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The Riccati Equation, Differential Transform, Rational Solutions and Applications

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Abstract

In this article, the Riccati Equation is considered. Various techniques of finding analytical solutions are explored. Those techniques consist mainly of making a change of variable or the use of Differential Transform. It is shown that the nonconstant rational functions whose numerator and denominator are of degree 1, cannot be solutions to the Riccati equation. Two applications of the Riccati equation are discussed. The first one deals with Quantum Mechanics and the second one deal with Physics.

Keywords

Riccati Equation, Differential Transform, Rational Solutions

1. Introduction

The Riccati Equation, named after the Italian mathematician Jacopo Fransesco Riccati [1], is perhaps one of the simplest and the most interesting first-order nonlinear ordinary differential equations. The Riccati equation has the general form

$$y' = c(x) + b(x)y + a(x)y^2, \tag{1}$$

where a, b and c are real value functions, y is the unknown variable and x is the free variable.

This equation arises in many fields of mathematics and physics. It has applications in random processes, optimal control, diffusion problems, stochastic realization theory, robust stabilization, network synthesis [2] [3], and quantum mechanics [4] [5], to name a few. Besides important engineering applications, the newer applications include areas such as financial mathematics.

One of the strengths of the Riccati equation is that it is a unifying link between linear quantum mechanics and other fields of physics, such as thermodynamics, and cosmology. For instance, information about the dynamics of a quantum mechanical wave packet can be obtained from a complex (nonlinear) NL Riccati equation [6]. It has also been shown that complex Riccati equations appear in time-dependent quantum mechanics, and the time-independent Schrodinger equation can be rewritten as a complex Riccati equation [6]. Moreover, the dynamics of the Bose-Einstein condensate and Friedman-Lemaitre equations can be described by a complex Riccati equation [7]. Also by rewriting the solution to the equation of motions, which has the Riccati form, and introducing a substitution, we can derive well-known expressions from statistical thermodynamics [8]. Other interesting characteristics of Riccati's equation include its possession of a superposition formula, the Painlev'e property, and its ability to be linearized [9].

The Riccati equation also plays an important role in financial mathematics [10] since most interest-rate models contain time-dependent functions. In the case of Cox-Ingersoll-Ross Interest-Rate Model (CIR), one must find time-dependent parameter that ensures equality between model prices and market prices, which in finance is known as calibration. Riccati equation is directly relevant to this calibration procedure, and once closed-formed solutions to a Riccati equation that corresponds to the CIR Model are found, these solutions can be utilized to show that there exists an analytical expression for any interest-rate derivative in the CIR interest-rate mode. For a full-scale reading of the fundamental theories of the Riccati equation, see the book of Reid [11].

It is well known that there is no general way to analytically solve a Riccati equation, and only special cases can be treated, especially if a particular solution is known. The solutions of this equation, for instance, can be obtained numerically by the forward Euler method and Runga-Kutta. An unconditional stable scheme was presented by the author in [12]. An analytic solution of the non-linear Riccati equation was obtained by El-Tawil *et al.* using Adomian decomposition method. Abbasbandy solved one example of the quadratic Riccati differential equation (with constant coefficient) by He's variational iteration method by using Adomain's polynomials [13]. In [14], the authors implemented the Homotopy Analysis Method (HAM) to solve a Ricatti equation.

The differential transform method (DTM) is a numerical method for solving both linear and nonlinear differential equations. It was first introduced by Zhou [15]. The method constructs a semi-analytical numerical technique that uses Taylor series method for the solution of differential equations in the form of a polynomial. Unlike the traditional Taylor series method that takes a long time for computation of higher order derivatives, the differential transform method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence.

Moreover, this method can be applied directly to linear and nonlinear ODEs without requiring linearization, discretization, or perturbation.

The article is divided into six chapters.

In chapter 2, several existing methods of solving the Riccati Equation are explored. In theorem 1, a new integrability condition is given.

