varphi\dot{C}\right) = 0,$$

$$s_{ad}\left(zp_\varphi - i\bar{C}C\right) = 0,$$

$$s_{ad}\left(\varphi\dot{p}_\varphi + i\dot{\bar{C}}\dot{C}\right) = 0,$$

$$s_{ad}(\dot{z} + \varphi) = 0.$$

$$(69)$$

The generalization of these anti-co-BRST invariant restrictions onto the $(1, 1)$-dimensional super-submanifold (of the most *common* $(1, 2)$-dimensional supermanifold) is as follows:

$$R(\tau, \bar{\vartheta}) = r(\tau), P_r(\tau, \bar{\vartheta}) = p_r(\tau), P_\varphi(\tau, \bar{\vartheta}) = p_\varphi(\tau),$$

$$\tilde{\mathcal{B}}(\tau, \bar{\vartheta}) = \mathcal{B}(\tau), F(\tau, \bar{\vartheta}) = C(\tau),$$

$$\mathcal{Z}(\tau, \bar{\vartheta})\, F(\tau, \bar{\vartheta}) = z(\tau)C(\tau), \Theta(\tau, \bar{\vartheta})\dot{F}(\tau, \bar{\vartheta}) = \varphi(\tau)\dot{C}(\tau),$$

$$\mathcal{Z}(\tau, \bar{\vartheta})P_\varphi(\tau, \bar{\vartheta}) - i\bar{F}(\tau, \bar{\vartheta})F(\tau, \bar{\vartheta}) = z(\tau)p_\varphi(\tau) - i\bar{C}(\tau)C(\tau),$$

$$\Theta(\tau, \bar{\vartheta})\dot{P}_\varphi(\tau, \bar{\vartheta}) + i\dot{\bar{F}}(\tau, \bar{\vartheta})\dot{F}(\tau, \bar{\vartheta}) = \varphi(\tau)p_\varphi(\tau) + i\dot{\bar{C}}(\tau)\dot{C}(\tau),$$

$$\dot{\mathcal{Z}}(\tau, \bar{\vartheta}) + \Theta(\tau, \bar{\vartheta}) = \dot{z}(\tau) + \varphi(\tau). \quad (70)$$

The above generalizations of the anti-co-BRST invariant restrictions, finally, lead to the derivation of the derived variables as follows:

$$b_1 = 0,$$

$$b_2 = -p_\varphi,$$

$$f_1 = C,$$

$$f_2 = -\dot{C},$$

$$f_3 = 0,$$

$$f_4 = 0,$$

$$f_5 = 0,$$

$$f_6 = 0.$$

$$(71)$$

Thus, we have determined all the derived variables using the same technique as we have used in the derivation of the co-BRST symmetry transformations. Finally, after substituting the value of derived variables into Equation (15), we obtain the following expressions for the *antichiral* expansions of the supervariables, namely,

$$z(\tau) \longrightarrow \mathcal{Z}^{(ad)}(\tau, \bar{\vartheta}) = z(\tau) + \bar{\vartheta}\,(C) \equiv z(\tau) + \bar{\vartheta}\,[s_{ad}\, z(\tau)],$$

$$\varphi(\tau) \longrightarrow \Theta^{(ad)}(\tau, \bar{\vartheta}) = \varphi(\tau) + \bar{\vartheta}\left(-\dot{C}\right) \equiv \varphi(\tau) + \bar{\vartheta}\,[s_{ad}\varphi(\tau)],$$

$$C(\tau) \longrightarrow F^{(ad)}(\tau, \bar{\vartheta}) = C(\tau) + \bar{\vartheta}\,(0) \equiv C(\tau) + \bar{\vartheta}\,[s_{ad}C(\tau)],$$

$$\bar{C}(\tau) \longrightarrow \bar{F}^{(ad)}(\tau, \bar{\vartheta}) = \bar{C}(\tau) + \bar{\vartheta}\left(-i\, p_\varphi\right) \equiv \bar{C}(\tau) + \bar{\vartheta}\,[s_{ad}\bar{C}(\tau)],$$

