Research Article

A Stochastic Holling-Type II Predator-Prey Model with Stage Structure and Refuge for Prey

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In this paper, we study a stochastic Holling-type II predator-prey model with stage structure and refuge for prey. Firstly, the existence and uniqueness of the global positive solution of the system are proved. Secondly, the stochastically ultimate boundedness of the solution is discussed. Next, sufficient conditions for the existence and uniqueness of ergodic stationary distribution of the positive solution are established by constructing a suitable stochastic Lyapunov function. Then, sufficient conditions for the extinction of predator population in two cases and that of prey population in one case are obtained. Finally, some numerical simulations are presented to verify our results.

1. Introduction

Population ecology is one of the most important research fields in biomathematics. There are three basic relationships among different species living in the same natural environment: (i) competitive relationship [1, 2], for example, crops and weeds compete for fertilizer, sunlight, and other resources for their own growth; (ii) reciprocal relationship [3, 4], such as rhinoceros birds feed on the parasites from rhinoceroses. For rhinoceroses, rhinoceros birds can get rid of the parasites and warn them when they are in danger; (iii) predation relationship [5, 6], as the saying goes, "big fish eat small fish, small fish eat shrimps." Among the relationships of population interaction, the predator-prey system has been extensively studied. Lotka and Volterra put forward the predator-prey model in 1925 and 1926, respectively. Since then, many scholars have devoted themselves to the study of various predator-prey models and obtained abundant research results [7–10].

In all the above predator-prey systems, it is generally assumed that predators in the same population have identical predation ability, and the prey in the same population has identical viability and fertility. However, in the real world, the life of many creatures should be divided into two stages: immature and mature. There are recognizable morphological

and behavioral differences that may exist between these stages. For example, the immature creatures have no fertility and predation ability, and their survival ability and defense ability are relatively weak. On the contrary, the mature creatures not only have reproductive ability and predator ability but also have strong survival ability and defensive ability. These will have a great impact on the dynamic behavior of the population. Thus, it is more practical to study the predator-prey model with stage structure. In view of this, many scholars have studied assorted predator-prey systems with stage structure and given corresponding dynamic analysis [11–17]. In particular, the following predator-prey model with stage structure for prey has been considered in [16].

$$
\begin{cases}
\dot{x}_1 = \alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \beta_1 x_1 x_3, \\
\dot{x}_2 = \beta x_1 - r_2 x_2, \\
\dot{x}_3 = x_3(-r + k\beta_1 x_1 - \eta_1 x_3),
\end{cases}
\tag{1}
$$

where $\alpha, r_1, \beta, \eta, m, b, r_2, r, k$, and η_1 are positive constants and $\dot{x}_i = dx_i/dt, i = 1, 2, 3. x_1, x_2$ and x_3 represent the size of the immature prey population, mature prey population, and predator population, respectively. And r_1, r_2 and r indicate the mortality rates of immature prey population, mature prey

population, and predator population, respectively. The parameter α denotes the birth rate of the immature prey population; β is the transformation rate of the mature prey population. η and η_1 denote the intraspecific competition coefficient of the immature prey population and predator population, respectively. k is the digestion constant. Predators only prey on immature prey.

Functional response plays an important role in describing the predator-prey model, which refers to the response of predators' predation rate to the density of prey, that is, the predation effect of predators on prey. Model (1) uses bilinear functional response to describe the interaction between predator and prey. However, many researchers [18–20] have pointed out that nonlinear functional response can more accurately describe the trend of population density. Specifically, the growth rate of predators described by Holling-type II functional response is increasing with the increase of the number of prey [21]. When the survival space is far from saturated and the resources are sufficient, the growth rate of predators is relatively fast, but when the survival space and resources are limited, the growth rate changes less significantly and finally approaches a fixed value. Holling-type II functional response is according to some practical phenomena and takes into account the influence of density constraint. Therefore, it has been discussed by many scholars and extensively applied [13, 22–26]. In addition, it is well known that in many cases, prey can avoid predators through refuges. For example, crabs on the beach hide under sand or stones to avoid seabirds, thus increasing their survival chance. In recent years, different predator-prey systems with refuges have attracted much attention [6, 27–30]. In [27], Qi and Meng studied the threshold behavior of a stochastic predator-prey system with prey refuge and fear effect. They concluded that the survival rate of prey can be improved by increasing the strength of refuge. In [29], Ghosh et al. have studied a prey-predator model incorporating prey refuge and additional food for predators and found that slightly higher refuge is beneficial to the coexistence of species, but strong refuge will lead to the extinction of the predator population.

Inspired by the above motivations, we consider the following Holling-type II predator-prey model with stage struc-ture and refuge for prey:

$$
\begin{cases}
\dot{x}_1 = \alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \dfrac{m(1-b)}{a + x_1(1-b)} x_1 x_3, \\[2mm]
\dot{x}_2 = \beta x_1 - r_2 x_2, \\[2mm]
\dot{x}_3 = x_3 \left(-r + \dfrac{km(1-b)}{a + x_1(1-b)} x_1 - \eta_1 x_3 \right),
\end{cases}
\tag{2}
$$

where m denotes the influence of predators on prey, a represents the half-saturation constant, $b \in [0, 1]$ denotes the refuge rate of immature prey, $(1-b)x_1$ is the number of immature prey that predators can capture, and $m(1-b)$ is the capture rate of predators.

On the other hand, in nature, the biological population is inevitably affected by environmental noise. May [31] pointed out that due to the continuous fluctuation of the environment, the birth rate, mortality, environmental capacity, and other parameters in the model will show random fluctuations in varying degrees. However, the deterministic model does not consider the impact of environmental disturbance, so to some extent, it cannot accurately predict the dynamic behavior of the population. In order to describe the environmental noise better, many scholars consider the environmental noise as white noise, for instance, [22–27, 32, 33]. Considering the sensitivity of mortality to environmental noise, in this paper, we assume that the mortality rates r_1, r_2 and r are disturbed by environmental noise, i.e.,

$$
\begin{aligned}
-r_1 &\longrightarrow -r_1 + \sigma_1 \dot{B}_1(t), \\
-r_2 &\longrightarrow -r_2 + \sigma_2 \dot{B}_2(t), \\
-r &\longrightarrow -r + \sigma_3 \dot{B}_3(t),
\end{aligned}
\tag{3}
$$

where $\sigma_i (i = 1, 2, 3)$ are the intensities of the white noise and $B_i(t) (i = 1, 2, 3)$ represent independent standard Brownian motions which are defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{F}\}_{t \geq 0}$ satisfying the usual conditions (that means it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Then, the stochastic model corresponding to the deterministic model (2) is derived in the following form:

$$
\begin{cases}
dx_1 = \left(\alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \dfrac{m(1-b)}{a + x_1(1-b)} x_1 x_3 \right) dt + \sigma_1 x_1 dB_1(t), \\[2mm]
dx_2 = (\beta x_1 - r_2 x_2) dt + \sigma_2 x_2 dB_2(t), \\[2mm]
dx_3 = x_3 \left(-r + \dfrac{km(1-b)}{a + x_1(1-b)} x_1 - \eta_1 x_3 \right) dt + \sigma_3 x_3 dB_3(t).
\end{cases}
\tag{4}
$$

Based on the analysis above, the paper focuses on the analysis of some dynamic behaviors of the stochastic Holling-type II predator-prey model with stage structure and refuge for prey. The major contributions are presented as follows. First, for the first time, the Holling-type II functional response is applied to establish a predator-prey model with stage structure and refuge for prey. Second, the existence and uniqueness as well as the stochastically ultimate

boundedness of positive solution of the stochastic model are analyzed. Third, some conditions for the existence and uniqueness of ergodic stationary distribution of the stochastic model are established. Finally, some extinction conditions of the stochastic model are studied.

Moreover, in [26], Xu et al. studied a stochastic two-predator one-prey model with modified Leslie-Gower and Holling-type II schemes. The authors proved the existence and uniqueness of global positive solution of their stochastic model by using comparison theorem. In this paper, we consider it by constructing a Lyapunov function. Besides, in [4], Liu et al. discussed a mutualism system in random environments. They obtained the existence and uniqueness of a stable stationary distribution by means of Markov semigroup theory and Fokker-Planck equation. In this paper, we consider the existence and uniqueness of an ergodic stationary distribution of the stochastic model (4) by the stochastic Lyapunov function method.

The rest of this paper is arranged as follows. Some useful definitions and lemmas are given in Section 2. The existence and uniqueness of the global positive solution are discussed in Section 3. In Section 4, we obtain sufficient conditions for stochastically ultimate bounded of the prey and predator. In Section 5, the sufficient conditions for the existence of a unique ergodic stationary distribution of the positive solution are established. In Section 6, we establish sufficient conditions for the extinction of the predator and prey in two cases. The first case is the extinction of the predator, and another case is that both the prey and the predator are extinct. To verify our results, some numerical simulations are presented in Section 7. Finally, the conclusion is given in Section 8.

2. Preliminaries

In this section, some definitions and lemmas are given to prepare for further work.