In chapter 3, it is shown that nonconstant rational functions of the form: $y(t)=\frac{k_1t+k_2}{k_3t+k_4}$ cannot be solution to the Riccati equation.

In chapter 4, the Differential Transform Method is considered. This method seems to be very effective and promising for solving Riccati Differential equations.

In chapter 5, applications to Quantum Mechanics and Classical Mechanics are discussed.

Chapter 6 is dedicated to the conclusion.

2. Solving the Riccati Equation

It is well known that solutions to the general Riccati equation are not available and only special cases can be treated. However, it is always possible to write the general solution of the Riccati Equation if a particular solution is known. After a change of variable using the particular solution, we obtain the Bernoulli Equation, which can be solved by traditional methods.

2.1. Deriving the General Solution of the Riccati Equation Given a Particular Solution

Consider the general form of the Riccati equation (E):

$$\frac{\mathrm{d}y}{\mathrm{d}t}=A(t)y^2+B(t)y+C(t)$$

Let y_1 be a solution of (E). That is

$$\frac{\mathrm{d}y}{\mathrm{d}t}=A(t)y_1^2+B(t)y_1+C(t)$$

It is well known that if $w=y-y_1$ then w satisfies the Bernoulli Equation (E_1):

$$w'+\left(-2A(t)y_1-B(t)\right)w=A(t)w^2.$$

which we know how to solve see [16]

2.2. Methods a Particular Solution to the Riccati Equation Solution

There are many ways of finding solutions. The techniques are based on a change of variable.

2.2.1. Method 1

Let (E): $y'=A(t)y^2+B(t)y+C(t)$ be the Riccati Equation. We use the transformation:

$$y=uv-B/A$$

where u is the unknown variable and v is a function of t. We want to find conditions on $A(t)$, $B(t)$, and $C(t)$ so that (E) is separable equation.

Substituting y and y'into (E) gives:

$$A^2u'v = M - A^2u(v' + Bv) + A^3u^2v^2$$

where

$$M = B'A - BA' + A^2C$$

If $M = 0$ and $v' + Bv = 0$ then the equation is separable, we obtain

$$\frac{u'}{u^2} = Av$$

After integrating:

$$u(t) = -\frac{1}{\int Av\mathrm{d}(t)}$$

Integrating $v' + Bv = 0$, we have $v = \mathrm{e}^{-\int B(t)\mathrm{d}(t)}$ which gives the particular solution:

$$u(t) = -\frac{1}{\int A\left(\mathrm{e}^{-\int B(t)\mathrm{d}(t)}\right)\mathrm{d}(t)}$$

2.2.2. Method 2

We use the transformation

$$y = \left(\frac{C}{A}\right)^{\frac{1}{2}} u$$

where $\frac{C}{A} \geq 0$ then the Riccati equation is reduced to the equation:

$$u' = (CA)^{\frac{1}{2}} \left(1 + \frac{B + \frac{A'}{2A} - \frac{C'}{2C}}{(CA)^{\frac{1}{2}}} u + u^2 \right)$$

If

$$\frac{B + \frac{A'}{2A} - \frac{C'}{2C}}{(CA)^{\frac{1}{2}}} = \text{constant} = K$$

then, $u' = (CA)^{\frac{1}{2}}(1 + Ku + u^2)$ is separable, and we can find an explicit solution.

2.2.3. Method 3

Consider the Riccati equation (E):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y^2 + B(t)y + C(t)$$

We consider the substitution $y(t) = uv - w$, were v and w are the known variables.