$$r(\tau) \longrightarrow R^{(ad)}(\tau, \bar{\vartheta}) = r(\tau) + \bar{\vartheta}\,(0) \equiv r(\tau) + \bar{\vartheta}\,[s_{ad}r(\tau)],$$

$$p_r(\tau) \longrightarrow P_r^{(ad)}(\tau, \bar{\vartheta}) = p_r(\tau) + \bar{\vartheta}\,(0) \equiv p_r(\tau) + \bar{\vartheta}\,[s_{ad}p_r(\tau)],$$

$$p_\varphi(\tau) \longrightarrow P_\varphi^{(ad)}(\tau, \bar{\vartheta}) = p_\varphi(\tau) + \bar{\vartheta}\,(0) \equiv p_\varphi(\tau) + \bar{\vartheta}\left[s_{ad}p_\varphi(\tau)\right],$$

$$\mathcal{B}(\tau) \longrightarrow \tilde{\mathcal{B}}^{(ad)}(\tau, \bar{\vartheta}) = \mathcal{B}(\tau) + \bar{\vartheta}\,(0) \equiv \mathcal{B}(\tau) + \bar{\vartheta}\,[s_{ad}\mathcal{B}(\tau)],$$

$$(72)$$

where superscript (ad) on the *antichiral* supervariables denotes the fact that *antichiral* supervariables are obtained after the application of quantum anti-co-BRST invariant restrictions (Equation (69)). Here, it is clear that the coefficient of Grassmannian variable $\bar{\vartheta}$ is simply the quantum anti-co-BRST symmetry transformation (s_{ad}). To be more clear, the anti-co-BRST symmetry transformation (s_{ad}) of any generic variable $\psi(\tau)$ is equal to the translation of the corresponding *antichiral* supervariable $\Psi^{(ad)}(\tau, \bar{\vartheta})$ along the $\bar{\vartheta}$-direction, i.e., $s_{ad}\psi(\tau) = \partial_{\bar{\vartheta}}\Psi^{(ad)}(\tau, \bar{\vartheta})$. This implies that the anti-co-BRST symmetry (s_{ad}) is connected with the Grassmannian translation generator $(\partial_{\bar{\vartheta}})$ as $s_{ad} \leftrightarrow \partial_{\bar{\vartheta}}$ [11–13].

5. Nilpotency and Absolute Anticommutativity of the Noether Conserved Charges: ACSA

In this section, we deduce the nilpotency and absolute anticommutativity properties of the conserved (anti-)BRST along with (anti-)co-BRST charges in the language of ACSA. For this, first of all, we show the nilpotency of the (anti-)BRST together with (anti-)co-BRST conserved charges. It is straightforward to express the expressions of the (anti-)BRST together with (anti-)co-BRST charges in terms of the (anti-)chiral supervariables and partial derivatives ($\partial_{\bar{\vartheta}}$, ∂_{ϑ}) with an equivalent integral form as follows:

$$Q_b = \frac{\partial}{\partial \bar{\vartheta}} \left[i \dot{\bar{F}}^{(b)}(\tau, \bar{\vartheta}) F^{(b)}(\tau, \bar{\vartheta}) - i \bar{F}^{(b)}(\tau, \bar{\vartheta}) \dot{F}^{(b)}(\tau, \bar{\vartheta}) \right]$$

$$\equiv \int d\bar{\vartheta} \left[i \dot{\bar{F}}^{(b)}(\tau, \bar{\vartheta}) F^{(b)}(\tau, \bar{\vartheta}) - i \bar{F}^{(b)}(\tau, \bar{\vartheta}) \dot{F}^{(b)}(\tau, \bar{\vartheta}) \right],$$

$$Q_{ab} = \frac{\partial}{\partial \vartheta} \left[i \bar{F}^{(ab)}(\tau, \vartheta) \dot{F}^{(ab)}(\tau, \vartheta) - i \dot{\bar{F}}^{(ab)}(\tau, \vartheta) F^{(ab)}(\tau, \vartheta) \right]$$