Lemma 1 (Itô formula) (see [34]). *Let $X(t)$ be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential*

$$dX(t) = f(t)dt + g(t)dB(t), \qquad (5)$$

where $f \in L^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in L^2(\mathbb{R}_+; \mathbb{R}^{d \times m})$. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then, $V(X(t), t)$ is again an Itô process with the stochastic differential given by

$$
\begin{aligned}
dV(X(t)z, t) = &[V_t(X(t), t) + V_X(X(t), t)f(t) \\
&+ \frac{1}{2} trace(g^T(t)V_{XX}(X(t), t)g(t))]dt \\
&+ V_X(X(t), t)g(t)dB(t) = LV(X(t), t)dt \\
&+ V_X(X(t), t)g(t)dB(t),
\end{aligned}
$$

$$(6)$$

where $V_t = \partial V / \partial t$, $V_X = (\partial V / \partial X_1, \partial V / \partial X_2, \cdots, \partial V / \partial X_d)$, $V_{XX} = (\partial^2 V / \partial X_i \partial X_j)_{d \times d}$.

Definition 2 (see [35]). With respect to system (4), the solution is said to be stochastically ultimate bounded, if for $\varepsilon \in (0, 1)$, there is a positive constant $H = H(\varepsilon)$ such that for any initial data $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^3_+$, the solution $(x_1(t), x_2(t), x_3(t))$ has the property that

$$\lim_{t \to \infty} \sup \mathbb{P}\{|x(t)| \geq H\} \leq \varepsilon, \qquad (7)$$

where $|x(t)| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Definition 3 (see [36]). The transition probability function $P(s, x, t, A)$ is said to be time-homogeneous (and the corresponding Markov process is called time-homogeneous) if the function $P(s, x, t + s, A)$ is independent of s, where $0 \leq s \leq t, x \in \mathbb{R}^d$ and $A \in \mathscr{B}$, $\mathscr{B}$ denotes the σ-algebra of Borel sets in $\mathbb{R}^d$.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^d$ described by the stochastic differential equation

$$dX(t) = f(X(t))dt + g(X(t))dB(t). \qquad (8)$$

The diffusion matrix of the process $X(t)$ is defined as follows:

$$
\begin{aligned}
A(x) &= (a_{ij}(x)), \\
a_{ij} &= g^i(x)g^j(x).
\end{aligned}
\qquad (9)
$$

Lemma 4 (see [36]). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded open domain $U \subset \mathbb{R}^d$ with regular boundary Γ, having the following properties:*

H_1: *the diffusion matrix $A(x)$ is strictly positive definite for all $x \in U$.*

H_2: *there exists a nonnegative C^2-function V such that LV is negative for any $\mathbb{R}^d \setminus U$.*

3. Existence and Uniqueness of the Global Positive Solution

The existence and uniqueness of global positive solution are the premises of studying the properties of population dynamics. From [34], for any given initial data $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^3_+$, if the coefficients of SDE model (4) satisfy the linear growth condition and the local Lipschitz condition, then there exists a unique global positive solution $(x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3_+$. But the coefficients of model (4) only satisfy the local Lipschitz condition; then, the solution of the system will explode in a finite time. In this section, we will prove that SDE model (4) has a unique global positive solution.

Theorem 5. *For any given initial data $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^3_+$, there is a unique solution $(x_1(t), x_2(t), x_3(t))$ to system (4) on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ with probability 1.*

Proof. Since the coefficients of system (4) satisfy the local Lipschitz condition, then for any given initial value $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^3$, there is a unique local solution $(x_1(t), x_2(t), x_3(t)) \in \mathbb{R}_+^3$ on $t \in (0, \tau_e)$ (see [34]), where τ_e denotes the explosion time. To show this solution is global, we only need to prove $\tau_e = \infty$ a.s. Let $n_0 > 0$ be sufficiently large such that $x_i(0) \in [1/n_0, n_0] (i = 1, 2, 3)$. For any integer $n \geq n_0$, we define the following stopping time:

$$\tau_n = \inf \left\{ t \in [0, \tau_e): \min \{x_1(t), x_2(t), x_3(t)\} \right.$$
$$\left. \leq \frac{1}{n} \text{ or } \max \{x_1(t), x_2(t), x_3(t)\} \geq n \right\}. \tag{10}$$

We let $\inf \varnothing = \infty$ ($\varnothing$ denotes the empty set). From the definition of stopping time, it is easy to know that τ_n is increasing as $n \longrightarrow \infty$. Set $\tau_\infty = \lim_{n \to \infty} \tau_n$; hence, $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^3$ a.s. for all $t \geq 0$. Thus, to complete the proof, we only need to prove $\tau_\infty = \infty$ a.s. If this statement is false, for any $\varepsilon \in (0, 1)$, there exists a constant $T > 0$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \tag{11}$$

Therefore, there is an integer $n_1 \geq n_0$, for all $n \geq n_1$, we have

$$\mathbb{P}\{\tau_n \leq T\} \geq \varepsilon. \tag{12}$$

Now, we define a C^2-function V:

$$V(x_1, x_2, x_3) = \left(x_1 - c - c \ln \frac{x_1}{c}\right) + \frac{\alpha}{r_2}(x_2 - 1 - \ln x_2)$$
$$+ \frac{1}{k}(x_3 - 1 - \ln x_3), \tag{13}$$

where c is a positive constant that will be determined later. Since

$$u - c - c \ln \frac{u}{c}, u - 1 - \ln u \geq 0, \quad \forall u > 0, \tag{14}$$

thus, V is a nonnegative function. Applying Itô formula to V, we have

$$dV(x_1, x_2, x_3) = LV(x_1, x_2, x_3)dt + \sigma_1(x_1 - c)dB_1(t)$$
$$+ \sigma_2 \frac{\alpha}{r_2}(x_2 - 1)dB_2(t) + \sigma_3 \frac{1}{k}(x_3 - 1)dB_3(t), \tag{15}$$

where $LV : \mathbb{R}_+^3 \longrightarrow \mathbb{R}$ is defined by

$$LV(x_1, x_2, x_3)$$
$$= \left(1 - \frac{c}{x_1}\right)\left(\alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \frac{m(1-b)}{a + x_1(1-b)}x_1 x_3\right)$$
$$+ \frac{\alpha}{r_2}\left(1 - \frac{1}{x_2}\right) \times (\beta x_1 - r_2 x_2) + \frac{1}{k}(x_3 - 1)$$
$$\cdot \left(-r + k\frac{m(1-b)}{a + x_1(1-b)}x_1 - \eta_1 x_3\right) + \frac{c}{2}\sigma_1^2 + \frac{\alpha}{2r_2}\sigma_2^2 + \frac{1}{2k}\sigma_3^2$$
$$= \alpha x_2 - (r_1 + \beta)x_1 - \eta x_1^2 - \frac{m(1-b)}{a + x_1(1-b)}x_1 x_3 - c\alpha\frac{x_2}{x_1}$$
$$+ c(r_1 + \beta) + c\eta x_1 + \frac{cm(1-b)}{a + x_1(1-b)}x_3 + \frac{\alpha\beta}{r_2}x_1 - \alpha x_2$$
$$- \frac{\alpha\beta x_1}{r_2 x_2} + \alpha - \frac{r}{k}x_3 + \frac{m(1-b)}{a + x_1(1-b)}x_1 x_3 - \frac{\eta_1}{k}x_3^2 + \frac{r}{k}$$
$$- \frac{m(1-b)}{a + x_1(1-b)}x_1 + \frac{\eta_1}{k}x_3 + \frac{c}{2}\sigma_1^2 + \frac{\alpha}{2r_2}\sigma_2^2 + \frac{1}{2k}\sigma_3^2$$
$$= -\eta x_1^2 + \left(\frac{\alpha\beta}{r_2} + c\eta\right)x_1 - (r_1 + \beta)x_1 - \frac{\eta_1}{k}x_3^2 + \frac{\eta_1}{k}x_3 - \frac{r}{k}x_3$$
$$+ \frac{cm(1-b)}{a + x_1(1-b)}x_3 - c\alpha\frac{x_2}{x_1} - \frac{\alpha\beta x_1}{r_2 x_2} + c(r_1 + \beta) + \alpha$$
$$- \frac{m(1-b)}{a + x_1(1-b)}x_1 + \frac{r}{k} + \frac{c}{2}\sigma_1^2 + \frac{\alpha}{2r_2}\sigma_2^2 + \frac{1}{2k}\sigma_3^2$$
$$\leq -\eta x_1^2 + \left(\frac{\alpha\beta}{r_2} + c\eta\right)x_1 - \frac{\eta_1}{k}x_3^2 + \frac{\eta_1}{k}x_3 + \left(\frac{cm(1-b)}{a} - \frac{r}{k}\right)x_3$$
$$+ c(r_1 + \beta) + \alpha + \frac{r}{k} + \frac{c}{2}\sigma_1^2 + \frac{\alpha}{2r_2}\sigma_2^2 + \frac{1}{2k}\sigma_3^2. \tag{16}$$

Choose $c = ar/km(1-b)$, then

$$LV \leq -\left[\eta x_1^2 - \left(\frac{\alpha\beta}{r_2} + c\eta\right)x_1 + \left(\frac{1}{2\sqrt{\eta}}\left(\frac{\alpha\beta}{r_2} + c\eta\right)\right)^2\right]$$
$$+ \frac{1}{4\eta}\left(\frac{\alpha\beta}{r_2} + c\eta\right)^2 - \frac{\eta_1}{k}\left(x_3^2 - x_3 + \frac{1}{4}\right) + \frac{\eta_1}{4k}$$
$$+ c(r_1 + \beta) + \alpha + \frac{r}{k} + \frac{c}{2}\sigma_1^2 + \frac{\alpha}{2r_2}\sigma_2^2 + \frac{1}{2k}\sigma_3^2$$
$$\leq \frac{1}{4\eta}\left(\frac{\alpha\beta}{r_2} + \frac{ar}{km(1-b)}\right)^2 + \frac{\eta_1}{4k} + \frac{ar(r_1 + \beta)}{km(1-b)} + \alpha + \frac{r}{k}$$
$$+ \frac{ar}{2km(1-b)}\sigma_1^2 + \frac{\alpha}{2r_2}\sigma_2^2 + \frac{1}{2k}\sigma_3^2 =: K. \tag{17}$$

where K is a positive constant. The rest of the proof is similar to [37] and hence is omitted here. The proof is completed. ∎

4. Stochastically Ultimate Boundedness

Theorem 5 shows that the solution of system (4) remains in the positive cone $\mathbb{R}_+^3$. However, this nonexplosion property in a population dynamical system is often not good enough. Hence, we need to consider another important asymptotic

property: stochastically ultimate boundedness, which means that the solution will be ultimately bounded with large probability.