Taking the derivative of $y(t)$:

Replacing in (E):

We obtain

$$u'v = -\left(v' + 2A(t)vw + Bv\right)u + A(t)w^2 - Bw + w' + C + A(t)u^2v^2$$

Assuming that $v' + 2Avw + Bv = -bAv^2$ and $Aw^2 - Bw + w' + C = aAv^2$, then we obtain the following separable equation that we can solve:

$$u'v = \left(bu + a + u^2\right)Av^2$$

2.2.4. Method 4: Changing the Riccati Equation into a Second Order Linear Differential Equations

Consider the Riccati equation (E):

$$y' = A(t)y^2 + B(t)y + C$$

We can turn the Riccati equation into a second order linear differential equation using the transformation $y = -\frac{u'}{Au}$, where u is the new variable. This leads to the second order equation:

$$u'' + \left(-B - \frac{A'}{A}\right)u' + ACu = 0$$

We also can use the transformation $y = \frac{Cu}{u'}$, which leads to the second order linear differential equation

$$u'' + \left(B + \frac{C'}{C}\right)u' + ACu = 0$$

If AC is a constant and $B + \frac{C'}{C}$ or $B + \frac{A'}{A}$ is a constant, then explicit solutions to the Riccati Equation can be derived.

2.2.5. Method 5

Lemma 1. *The Riccati equation* (E):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y^2 + b(t) + c(t), \text{ where } a(t) \neq 0$$

can be reduced into the following equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + m(t)y + n(t)$$

Proof. Using the change of variable $y = \frac{v}{a(t)}$ and replacing in (E)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{v'a(t) - va(t)'}{a(t)^2}$$

$$\frac{v'a(t) - va(t)'}{a(t)^2} = a(t)\frac{v^2}{\left(a(t)\right)^2} + b(t)\frac{v}{a(t)} + c(t) \Rightarrow$$

$$v' - \frac{a(t)'}{a(t)'} = v^2 + bv + c(t) \Rightarrow$$

$$v' = v^2 + \left(b(t) + \frac{a(t)'}{a(t)} \right) v + c(t)$$

Lemma 2. *The Riccati equation* (E) *can be turned into the equation*:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \gamma(t) y^2 + f(t) \text{, where } \gamma(t) = |a(t)| \mathrm{e}^{-\int b(t)\mathrm{d}t} \text{, where}$$

$$f(t) = \frac{\left(b + \frac{a'}{a} \right)' + c}{\gamma}$$

Proof. From lemma 1 (E) can be transformed into the equation

$$v' = v^2 + \left(b + \frac{a'}{a} \right) v + c$$

Let $v = -\left(b + \frac{a'}{a} \right) + \gamma w$, where $\gamma(t) = |a(t)| \mathrm{e}^{-\int b(t)\mathrm{d}t}$. Then w satisfies the Riccati equation:

$$w' = \gamma w^2 + \frac{\left(b + \frac{a'}{a} \right)' + c}{\gamma}$$

Theorem 1. *If* $c(t) = -\left(b(t) + \frac{a'(t)}{a(t)} \right)'$, *then the function*

$$y(t) = -\left(\frac{b(t)}{a(t)} + \frac{a'(t)}{a^2(t)} \right) - \frac{|a(t)|}{a(t)} \frac{\mathrm{e}^{-\int b(t)\mathrm{d}t}}{\int |a(t)| \mathrm{e}^{-\int b(t)\mathrm{d}t} \mathrm{d}t}$$

is a solution is of the Riccati equation (E):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t) y^2 + b(t) y + c(t)$$

If $c(t) = -\left(b(t) + \frac{a'(t)}{a'(t)} \right)' + (\gamma(t))^2$, where $\gamma(t) = \left| a(t) \mathrm{e}^{-\int b(t)\mathrm{d}t} \right|$, then the function

$$y(t) = -\left(\frac{b(t)}{a(t)} + \frac{a'(t)}{a^2(t)} \right) - \frac{|a(t)|}{a(t)} \mathrm{e}^{-\int b(t)\mathrm{d}t} \tan\left(\int \gamma(t) \mathrm{d}t \right)$$

is a solution to the Riccati equation (E):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t) y^2 + b(t) y + c(t)$$

Proof. According to Lemma (1) the Riccati Equation, $\frac{\mathrm{d}y}{\mathrm{d}t} = a(t) y^2 + b(t) y + c(t)$ can be turned into (E_1) $v' = v^2 + \left(b + \frac{a'}{a} \right) v + c$ using the change $y = \frac{v}{a}$. According to Lemma (2), (E_1) can be turned into (E_2)

$v' = \gamma w^2 + \frac{\left(b + \frac{a'}{a}\right)'}{\gamma}$ using the change $v = -\left(b + \frac{a'}{a}\right) + \gamma w$, where $\gamma = |a(t)| \mathrm{e}^{-\int b(t)\mathrm{d}t}$.