$$\equiv \int d\vartheta \left[i \bar{F}^{(ab)}(\tau, \vartheta) \dot{F}^{(ab)}(\tau, \vartheta) - i \dot{\bar{F}}^{(ab)}(\tau, \vartheta) F^{(ab)}(\tau, \vartheta) \right],$$

$$Q_d = \frac{\partial}{\partial \vartheta} \left[i \bar{F}^{(d)}(\tau, \vartheta) \dot{F}^{(d)}(\tau, \vartheta) - i \dot{\bar{F}}^{(d)}(\tau, \vartheta) F^{(d)}(\tau, \vartheta) \right]$$

$$\equiv \int d\vartheta \left[i \bar{F}^{(d)}(\tau, \vartheta) \dot{F}^{(d)}(\tau, \vartheta) - i \dot{\bar{F}}^{(d)}(\tau, \vartheta) F^{(d)}(\tau, \vartheta) \right],$$

$$Q_{ad} = \frac{\partial}{\partial \bar{\vartheta}} \left[i \dot{\bar{F}}^{(ad)}(\tau, \bar{\vartheta}) F^{(ad)}(\tau, \bar{\vartheta}) - i \bar{F}^{(ad)}(\tau, \bar{\vartheta}) \dot{F}^{(ad)}(\tau, \bar{\vartheta}) \right]$$

$$\equiv \int d\bar{\vartheta} \left[i \dot{\bar{F}}^{(ad)}(\tau, \bar{\vartheta}) F^{(ad)}(\tau, \bar{\vartheta}) - i \bar{F}^{(ad)}(\tau, \bar{\vartheta}) \dot{F}^{(ad)}(\tau, \bar{\vartheta}) \right],$$

$$(73)$$

where the superscripts (b) and (ab) stand for the *antichiral* and *chiral* supervariables that have been obtained after the application of the BRST and anti-BRST invariant restrictions, respectively. The superscripts (d) and (ad) show the *chiral* and *antichiral* supervariables that are obtained after the application of co-BRST and anti-co-BRST invariant restrictions, respectively. It is clear that the nilpotency ($\partial_{\bar{\vartheta}}^2 = 0, \partial_{\vartheta}^2 = 0$) of the translational generators ($\partial_{\bar{\vartheta}}, \partial_{\vartheta}$) implies that

$$\partial_{\bar{\vartheta}} Q_b = 0 \Leftrightarrow s_b Q_b = -i\{Q_b, Q_b\} = 0, \quad (74)$$

$$\partial_{\vartheta} Q_{ab} = 0 \Leftrightarrow s_{ab} Q_{ab} = -i\{Q_{ab}, Q_{ab}\} = 0, \quad (75)$$

$$\partial_{\vartheta} Q_d = 0 \Leftrightarrow s_d Q_d = -i\{Q_d, Q_d\} = 0, \quad (76)$$

$$\partial_{\bar{\vartheta}} Q_{ad} = 0 \Leftrightarrow s_{ad} Q_{ad} = -i\{Q_{ad}, Q_{ad}\} = 0, \quad (77)$$

which show the nilpotency $[Q_{(a)b}^2 = Q_{(a)d}^2 = 0]$ of the conserved charges within the ambit of ACSA to BRST formalism. Thus, we have shown that there is a deep connection

between the nilpotency ($\partial_{\bar{\vartheta}}^2 = 0, \partial_{\vartheta}^2 = 0$) of the translational generator ($\partial_{\bar{\vartheta}}, \partial_{\vartheta}$) and the nilpotency (i.e., $Q_{(a)b}^2 = Q_{(a)d}^2 = 0$) of the (anti-)BRST and (anti-)co-BRST charges [$Q_{(a)b}, Q_{(a)d}$]. The above nilpotency property can be also captured in an ordinary space where we use the (anti-)BRST *exact* as well as (anti-)co-BRST *exact* forms of the charges, namely,

$$Q_b = -is_b \left(\bar{C}\dot{C} - \dot{\bar{C}}C \right),$$

$$Q_{ab} = +is_{ab} \left(\bar{C}\dot{C} - \dot{\bar{C}}C \right),$$

$$Q_d = is_d \left(\bar{C}\dot{C} - \dot{\bar{C}}C \right),$$

$$Q_{ad} = -is_{ad} \left(\bar{C}\dot{C} - \dot{\bar{C}}C \right).$$

$$(78)$$

The above expressions show the nilpotency property of the (anti-)BRST along with (anti-)co-BRST conserved charges, in a simpler way, in an ordinary space (cf. (74)).