Theorem 6. *If the conditions $\sigma_1^2 < 2r_1 + 2\beta - 1$, $\sigma_2^2 < 2r_2 - 1$, and $\sigma_3^2 < 2r - 2km - 1$ hold, then, the solution of system (4) is stochastically ultimately bounded for any initial data $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^3$.*

Proof. For $(x_1(t), x_2(t), x_3(t)) \in \mathbb{R}_+^3$, define

$$V(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2. \tag{18}$$

By Itô formula, we have

$$dV(x_1, x_2, x_3) = LV(x_1, x_2, x_3)dt + 2\sigma_1 x_1^2 dB_1(t) + 2\sigma_2 x_2^2 dB_2(t) + 2\sigma_3 x_3^2 dB_3(t), \tag{19}$$

where

$$
\begin{aligned}
LV &= 2x_1\left(\alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \frac{m(1-b)}{a + x_1(1-b)} x_1 x_3\right) \\
&\quad + 2x_2(\beta x_1 - r_2 x_2) + 2x_3^2\left(-r + k\frac{m(1-b)}{a + x_1(1-b)} x_1 \right. \\
&\quad \left. - \eta_1 x_3\right) + \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 \\
&= -2\eta x_1^3 + 2\alpha x_1 x_2 + (\sigma_1^2 - 2r_1 - 2\beta)x_1^2 \\
&\quad - 2\frac{m(1-b)}{a + x_1(1-b)} x_1^2 x_3 + 2\beta x_1 x_2 + (\sigma_2^2 - 2r_2)x_2^2 \\
&\quad - 2\eta_1 x_3^3 + (\sigma_3^2 - 2r)x_3^2 + \frac{2km(1-b)}{a + x_1(1-b)} x_1 x_3^2 \\
&\leq (\sigma_1^2 - 2r_1 - 2\beta + 1)x_1^2 + (\sigma_2^2 - 2r_2 + 1)x_2^2 \\
&\quad + (\sigma_3^2 - 2r + 2km + 1)x_3^2 + 2(\alpha + \beta)x_1 x_2 \\
&\quad - x_1^2 - x_2^2 - x_3^2.
\end{aligned}
\tag{20}
$$

Let

$$
f(x_1, x_2, x_3) = (\sigma_1^2 - 2r_1 - 2\beta + 1)x_1^2 + (\sigma_2^2 - 2r_2 + 1)x_2^2 \\
+ (\sigma_3^2 - 2r + 2km + 1)x_3^2 + 2(\alpha + \beta)x_1 x_2. \tag{21}
$$

According to the conditions, we can find that function $f(x_1, x_2, x_3)$ has an upper bound. We assume that its upper bound is as follows:

$$M_1 = \sup_{(x_1, x_2, x_3) \in \mathbb{R}_+^3} f(x_1, x_2, x_3). \tag{22}$$

Let $M = M_1 + 1$ and noticing $f(0, 0, 0) = 0$, thus, $M > 0$. According to (19), we can obtain

$$dV(x_1, x_2, x_3) \leq \left[N - \left(x_1^2 + x_2^2 + x_3^2\right)\right]dt + 2\sigma_1 x_1^2 dB_1(t) + 2\sigma_2 x_2^2 dB_2(t) + 2\sigma_3 x_3^2 dB_3(t). \tag{23}$$

By Itô formula, we have

$$
\begin{aligned}
d\left(e^t V(x_1, x_2, x_3)\right) \\
&= e^t V(x_1, x_2, x_3)dt + e^t dV(x_1, x_2, x_3) \\
&\leq e^t\left(x_1^2 + x_2^2 + x_3^2\right)dt + \left[e^t M - e^t\left(x_1^2 + x_2^2 + x_3^2\right)\right] \\
&\quad \cdot dt + 2\sigma_1 e^t x_1^2 dB_1(t) + 2\sigma_2 e^t x_2^2 dB_2(t) \\
&\quad + 2\sigma_3 e^t x_3^2 dB_3(t) = e^t M dt + 2\sigma_1 e^t x_1^2 dB_1(t) \\
&\quad + 2\sigma_2 e^t x_2^2 dB_2(t) + 2\sigma_3 e^t x_3^2 dB_3(t).
\end{aligned}
\tag{24}
$$

Integrating both sides of (24) from 0 to t and then taking expectations, we get

$$e^t \mathbb{E}(V(x_1, x_2, x_3)) \leq V(x_1(0), x_2(0), x_3(0)) + Me^t - M. \tag{25}$$

Hence, we have

$$\lim_{t \to \infty} \mathbb{E}(V(x_1, x_2, x_3)) \leq M, \tag{26}$$

namely,

$$\lim_{t \to \infty} \mathbb{E}\left(x_1^2 + x_2^2 + x_3^2\right) \leq M. \tag{27}$$

For any $\varepsilon > 0$, let $H = \sqrt{M}/\sqrt{\varepsilon}$. By Chebyshev's inequality, we can obtain

$$\mathbb{P}\{|x(t)| \geq H\} \leq \frac{\mathbb{E}\left(|x(t)|^2\right)}{H^2}. \tag{28}$$

Then,

$$\lim_{t \to \infty} \sup \mathbb{P}\{|x(t)| \geq H\} \leq \frac{M}{H^2} = \varepsilon. \tag{29}$$

The proof of Theorem 6 is completed.□

5. Existence of Ergodic Stationary Distribution

In this section, we will study the sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the positive solutions of system (4). The existence of an ergodic stationary distribution means that all species can coexist for a long time and are stochastic weakly persistent [17].

Theorem 7. *Assume that $r_2 > 6(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)$, $r > 4(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)$ and $0 \leq b < 1$, then for any initial $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^3$, system (4) has a unique ergodic stationary distribution.*

Proof. According to Lemma 4, we first prove that the condition H_1 is true. The diffusion matrix of system (4) is given by

$$A = \begin{pmatrix} \sigma_1^2 x_1^2 & 0 & 0 \\ 0 & \sigma_2^2 x_2^2 & 0 \\ 0 & 0 & \sigma_3^2 x_3^2 \end{pmatrix}. \tag{30}$$

Clearly, the matrix A is strictly positive definite for all $(x_1, x_2, x_3) \in \mathbb{R}_+^3$. Hence, the condition H_1 in Lemma 4 holds.

Next, we verify that the condition H_2 holds. Let $V_1 = x_1 + (\alpha/r_2)x_2 - \ln x_1$. By Itô formula, we have

$$
\begin{aligned}
LV_1 &\le \alpha x_2 - \eta x_1^2 + \frac{\alpha\beta}{r_2}x_1 - \alpha x_2 - \frac{\alpha x_2}{x_1} + r_1 + \beta + \eta x_1 \\
&\quad + \frac{m(1-b)}{a}x_3 + \frac{1}{2}\sigma_1^2 = -\eta x_1^2 + \left(\frac{\alpha\beta}{r_2} + \eta\right)x_1 \\
&\quad - \frac{\alpha x_2}{x_1} + r_1 + \beta + \frac{m(1-b)}{a}x_3 + \frac{1}{2}\sigma_1^2 \\
&= -\eta\left[x_1^2 - \frac{1}{\eta}\left(\frac{\alpha\beta}{r_2} + \eta\right)x_1 + \left(\frac{1}{2\eta}\left(\frac{\alpha\beta}{r_2} + \eta\right)\right)^2 - \frac{1}{4\eta^2}\right. \\
&\quad \left.\cdot\left(\frac{\alpha\beta}{r_2} + \eta\right)^2\right] - \frac{\alpha x_2}{x_1} + r_1 + \beta + \frac{m(1-b)}{a}x_3 + \frac{1}{2}\sigma_1^2 \\
&\le \frac{1}{4\eta}\left(\frac{\alpha\beta}{r_2} + \eta\right)^2 - \frac{\alpha x_2}{x_1} + r_1 + \beta + \frac{m}{a}x_3 + \frac{1}{2}\sigma_1^2.
\end{aligned}
\tag{31}
$$