Now, we are in a stage to show the absolute anticommutativity of the (anti-)BRST along with (anti-)co-BRST charges. For this purpose, we write the charges in terms of the (anti-)chiral supervariables and the derivatives ($\partial_{\vartheta}, \partial_{\bar{\vartheta}}$) of the Grassmannian variables ($\bar{\vartheta}, \vartheta$)

$$Q_b = -i\frac{\partial}{\partial \vartheta} \left[\dot{F}^{(ab)}(\tau, \vartheta) F^{(ab)}(\tau, \vartheta) \right] \equiv -i \int d\vartheta \left[\dot{F}^{(ab)}(\tau, \vartheta) F^{(ab)}(\tau, \vartheta) \right],$$

$$Q_{ab} = i\frac{\partial}{\partial \bar{\vartheta}} \left[\dot{\bar{F}}^{(b)}(\tau, \bar{\vartheta}) \bar{F}^{(b)}(\tau, \bar{\vartheta}) \right] \equiv i \int d\bar{\vartheta} \left[\dot{\bar{F}}^{(b)}(\tau, \bar{\vartheta}) \bar{F}^{(b)}(\tau, \bar{\vartheta}) \right],$$

$$Q_d = i\frac{\partial}{\partial \bar{\vartheta}} \left[\dot{\bar{F}}^{(ad)}(\tau, \bar{\vartheta}) \bar{F}^{(ad)}(\tau, \bar{\vartheta}) \right] \equiv i \int d\bar{\vartheta} \left[\dot{\bar{F}}^{(ad)}(\tau, \bar{\vartheta}) \bar{F}^{(ad)}(\tau, \bar{\vartheta}) \right],$$

$$Q_{ad} = -i\frac{\partial}{\partial \vartheta} \left[\dot{F}^{(d)}(\tau, \vartheta) F^{(d)}(\tau, \vartheta) \right] \equiv -i \int d\vartheta \left[\dot{F}^{(d)}(\tau, \vartheta) F^{(d)}(\tau, \vartheta) \right],$$

$$(79)$$

where the superscripts $(a)b$ and $(a)d$ denote the same meaning as explained earlier. Here, it is straightforward to check that the nilpotency ($\partial_{\bar{\vartheta}}^2 = 0, \partial_{\vartheta}^2 = 0$) of the translational generators ($\partial_{\bar{\vartheta}}, \partial_{\vartheta}$) implies the following relations:

$$\partial_{\vartheta} Q_b = 0 \Leftrightarrow s_{ab} Q_b = -i\{Q_b, Q_{ab}\} = 0,$$

$$\partial_{\bar{\vartheta}} Q_{ab} = 0 \Leftrightarrow s_b Q_{ab} = -i\{Q_{ab}, Q_b\} = 0,$$

$$\partial_{\bar{\vartheta}} Q_d = 0 \Leftrightarrow s_{ad} Q_d = -i\{Q_d, Q_{ad}\} = 0,$$

$$\partial_{\vartheta} Q_{ad} = 0 \Leftrightarrow s_d Q_{ad} = -i\{Q_{ad}, Q_d\} = 0,$$

$$(80)$$

which show the absolute anticommutativity property of the (anti-)BRST as well as (anti-)co-BRST conserved charges. The property of absolute anticommutativity of conserved charges can also be captured explicitly in an ordinary space by using the following (anti-)BRST *exact* and (anti-)co-BRST *exact* forms of the charges, namely,

$$Q_b = -is_{ab}\left(\dot{C}C\right), \; Q_{ab} = +is_b\left(\dot{\bar{C}}\bar{C}\right),$$
$$Q_d = is_{ad}\left(\dot{\bar{C}}\bar{C}\right), \; Q_{ad} = -is_d\left(\dot{C}C\right). \tag{81}$$

6. Invariances of Lagrangian in ACSA

In this section, we discuss the (anti-)BRST along with (anti-)co-BRST invariances of the Lagrangian (7) within the scope of ACSA to BRST formalism. For this purpose, foremost, we generalize the ordinary Lagrangian of $(0+1)$-dimensional onto the suitably chosen $(1, 1)$-dimensional (anti-)chiral super-submanifold of the most *common* $(1, 2)$-dimensional supermanifold. The expressions of the (anti-)chiral super Lagrangian are

$$L(\tau) \longrightarrow \tilde{L}^{(ac)}\left(\tau, \bar{\vartheta}\right) = \dot{r}(\tau)\,p_r(\tau) + \dot{\Theta}^{(b)}\left(\tau, \bar{\vartheta}\right)p_\varphi(\tau)$$
$$- \frac{1}{2}\,p_r^2(\tau) - \frac{1}{2r^2}\,p_\varphi^2(\tau)$$
$$- \mathcal{Z}^{(b)}\left(\tau, \bar{\vartheta}\right)p_\varphi(\tau) - V(r)$$
$$+ \frac{1}{2}\,\mathcal{B}^2(\tau) + \mathcal{B}(\tau)$$
$$\cdot \left[\dot{\mathcal{Z}}^{(b)}\left(\tau, \bar{\vartheta}\right) + \Theta^{(b)}\left(\tau, \bar{\vartheta}\right)\right]$$
$$- i\dot{\bar{F}}^{(b)}\left(\tau, \bar{\vartheta}\right)\dot{C}(\tau) + i\bar{F}^{(b)}\left(\tau, \bar{\vartheta}\right)C(\tau),$$

$$L(\tau) \longrightarrow \tilde{L}^{(c)}(\tau, \vartheta) = \dot{r}(\tau)p_r(\tau) + \dot{\Theta}^{(ab)}(\tau, \vartheta)p_\varphi(\tau)$$
$$- \frac{1}{2}\,p_r^2(\tau) - \frac{1}{2r^2}\,p_\varphi^2(\tau)$$
$$- \mathcal{Z}^{(ab)}(\tau, \vartheta)\,p_\varphi(\tau) - V(r)$$
$$+ \frac{1}{2}\,\mathcal{B}^2(\tau) + \mathcal{B}(\tau)$$
$$\cdot \left[\dot{\mathcal{Z}}^{(ab)}(\tau, \vartheta) + \Theta^{(ab)}(\tau, \vartheta)\right]$$
$$- i\,\dot{\bar{C}}(\tau)\,\dot{F}^{(ab)}(\tau, \vartheta) + i\,\bar{C}(\tau)\,F^{(ab)}(\tau, \vartheta), \tag{82}$$

where the superscripts (ac) and (c) on the super Lagrangians denote the *antichiral* and *chiral* super Lagrangians (containing *antichiral* and *chiral* supervariables), respectively. It is evident that under the application of translational generators $(\partial_{\bar{\vartheta}}, \partial_{\vartheta})$, we get the (anti-)BRST invariance of Lagrangian (L) with the following results:

$$\frac{\partial}{\partial \bar{\vartheta}}\left[\tilde{L}^{(ac)}\left(\tau, \bar{\vartheta}\right)\right] = \frac{d}{d\tau}\left[\mathcal{B}(\tau)\dot{C}(\tau)\right],$$
$$\frac{\partial}{\partial \vartheta}\left[\tilde{L}^{(c)}\left(\tau, \vartheta\right)\right] = \frac{d}{d\tau}\left[\mathcal{B}(\tau)\dot{\bar{C}}(\tau)\right], \tag{83}$$

which imply that the generalized version of super Lagrangians remains quasi-invariant (i.e., up to a total time derivative) under the translational generators $(\partial_{\bar{\vartheta}}, \partial_{\vartheta})$ within the scope of ACSA which are consistent with Equation (10).