Define $V_2 = V_1 - (4\alpha\beta/(r_2 + (\sigma_2^2/2))^2)\ln x_2 - \ln x_3 + ((m + a\eta_1)/ar)x_3$, then we have

$$
\begin{aligned}
LV_2 &\le \frac{1}{4\eta}\left(\frac{\alpha\beta}{r_2} + \eta\right)^2 - \frac{\alpha x_2}{x_1} + r_1 + \beta + \frac{m}{a}x_3 \\
&\quad + \frac{1}{2}\sigma_1^2 + r + \eta_1 x_3 + \frac{1}{2}\sigma_3^2 - \frac{4\alpha\beta^2 x_1}{x_2\left(r_2 + (\sigma_2^2/2)\right)^2} \\
&\quad + \frac{4\alpha\beta}{r_2 + (\sigma_2^2/2)} - \frac{m + a\eta_1}{a}x_3 + \frac{km(m + a\eta_1)}{a^2 r}x_1 x_3 \\
&\le -2\sqrt{\frac{4\alpha^2\beta^2}{\left(r_2 + (\sigma_2^2/2)\right)^2}} + \frac{1}{4\eta}\left(\frac{\alpha\beta}{r_2} + \eta\right)^2 + r_1 \\
&\quad + \beta + \frac{1}{2}\sigma_1^2 + r + \frac{1}{2}\sigma_3^2 + \frac{4\alpha\beta}{r_2 + (\sigma_2^2/2)} \\
&\quad + \frac{km(m + a\eta_1)}{a^2 r}x_1 x_3 = \frac{1}{4\eta}\left(\frac{\alpha\beta}{r_2} + \eta\right)^2 + r_1 + \beta \\
&\quad + \frac{1}{2}\sigma_1^2 + r + \frac{1}{2}\sigma_3^2 + \frac{km(m + a\eta_1)}{a^2 r}x_1 x_3 \\
&= \lambda + \frac{km(m + a\eta_1)}{a^2 r}x_1 x_3,
\end{aligned}
\tag{32}
$$

where $\lambda = 1/4\eta((\alpha\beta/r_2) + \eta)^2 + r_1 + \beta + (1/2)\sigma_1^2 + r + (1/2)\sigma_3^2 > 0$. Define

$$V_3 = -\ln x_2 + Mx_3, \tag{33}$$

where $M > 0$ is a sufficiently large number. Then, we obtain

$$L(V_3) \le -\beta\frac{x_1}{x_2} + r_2 + \frac{1}{2}\sigma_2^2 + \frac{kmM}{a}x_1 x_3. \tag{34}$$

Define

$$V_4 = \frac{1}{\theta + 1}\left(x_1 + \frac{3\alpha}{r_2}x_2 + \frac{x_3}{k}\right)^{\theta+1}, \tag{35}$$

where $\theta = 4$. Let

$$U = x_1 + \frac{3\alpha}{r_2}x_2 + \frac{x_3}{k}, \tag{36}$$

then

$$
\begin{aligned}
LV_4 &= U^\theta\left(\alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \frac{m(1-b)}{a + x_1(1-b)}x_1 x_3\right. \\
&\quad \left. + \frac{3\alpha\beta}{r_2}x_1 - 3\alpha x_2 - \frac{r}{k}x_3\right) + U^\theta\left(\frac{m(1-b)}{a + x_1(1-b)}x_1 x_3\right. \\
&\quad \left. - \frac{\eta_1}{k}x_3\right) + \frac{\theta}{2}U^{\theta-1}\left(\sigma_1^2 x_1^2 + \left(\frac{3\alpha}{r_2}\right)^2\sigma_2^2 x_2^2 + \frac{1}{k^2}\sigma_3^2 x_3^2\right) \\
&\le \frac{3\alpha}{r_2}x_1\left(x_1 + \frac{3\alpha}{r_2}x_2 + \frac{x_3}{k}\right)^\theta - 2\alpha\left(\frac{3\alpha}{r_2}\right)^\theta x_2^{\theta+1} - \eta x_1^{\theta+2} \\
&\quad - \frac{r}{k^{\theta+1}}x_3^{\theta+1} + \frac{\theta}{2}U^{\theta+1}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right) \\
&= -\frac{\eta}{2}x_1^{\theta+2} - \alpha\left(\frac{3\alpha}{r_2}\right)^\theta x_2^{\theta+1} - \frac{r}{2k^{\theta+1}}x_3^{\theta+1} + B_1,
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
B_1 &= -\frac{\eta}{2}x_1^{\theta+2} - \alpha\left(\frac{3\alpha}{r_2}\right)^\theta x_2^{\theta+1} - \frac{r}{2k^{\theta+1}}x_3^{\theta+1} \\
&\quad + \frac{3\alpha}{r_2}x_1\left(x_1 + \frac{3\alpha}{r_2}x_2 + \frac{x_3}{k}\right)^\theta + \frac{\theta}{2}U^{\theta+1}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right).
\end{aligned}
\tag{38}
$$

From the assumptions that B_1 has an upper bound. Let $B = \sup_{(x_1, x_2, x_3) \in \mathbb{R}_+^3} B_1$. Then,

$$LV_4 \le -\frac{\eta}{2}x_1^{\theta+2} - \alpha\left(\frac{3\alpha}{r_2}\right)^\theta x_2^{\theta+1} - \frac{r}{2k^{\theta+1}}x_3^{\theta+1} + B. \tag{39}$$

Define a C^2-function $\bar{V} : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$\bar{V} = NV_2(x_1, x_2, x_3) + V_3(x_2) + V_4(x_1, x_2, x_3), \tag{40}$$

where $N < 0$ is a sufficiently small number satisfying the following condition:

$$N\lambda + B + r_2 + \frac{\sigma_2^2}{2} \leq -K - 1. \tag{41}$$

K is a positive constant. Since $\bar{V}$ is continuous and tends to ∞ as (x_1, x_2, x_3) approaches the boundary of $\mathbb{R}_+^3$. Hence, it must have a lower bound. Now, we assume that it gets this lower bound at point $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathbb{R}_+^3$. Let

$$V = NV_2(x_1, x_2, x_3) + V_3(x_2) + V_4(x_1, x_2, x_3) - \bar{V}(\bar{x}_1, \bar{x}_2, \bar{x}_3). \tag{42}$$

By (32), (34), (39), and (41), we have

$$
\begin{aligned}
LV &\leq N\lambda + \frac{kmN(a\eta_1 + m)}{a^2 r} x_1 x_3 \\
&\quad - \beta \frac{x_1}{x_2} - \frac{\eta}{2} x_1^{\theta+2} - \alpha \left(\frac{3\alpha}{r_2} \right)^{\theta} x_2^{\theta+1} \\
&\quad - \frac{r}{2k^{\theta+1}} x_3^{\theta+1} + \frac{kmM}{a} x_1 x_3 + B + r_2 + \frac{\sigma_2^2}{2} \\
&\leq N\lambda - \beta \frac{x_1}{x_2} - \frac{\eta}{2} x_1^{\theta+2} - \alpha \left(\frac{3\alpha}{r_2} \right)^{\theta} x_2^{\theta+1} \\
&\quad - \frac{r}{2k^{\theta+1}} x_3^{\theta+1} + \frac{kmM}{a} x_1 x_3 + B + r_2 + \frac{\sigma_2^2}{2}.
\end{aligned}
\tag{43}
$$

Define a bounded open set U_ε as follows:

$$U_\varepsilon = \left\{ (x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid \varepsilon < x_1 < \frac{1}{\varepsilon}, \varepsilon < x_3 < \frac{1}{\varepsilon}, \varepsilon^2 < x_2 < \frac{1}{\varepsilon^2} \right\}. \tag{44}$$

$0 < \varepsilon < 1$ is a sufficiently small number. In the set $\mathbb{R}_+^3 \setminus U_\varepsilon$, we can choose ε sufficiently small such that the following conditions hold:

$$\varepsilon \leq \min \left\{ \frac{a(\theta+1)}{kmM}, \frac{ar(\theta+1)}{2k^{\theta+2}mM}, \frac{a(\theta+2)}{kmM(\theta+1)}, \frac{a(\theta+2)\eta}{2kmM} \right\}, \tag{45}$$

$$\max \left\{ -\frac{\beta}{\varepsilon}, -\frac{\eta}{4} \frac{1}{\varepsilon^{\theta+2}}, -\frac{r}{4k^{\theta+1}} \frac{1}{\varepsilon^{\theta+1}}, -\frac{\alpha}{2} \left(\frac{3\alpha}{r_2} \right)^{\theta} \varepsilon_2^{\theta+1} \right\} \leq -D - K, \tag{46}$$

where D is a positive constant which will be determined later. We can divide $\mathbb{R}_+^3 \setminus U_\varepsilon$ into six domains:

$$
\begin{aligned}
U_1 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \leq \varepsilon \right\}, \\
U_2 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \leq \varepsilon \right\}, \\
U_3 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > \varepsilon, x_2 \leq \varepsilon^2 \right\}, \\
U_4 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq \frac{1}{\varepsilon} \right\}, \\
U_5 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > \frac{1}{\varepsilon} \right\}, \\
U_6 &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 > \frac{1}{\varepsilon} \right\}.
\end{aligned}
\tag{47}
$$

Clearly, $U_\varepsilon^c = \mathbb{R}_+^3 \setminus U_\varepsilon = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6$. Next, we will prove that $LV(x_1, x_2, x_3) \leq -K$ for any $(x_1, x_2, x_3) \in U_\varepsilon^c$.