Now, we capture the (anti-)co-BRST invariance of the Lagrangian (7) within the scope of (anti-)chiral supervariable approach. For this, we generalize the ordinary Lagrangian into (anti-)co-BRST super Lagrangian where $(0+1)$-dimensional theory is generalized onto the $(1, 1)$-dimensional (anti-)chiral super-submanifold of the *common* $(1, 2)$-dimensional supermanifold as follows:

$$L(\tau) \longrightarrow \tilde{L}^{(c,d)}(\tau, \vartheta) = \dot{r}(\tau)\,p_r(\tau) + \dot{\Theta}^{(d)}(\tau, \vartheta)\,p_\varphi(\tau)$$
$$- \frac{1}{2}\,p_r^2(\tau) - \frac{1}{2r^2}\,p_\varphi^2(\tau)$$
$$- \mathcal{Z}^{(d)}(\tau, \vartheta)\,p_\varphi(\tau) - V(r)$$
$$+ \frac{1}{2}\,\mathcal{B}^2(\tau) + \mathcal{B}(\tau)$$
$$\cdot \left[\dot{\mathcal{Z}}^{(d)}(\tau, \vartheta) + \Theta^{(d)}(\tau, \vartheta)\right]$$
$$- i\dot{\bar{C}}(\tau)\,\dot{F}^{(d)}(\tau, \vartheta) + i\bar{C}(\tau)\,F^{(d)}(\tau, \vartheta),$$

$$L(\tau) \longrightarrow \tilde{L}^{(ac,ad)}\left(\tau, \bar{\vartheta}\right) = \dot{r}(\tau)p_r(\tau) + \dot{\Theta}^{(ad)}\left(\tau, \bar{\vartheta}\right)p_\varphi(\tau)$$
$$- \frac{1}{2}p_r^2(\tau) - \frac{1}{2r^2}\,p_\varphi^2(\tau)$$
$$- \mathcal{Z}^{(ad)}\left(\tau, \bar{\vartheta}\right)p_\varphi(\tau) - V(r)$$
$$+ \frac{1}{2}\mathcal{B}^2(\tau) + \mathcal{B}(\tau)$$
$$\cdot \left[\dot{\mathcal{Z}}^{(ad)}\left(\tau, \bar{\vartheta}\right) + \Theta^{(ad)}\left(\tau, \bar{\vartheta}\right)\right]$$
$$- i\dot{\bar{F}}^{(ad)}\left(\tau, \bar{\vartheta}\right)\dot{C}(\tau) + i\bar{F}^{(ad)}\left(\tau, \bar{\vartheta}\right)C(\tau), \tag{84}$$

where the superscripts (c, d) and (ac, ad) denote the super Lagrangians (containing the chiral and *antichiral* supervariables) obtained after the application of the co-BRST and anti-co-BRST invariant restrictions, respectively. It is straightforward to check that

$$\frac{\partial}{\partial \vartheta}\left[\tilde{L}^{(c,d)}(\tau, \vartheta)\right] = -\frac{d}{d\tau}\left[p_\varphi(\tau)\dot{\bar{C}}(\tau)\right],$$
$$\frac{\partial}{\partial \bar{\vartheta}}\left[\tilde{L}^{(ac,ad)}\left(\tau, \bar{\vartheta}\right)\right] = -\frac{d}{d\tau}\left[p_\varphi(\tau)\dot{C}(\tau)\right], \tag{85}$$

which show the (anti-)co-BRST invariance of the Lagrangian L within the ambit of ACSA to BRST formalism. At the end of this section, we have the following concluding remarks. There are deep connections between the (anti-)BRST symmetries $(s_{(a)b})$ and derivatives $(\partial_{\bar{\vartheta}}, \partial_{\vartheta})$ of the Grassmannian variables $(\bar{\vartheta}, \vartheta)$ with the following mappings: $s_b \leftrightarrow \partial_{\bar{\vartheta}}$ and $s_{ab} \leftrightarrow \partial_\vartheta$. Similarly, in the case of (anti-)co-BRST symmetry transformations, it is clear that these symmetry transformations are also connected with the derivatives $(\partial_{\bar{\vartheta}}, \partial_{\vartheta})$ of Grassmannian variables with the mappings: $s_d \leftrightarrow \partial_\vartheta$ and $s_{ad} \leftrightarrow \partial_{\bar{\vartheta}}$ (cf. Sections 4 and 5).