For any $(x_1, x_2, x_3) \in U_1$, due to $x_1 x_3 \leq \varepsilon x_3 \leq \varepsilon((\theta + x_3^{\theta+1})/(\theta+1)) = (\varepsilon\theta/(\theta+1)) + (\varepsilon x_3^{\theta+1}/(\theta+1))$, by (43) and (45), we have

$$
\begin{aligned}
LV &\leq N\lambda + \frac{kmM\varepsilon\theta}{a(\theta+1)} - \left[\frac{r}{2k^{\theta+1}} - \frac{kmM\varepsilon}{a(\theta+1)} \right] x_3^{\theta+1} + B + r_2 + \frac{\sigma_2^2}{2} \\
&\leq -K - 1 + 1 = -K < 0.
\end{aligned}
\tag{48}
$$

For any $(x_1, x_2, x_3) \in U_2$, since $x_1 x_3 \leq \varepsilon x_1 \leq \varepsilon((\theta + 1 + x_1^{\theta+2})/(\theta+2)) = (\varepsilon(\theta+1)/(\theta+2)) + (\varepsilon x_1^{\theta+2}/(\theta+2))$, by (43) and (45), we can get

$$
\begin{aligned}
LV &\leq N\lambda + \frac{kmM(\theta+1)\varepsilon}{a(\theta+2)} - \left(\frac{\eta}{2} - \frac{kmM\varepsilon}{a(\theta+2)} \right) \\
&\quad \cdot x_1^{\theta+2} + B + r_2 + \frac{\sigma_2^2}{2} \leq -K - 1 + 1 = -K < 0.
\end{aligned}
\tag{49}
$$

From (43), we have

$$
\begin{aligned}
LV &\leq -\beta \frac{x_1}{x_2} - \frac{\eta}{4} x_1^{\theta+2} - \frac{\alpha}{2} \left(\frac{3\alpha}{r_2} \right)^{\theta} x_2^{\theta+1} - \frac{r}{4k^{\theta+1}} x_3^{\theta+1} \\
&\quad - \frac{\eta}{4} x_1^{\theta+2} - \frac{\alpha}{2} \left(\frac{3\alpha}{r_2} \right)^{\theta} x_2^{\theta+1} - \frac{r}{4k^{\theta+1}} x_3^{\theta+1} \\
&\quad + \frac{kmM}{a} x_1 x_3 + B + r_2 \leq -\beta \frac{x_1}{x_2} - \frac{\eta}{4} x_1^{\theta+2} \\
&\quad - \frac{\alpha}{2} \left(\frac{3\alpha}{r_2} \right)^{\theta} x_2^{\theta+1} - \frac{r}{4k^{\theta+1}} x_3^{\theta+1} + D,
\end{aligned}
\tag{50}
$$

$$
D = \sup_{(x_1, x_2, x_3) \in \mathbb{R}_+^3} \left\{ -\frac{\eta}{4} x_1^{\theta+2} - \frac{\alpha}{2} \left(\frac{3\alpha}{r_2} \right)^{\theta} x_2^{\theta+1} - \frac{r}{4k^{\theta+1}} x_3^{\theta+1} + \frac{kmM}{a} x_1 x_3 + B + r_2 + \frac{\sigma_2^2}{2} \right\}.
\tag{51}
$$

For any $(x_1, x_2, x_3) \in U_3$, by (46) and (50), we obtain

$$LV \leq -\beta \frac{x_1}{x_2} + D \leq -\beta \frac{\varepsilon}{\varepsilon^2} + D = -\frac{\beta}{\varepsilon} + D \leq -K < 0. \quad (52)$$

For any $(x_1, x_2, x_3) \in U_4$, by (46) and (50), we have

$$LV \leq -\frac{\eta}{4} x_1^{\theta+2} + D \leq -\frac{\eta}{4} \frac{1}{\varepsilon^{\theta+2}} + D \leq -K < 0. \quad (53)$$

For any $(x_1, x_2, x_3) \in U_5$, by (46) and (50), we obtain

$$LV \leq -\frac{r}{4k^{\theta+1}} x_3^{\theta+1} + D \leq -\frac{r}{4k^{\theta+1}} \frac{1}{\varepsilon^{\theta+1}} + D \leq -K < 0. \quad (54)$$

For any $(x_1, x_2, x_3) \in U_6$, by (46) and (50), we get

$$LV \leq -\frac{\alpha}{2} \left(\frac{3\alpha}{r_2}\right)^\theta x_2^{\theta+1} + D \leq -\frac{\alpha}{2} \left(\frac{3\alpha}{r_2}\right)^\theta \varepsilon_2^{\theta+1} + D \leq -K < 0. \quad (55)$$

Therefore, there is a sufficiently small ε, s.t. $LV(x_1, x_2, x_3) \leq -K < 0$ for any $(x_1, x_2, x_3) \in U_\varepsilon^c$. So the condition H_2 in Lemma 4 is satisfied. Thus, system (4) has a unique stationary distribution and it has the ergodic property. We have completed the proof. □

6. Stochastic Extinction

In this section, we will establish sufficient conditions for the extinction of the prey and predator populations.

Theorem 8. *Let $(x_1(t), x_2(t), x_3(t))$ be the solution of system (4) with an initial value $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^3$. If $(\alpha\beta/r_2) > (r_1 + \beta)$ and $((km(1-b)((\alpha\beta/r_2) - r_1 - \beta))/a\eta) < r + (\sigma_3/2)$, then the predator population will die out exponentially with probability one, i.e.,*

$$\lim_{t \to \infty} x_3(t) = 0 \; a.s. \quad (56)$$

Proof. Define

$$V = \ln x_3 + C\left(x_1 + \frac{\alpha}{r_2} x_2\right), \quad (57)$$

where C is a positive constant that will be determined later. By Itô formula, we have

$$
\begin{aligned}
LV &= C\left(\alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \frac{m(1-b)}{a + x_1(1-b)} x_1 x_3 \right.\\
&\quad \left. + \frac{\alpha\beta}{r_2} x_1 - \alpha x_2\right) - r + k\frac{m(1-b)}{a + x_1(1-b)} x_1 - \eta_1 x_3 - \frac{1}{2}\sigma_3^2\\
&\leq C\left[-\eta x_1^2 + \left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right) x_1\right] - r - \frac{1}{2}\sigma_3^2 + k\frac{m(1-b)}{a} x_1\\
&= C\left[-\eta\left(x_1^2 - \frac{2}{\eta}\left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right) x_1 + \frac{1}{\eta^2}\left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right)^2\right)\right]\\
&\quad + \frac{C}{\eta}\left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right)^2 - C\left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right) x_1 - r - \frac{1}{2}\sigma_3^2\\
&\quad + k\frac{m(1-b)}{a} x_1 \leq \frac{C}{\eta}\left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right)^2\\
&\quad - C\left(\frac{\alpha\beta}{r_2} - r_1 - \beta\right) x_1 - r - \frac{1}{2}\sigma_3^2 + k\frac{m(1-b)}{a} x_1.
\end{aligned}
$$
$$(58)$$

Let $C = km(1-b)/a((\alpha\beta/r_2) - r_1 - \beta)$, then

$$LV \leq -r - \frac{1}{2}\sigma_3^2 + \frac{km(1-b)((\alpha\beta/r_2) - r_1 - \beta))}{a\eta}. \quad (59)$$

By Itô formula,

$$
\begin{aligned}
dV(x_1, x_2, x_3) &= LV(x_1, x_2, x_3)dt + \sigma_3 dB_3(t)\\
&\quad + C\sigma_1 x_1 dB_1(t) + C\frac{\alpha}{2}\sigma_2 x_2 dB_2(t).
\end{aligned}
$$
$$(60)$$

And hence,

$$
\begin{aligned}
dV(x_1, x_2, x_3) &\leq \left[-r - \frac{1}{2}\sigma_3^2 + \frac{km(1-b)((\alpha\beta/r_2) - r_1 - \beta)}{a\eta}\right]dt\\
&\quad + \sigma_3 dB_3(t) + C\sigma_1 x_1 dB_1(t) + C\frac{\alpha}{r_2}\sigma_2 x_2 dB_2(t).
\end{aligned}
$$
$$(61)$$

Integrating (61) from 0 to t and then dividing by t on both sides, we get

$$
\begin{aligned}
\frac{V(t) - V(0)}{t} &\leq \left[-r - \frac{1}{2}\sigma_3^2 + \frac{km(1-b)((\alpha\beta/r_2) - r_1 - \beta)}{a\eta}\right]\\
&\quad + \sigma_3 \frac{B_3(t)}{t} + C\sigma_1 \frac{1}{t}\int_0^t x_1(s)dB_1(s)\\
&\quad + C\frac{\alpha}{r_2}\sigma_2 \frac{1}{t}\int_0^t x_2(s)dB_2(s).
\end{aligned}
$$
$$(62)$$

Similar to [38], we can get $\lim_{t\to\infty}(1/t)\int_0^t x_1(s)dB_1(s) = 0$ and $\lim_{t\to\infty}(1/t)\int_0^t x_2(s)dB_2(s) = 0$. Then,

$$\lim_{t\to\infty} \sup \frac{V(t)}{t} \leq -r - \frac{1}{2}\sigma_3^2 + \frac{km(1-b)((\alpha\beta/r_2) - r_1 - \beta)}{a\eta} < 0. \quad (63)$$

Thus,

$$\lim_{t \longrightarrow \infty} \sup \frac{\ln x_3(t)}{t} < 0. \tag{64}$$

Then, we obtain $\lim_{t \longrightarrow \infty} x_3(t) = 0$.