7. Conclusions

In our present analysis, for the first time, we have derived the off-shell nilpotent *quantum* (anti-)BRST along with (anti-)co-BRST symmetry transformations within the scope of ACSA. We have also discussed the nilpotency along with absolute anticommutativity properties of the corresponding (anti-)BRST along with (anti-)co-BRST conserved charges of the *ordinary* $(0 + 1)$-dimensional gauge invariant Christ–Lee model within the ambit of (anti-)chiral super-variable approach (ACSA) to BRST formalism.

The *novel* remarks of our present endeavor are the derivation of the off-shell nilpotent (anti-)BRST along with (anti-)co-BRST symmetry transformations (cf. Section 4) and the proof of nilpotency and the anticommutativity properties of the (anti-)BRST and (anti-)co-BRST charges in spite of the fact that we have taken into account only the (anti-)chiral super expansions of the supervariables (cf. Section 5). The nilpotency and anticommutativity properties of the above conserved charges and derivation of the corresponding (anti-)BRST and (anti-)co-BRST symmetry transformations are obvious when the *full* super expansions of the supervariables (i.e., BT-supervariable formalism [29–32]) are taken into account. However, for the present study, we have shown these properties with the help of *only* (anti-)chiral super expansions of the (anti-)chiral supervariables.

It is worthwhile to mention that the nilpotency of the BRST as well as anti-BRST conserved charges is connected with the nilpotency ($\partial_{\bar{\vartheta}}^2 = \partial_{\vartheta}^2 = 0$) of the translational generators $\partial_{\bar{\vartheta}}$ and ∂_{ϑ}, respectively. On the other hand, nilpotency of the co-BRST and anti-co-BRST charges is connected with nilpotency ($\partial_{\vartheta}^2 = \partial_{\bar{\vartheta}}^2 = 0$) of the translational generators ∂_{ϑ} and $\partial_{\bar{\vartheta}}$, respectively. However, we have shown (cf. Section 5) that the absolute anticommutativity of the BRST charge with anti-BRST charge is connected with the nilpotency ($\partial_{\vartheta}^2 = 0$) of the translational generator (∂_{ϑ}) and absolute anticommutativity of anti-BRST charge with BRST charge is connected with the nilpotency ($\partial_{\bar{\vartheta}}^2 = 0$) of the translational generator ($\partial_{\bar{\vartheta}}$). On the contrary, the absolute anticommutativity of the co-BRST charge with anti-co-BRST charge is connected with the nilpotency of the translational generator ($\partial_{\bar{\vartheta}}$) and the absolute anticommutativity of the anti-co-BRST charge with co-BRST charge is deeply related with the nilpotency of the translational generator (∂_{ϑ}). These statements are completely novel for the present model. We have also captured the (anti-)BRST along with (anti-)co-BRST invariances of the Lagrangian within the scope of ACSA. In fact, the action corresponding to the (anti-)chiral super Lagrangian is independent of Grassmannian variables ($\vartheta, \bar{\vartheta}$) which is completely *novel* for the present CL model (cf. Section 6).

The above issues, within the scope of ACSA to BRST approach, would be discussed in our future investigations for the various gauge-invariant models/theories like ABJM theory [33–35], supersymmetric Chern-Simons theory [36], Jackiw-Pi model, Freedman-Townsend model, and Abelian gauge theory with higher derivative matter fields. In fact, our standard techniques of ACSA to BRST formalism are applicable wherever gauge invariance is present in the the-ory. Furthermore, there is an interesting and important work [37] that would be discussed in the future for different prospects of the theoretical and physical point of view in the domain of theoretical high-energy physics.

Data Availability

No data were used to support this study.

Additional Points

arXiv identifier. arXiv: 2102.03845 [hep-th]. A version of this work is available as a preprint at [38].

Conflicts of Interest

The authors of this study declare that there are no conflicts of interest.

Acknowledgments

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