Remark 9. Theorem 8 shows that the larger noise intensity or high mortality of predators as well as high refuge rate of immature prey can lead to the extinction of the predator population. However, under these conditions, we cannot judge whether the immature and mature prey populations are extinct or not.

Theorem 10. *Assume that $(x_1(t), x_2(t), x_3(t))$ is the solution of system (4) with initial data $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^3$, if $(2r_1 + \sigma_1^2)(2\alpha - 2r_2 - \sigma_2^2) < -(\alpha - r_1 - r_2)^2$, then, all the populations will die out exponentially with probability one, i.e.,*

$$\lim_{t \longrightarrow \infty} x_1(t) = 0, \lim_{t \longrightarrow \infty} x_2(t) = 0, \lim_{t \longrightarrow \infty} x_3(t) = 0 \, a.s. \tag{65}$$

Proof. Let $V = \ln(x_1 + x_2)$. By Itô formula, we have

$$
\begin{aligned}
LV(x_1, x_2) &= \frac{1}{x_1 + x_2} \left(\alpha x_2 - r_1 x_1 - \eta x_1^2 - r_2 x_2 \right. \\
&\quad \left. - \frac{m(1-b)}{a + x_1(1-b)} x_1 x_3 \right) - \frac{1}{2} \frac{\sigma_1^2 x_1^2}{(x_1 + x_2)^2} \\
&\quad - \frac{1}{2} \frac{\sigma_2^2 x_2^2}{(x_1 + x_2)^2} - \frac{\sigma_1 \sigma_2 x_1 x_2}{(x_1 + x_2)^2} \\
&\leq \frac{1}{2(x_1 + x_2)^2} [2(x_1 + x_2)(\alpha x_2 - r_1 x_1 \\
&\quad - r_2 x_2) - \sigma_1^2 x_1^2 - \sigma_2^2 x_2^2].
\end{aligned} \tag{66}
$$

Rewriting

$$2(x_1 + x_2)(\alpha x_2 - r_1 x_1 - r_2 x_2) - \sigma_1^2 x_1^2 - \sigma_2^2 x_2^2 \tag{67}$$

as the following form

$$(x_1, x_2) \begin{pmatrix} -2r_1 - \sigma_1^2 & \alpha - r_1 - r_2 \\ \alpha - r_1 - r_2 & 2\alpha - 2r_2 - \sigma_2^2 \end{pmatrix} (x_1, x_2)^{\mathrm{T}}. \tag{68}$$

Let the matrix

$$A = \begin{pmatrix} -2r_1 - \sigma_1^2 & \alpha - r_1 - r_2 \\ \alpha - r_1 - r_2 & 2\alpha - 2r_2 - \sigma_2^2 \end{pmatrix}, \tag{69}$$

it is clear that

$$
\begin{aligned}
|A_1| &= -2r_1 - \sigma_1^2 < 0, \\
|A_2| &= |A| = \left(-2r_1 - \sigma_1^2 \right) \left(2\alpha - 2r_2 - \sigma_2^2 \right) - (\alpha - r_1 - r_2)^2 > 0.
\end{aligned} \tag{70}
$$

Thus, A is negative definite. Define $\lambda_{\max}$ as the maximum eigenvalue of matrix A, then we have

$$LV(x_1, x_2) = (x_1, x_2) A (x_1, x_2)^T \leq -\frac{|\lambda_{\max}|}{2(x_1 + x_2)^2} (x_1^2 + x_2^2). \tag{71}$$

Thus, we obtain

$$
\begin{aligned}
dV(x_1, x_2) &\leq -\frac{|\lambda_{\max}|}{2(x_1 + x_2)^2} (x_1^2 + x_2^2) dt + \frac{\sigma_1 x_1}{x_1 + x_2} dB_1(t) \\
&\quad + \frac{\sigma_2 x_2}{x_1 + x_2} dB_2(t) \leq -\frac{1}{4} |\lambda_{\max}| dt \\
&\quad + \frac{\sigma_1 x_1}{x_1 + x_2} dB_1(t) + \frac{\sigma_2 x_2}{x_1 + x_2} dB_2(t).
\end{aligned} \tag{72}
$$

Integrating (72) from 0 to t and then dividing by t on both sides, we get

$$
\begin{aligned}
\frac{\ln(x_1 + x_2)}{t} &\leq \frac{\ln(x_1(0) + x_2(0))}{t} - \frac{1}{4} |\lambda_{\max}| + \frac{1}{t} \\
&\quad \cdot \int_0^t \frac{\sigma_1 x_1}{x_1 + x_2} dB_1(t) + \frac{1}{t} \int_0^t \frac{\sigma_2 x_2}{x_1 + x_2} dB_2(t).
\end{aligned} \tag{73}
$$

By the strong law of large number of the martingale, we have

$$\lim_{t \longrightarrow \infty} \sup \frac{\ln(x_1 + x_2)}{t} \leq -\frac{1}{4} |\lambda_{\max}| < 0. \tag{74}$$

According to the nonnegativity of x_1 and x_2, we can obtain that

$$
\begin{aligned}
\lim_{t \longrightarrow \infty} \sup \frac{\ln(x_1)}{t} &< 0, \\
\lim_{t \longrightarrow \infty} \sup \frac{\ln(x_2)}{t} &< 0,
\end{aligned} \tag{75}
$$

which implies that $\lim_{t \longrightarrow \infty} x_1(t) = 0$, $\lim_{t \longrightarrow \infty} x_2(t) = 0$. In other words, the prey population goes to extinction exponentially with probability one. Because of $\lim_{t \longrightarrow \infty} x_1(t) = 0$, thus, $\forall \varepsilon > 0$, $\exists T_0 = T_0(\omega)$ and a set $\Omega_\varepsilon \subset \Omega$ such that $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$ and

$$\frac{km(1-b)}{a + x_1(1-b)} x_1 \leq \frac{km(1-b)}{a} x_1 \leq \frac{km(1-b)}{a} \varepsilon. \tag{76}$$

By Itô formula, we have

$$
\begin{aligned}
d(\ln x_3) &= \left(-r + \frac{km(1-b)}{a + x_1(1-b)} x_1 - \eta_1 x_3 - \frac{1}{2} \sigma_3^2 \right) dt + \sigma_3 dB_3(t) \\
&\leq \left(-r + \frac{km(1-b)}{a} \varepsilon - \frac{1}{2} \sigma_3^2 \right) dt + \sigma_3 dB_3(t).
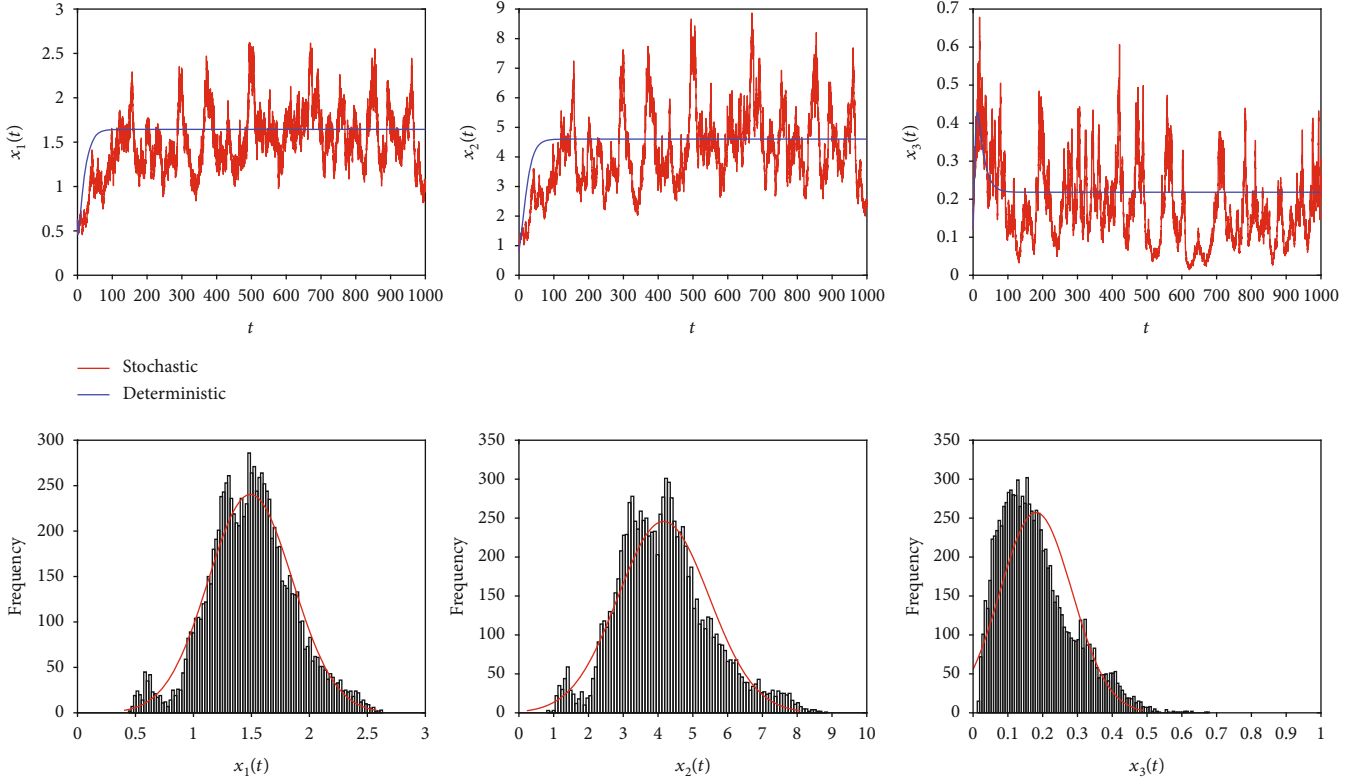\end{aligned} \tag{77}
$$

FIGURE 1: The upper column indicates the paths of x_1, x_2, and x_3 of system (4) with an initial value $(x_1(0), x_2(0), x_3(0)) = (0.63, 0.8, 0.12)$. The strengths of white noise as $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, and $\sigma_3 = 0.2$ and other parameter values are taken as (i). The red lines represent the solution of system (4), and the blue lines represent the solution of the corresponding undisturbed system (2). The under column displays the histogram of the probability density function of x_1, x_2, and x_3 populations.

Integrating (77) from 0 to t and then dividing by t on both sides, we get

$$\frac{\ln x_3(t) - \ln x_3(0)}{t} \leq -r - \frac{1}{2}\sigma_3^2 + \frac{km(1-b)}{a}\varepsilon + \frac{1}{t}\int_0^t \sigma_3 dB_3(t). \tag{78}$$

Then, we obtain

$$\lim_{t \to \infty} \frac{\ln x_3}{t} \leq -r - \frac{1}{2}\sigma_3^2 + \frac{km(1-b)}{a}\varepsilon. \tag{79}$$

Let $\varepsilon \longrightarrow 0$, then $\lim_{t \to \infty} \sup ((\ln x_3(t) - \ln x_3(0))/t) \leq -r - (1/2)\sigma_3^2 < 0$, which implies that $\lim_{t \to \infty} x_3 = 0.\square$

Remark 11. According to Theorem 10, we can find that the extinction of immature prey population leads to the extinction of predator population. This is true, since model (4) shows that the predator population has no additional food source.

7. Numerical Simulations

For the sake of confirming our theoretical results established in the previous sections, we numerically simulate stochastic system (4). We use Milstein's higher order method mentioned in [39] to give numerical simulations. The discretization transformation of system (4) is given as follows:

$$\begin{cases} x_1^{j+1} = x_1^j + \left(\alpha x_2^j - r_1 x_1^j - \beta x_1^j - \eta \left(x_1^j\right)^2 - \frac{m(1-b)}{a + x_1^j(1-b)}x_1^j x_3^j\right)\Delta t + \sigma_1 x_1^j \sqrt{\Delta t}\varepsilon_{1,j} + \frac{\sigma_1^2}{2}x_1^j\left(\varepsilon_{1,j}^2 - 1\right)\Delta t, \\ x_2^{j+1} = x_2^j + \left(\beta x_1^j - r_2 x_2^j\right)\Delta t + \sigma_2 x_2^j \sqrt{\Delta t}\varepsilon_{2,j} + \frac{\sigma_2^2}{2}x_2^j\left(\varepsilon_{2,j}^2 - 1\right)\Delta t, \\ x_3^{j+1} = x_3^j + x_3^j\left(-r + \frac{km(1-b)}{a + x_1^j(1-b)}x_1^j - \eta_1 x_3^j\right)\Delta t + \sigma_3 x_3^j \sqrt{\Delta t}\varepsilon_{3,j} + \frac{\sigma_3^2}{2}x_3^j\left(\varepsilon_{3,j}^2 - 1\right)\Delta t, \end{cases} \tag{80}$$
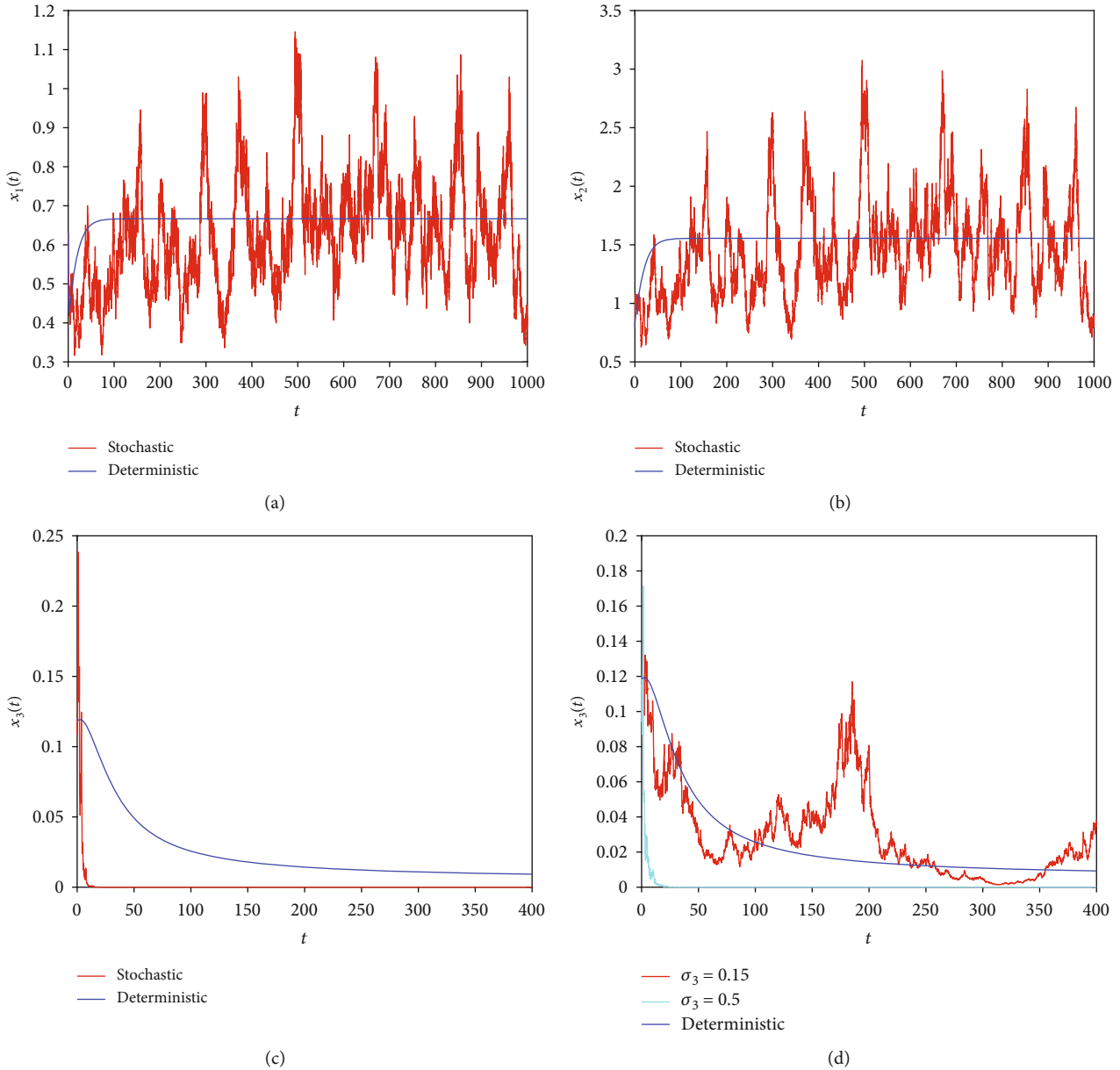
FIGURE 2: Trajectories of x_1, x_2, and x_3 of system (4) with an initial value $(x_1(0), x_2(0), x_3(0)) = (0.63, 0.8, 0.12)$. The red lines and cyan line represent the solution of system (4), and the blue lines represent the solution of the corresponding undisturbed system (2).

where the time increment $\Delta t > 0$. $\sigma_i^2 > 0 (i = 1, 2, 3)$ are the intensities of the white noise, and $\varepsilon_{i,j}(i = 1, 2, 3)$ denote mutually independent Gaussian random variables which follow distribution $N(0, 1)$.

First, we select the parameter values as follows: (i) $\alpha = 0.9$, $r_1 = 0.8$, $\beta = 0.7$, $\eta = 0.6$, $r = 0.2$, $m = 1$, $b = 0.3$, $a = 1$, $r_2 = 0.25$, $k = 0.95$, $\eta_1 = 0.5$. Here, we consider the strengths of white noise as $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, and $\sigma_3 = 0.2$. By simple calculation, we have that $r_2 = 0.25 > 6(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) = 0.24$, $r = 0.2 > 4(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) = 0.16, 0 < b = 0.3 < 1$. Thus, under these circumstances, Theorem 7 holds. Then, we can obtain that system (4) has a unique ergodic stationary distribution. Figure 1 illustrates this. And from Figure 1, we can directly see that immature prey population, mature prey population,

and predator population coexist for a long time and are stochastic persistent. In addition, Figure 1 shows that the trajectories of the solution for the perturbed system (4) will show random fluctuations (red lines) when the undisturbed system are smooth curves (blue lines). These results indicate that the population density does fluctuate more or less due to the influence of white noise.

Next, we choose the parameter values as (ii) $\alpha = 0.9$, $r_1 = 0.8$, $\beta = 0.7$, $\eta = 0.9$, $r = 0.45$, $m = 1$, $b = 0.3$, $a = 1$, $k = 0.95$, $\eta_1 = 0.5$, $r_2 = 0.3$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.7$. We can calculate that $\alpha\beta/r_2 = 2.1 > (r_1 + \beta) = 1.5$, $(km(1 - b)((\alpha\beta/r_2) - (r_1 + \beta)))/a\eta \approx 0.443 < r + (\sigma_3^2/2) = 0.695$. Therefore, the conditions of Theorem 8 are satisfied. By Theorem 8, we get that the predator population goes extinct exponentially with
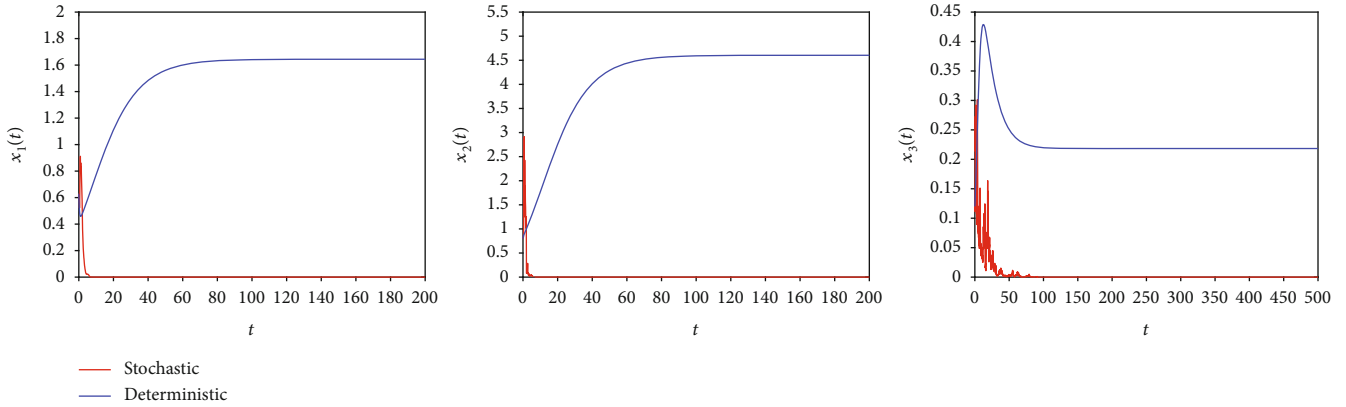
FIGURE 3: The paths of x_1, x_2, and x_3 of system (4) with an initial value $(x_1(0), x_2(0), x_3(0)) = (0.63, 0.8, 0.12)$. The red lines represent the solution of system (4), and the blue lines represent the solution of the corresponding undisturbed system (2).
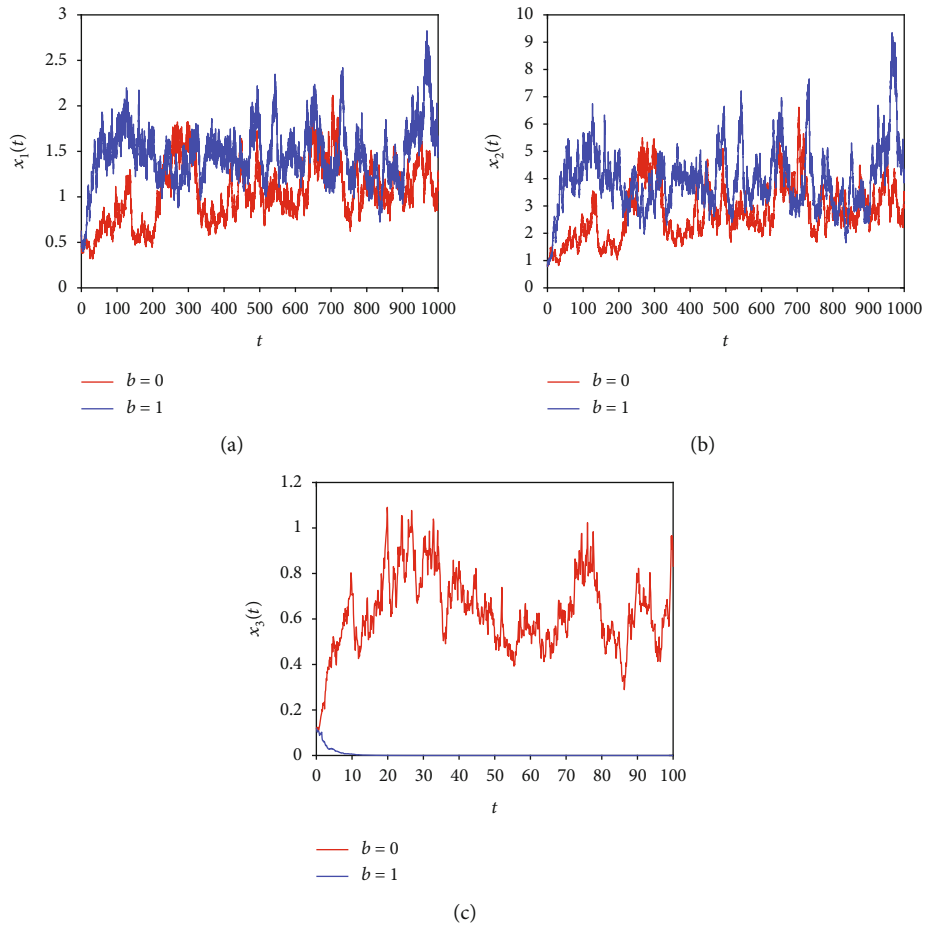

FIGURE 4: Trajectories of system (4) for $b = 0$ and $b = 1$.

probability one. Figures 2(a)–2(c) show the simulation results. In this case, we can see that the predator population is extinct, but the prey population is persistent. In addition, Theorem 8 shows that large noise intensity will lead to the extinction of the predator population. Therefore, in order to illustrate the effect of noise intensity on predator population, we choose $\sigma_3 = 0.15$ and $\sigma_3 = 0.5$, respectively, and other parameters are taken as (ii). The simulation result is

shown in Figure 2(d). From Figure 2(d), we can see that when the noise intensity is large, that is, when $\sigma_3 = 0.5$ (see the cyan line of Figure 2(d)), the predator population is extinct, which can be directly proved by the conclusion of Theorem 8.

In order to illustrate the conclusion of Theorem 10, we choose the noise intensities as $\sigma_1 = 0.1$, $\sigma_2 = 1.15$, and $\sigma_3 = 0.7$, respectively. And other parameters are taken as (i). By

simple calculation, we have that $(2r_1 + \sigma_1^2)(2\alpha - 2r_2 - \sigma_2^2) \approx -0.0362 < -(\alpha - r_1 - r_2)^2 = -0.0225$. Thus, the condition of Theorem 10 holds. By Theorem 10, we obtain that the prey and predator populations die out exponentially with probability one. The numerical simulation results are shown in Figure 3.

Finally, we use the numerical simulation method to analyze the effect of prey refuge on system (4). The refuge rate b of immature prey varies from 0 to 1. $b = 0$ means that the immature prey population has no refuge, that is to say, the immature prey population is completely exposed to the predatory environment of the predator population. On the contrary, $b = 1$ indicates that the immature prey is totally protected by refuge. Now, we take $b = 0$ and $b = 1$, respectively, and other parameters are as (i). The numerical simulation results are shown in Figure 4. Figure 4 shows that when $b = 0$, the immature prey population, mature prey population, and predator population are persistent, that is, the predator population and the prey population coexist, which can be verified by Theorem 7. But when $b = 1$, the predator population becomes extinct, because the predator population has no additional food source. Theorem 8 also proves this result.

8. Conclusion

In this paper, a stochastic Holling-type II predator-prey model with stage structure and refuge for prey is analyzed. Firstly, the existence and uniqueness of the global positive solution for system (4) have been proved. Due to its nonexplosive property, we obtain the conditions for the stochastically ultimate boundedness of positive solution. Next, the sufficient conditions for the existence of a unique ergodic stationary distribution of the positive solution are established. Then, we obtain sufficient conditions for the extinction of predator population in two cases and that of prey population in one case. The theoretical results are proved by numerical simulations.

The following conclusions are verified by numerical simulations. Figure 1 shows that the existence of ergodic stationary distribution can allow all populations are coexistent and stochastic persistent in a long time. In Figure 2, it is shown that the large noise intensity will lead to the extinction of predator population, as shown in Figures 2(c) and 2(d), but the prey population can survive, as shown in Figures 2(a) and 2(b). Figure 3 shows that the extinction of immature prey population will lead to the extinction of predator population. This is due to the fact that predators do not have additional food sources. Finally, it can be seen from Figure 4(c) that if the immature prey population is completely protected by refuge, it will lead predator population towards extinction.

In addition, based on our results, from the perspective of ecological protection, we can provide additional food to the predators to ensure that the prey population and predator population maintain a high-density level. On the one hand, it can reduce the predation rate of the predator population to the prey [30]; on the other hand, it can ensure that the predator population will not be completely extinct when the refuge rate of the prey population is high.

Data Availability

Please contact the author for data requests.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

SZ and CW suggested the model, helped in result interpretation and manuscript evaluation, and supervised the development of work. WS and YH helped to evaluate, revise, and edit the manuscript. All authors read and approved the final manuscript.

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