Now if $c = -\left(b + \frac{a'}{a}\right)'$, then $w' = \gamma w^2 \Rightarrow \frac{w'}{w} = \gamma \Rightarrow w = \mathrm{e}^{\int \gamma(t)\mathrm{d}t}$. Going backward we find

$$y(t) = -\left(\frac{b}{a} + \frac{a'}{a^2}\right) - \frac{|a|}{a} \frac{\mathrm{e}^{-\int b(t)\mathrm{d}t}}{\int |a| \mathrm{e}^{-\int b(t)\mathrm{d}t} \mathrm{d}t}$$

Same thing if $c = -\left(b + \frac{a'}{a}\right)' + \gamma^2$. We have $w' = \gamma w^2 + \gamma \quad \Rightarrow$

$$\frac{w'}{w^2 + 1} = \gamma \quad \Rightarrow$$

$$\int \frac{w'}{w^2 + 1} \mathrm{d}t = \int \gamma \mathrm{d}t \quad \Rightarrow$$

$$\arctan w = \int \gamma \mathrm{d}t \quad \Rightarrow$$

$$w = \tan\left(\int \gamma \mathrm{d}t\right)$$

Going backward we find that:

$$y(t) = -\left(\frac{b}{a} + \frac{a'}{a}\right) + \frac{|a|}{a} \mathrm{e}^{-\int b(t)\mathrm{d}(t)} \tan\left(\int \gamma(t) \mathrm{d}t\right)$$

Example 1

Consider the Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^t y^2 - \frac{1}{t} y - \frac{1}{t^2}$$

where

$$a(t) = \mathrm{e}^t,\ b(t) = -\frac{1}{t},$$

Substituting $a(t), b(t)$ into $-\left(b(t) + \frac{a'}{a}\right)$ gives:

$$-\left(b(t) + \frac{a'}{a}\right)' = -\left(-\frac{1}{t} + 1\right)' = -\left(-\frac{1}{t}\right)' = -\frac{1}{t^2} = c(t)$$

Using the function

$$y(t) = -\left(\frac{b}{a} + \frac{a'}{a}\right) - \frac{|a|}{a} \frac{\mathrm{e}^{-\int b(t)\mathrm{d}t}}{\int |a| \mathrm{e}^{\int b(t)\mathrm{d}t} \mathrm{d}t}$$

and substituting in $a(t), b(t)$ and $c(t)$ gives the solution:

$$y(t) = -\frac{1}{t\mathrm{e}^t} - \frac{1}{\mathrm{e}^t} - \frac{t}{\mathrm{e}^t (t-1)} \quad \Rightarrow$$

$$y(t) = -\mathrm{e}^{-t}\left(1+\frac{1}{t}\right)+\frac{t}{t-1} \quad \Rightarrow$$

$$y(t) = -\mathrm{e}^{-t}\frac{2t^2}{t(t-1)}$$

3. About the Rational Solution $y(t)=\dfrac{k_1t+k_2}{k_3t+k_4}$ of the Riccati Equation

Recall that if the coefficients of the Riccati equation are polynomials, then particular polynomials solutions can sometimes be found. For instance, in [17] the authors have developed a method by which polynomial solutions to the more general Riccati equation:

$$A(t)y' = P(t)y^2 + Q(t)y + R(t)$$

can be found, where $A(t), P(t), Q(t)$ and $R(t)$ are polynomial functions in *t*.

Now how about rational solution in the form

$$y(t) = \frac{k_1t+k_2}{k_3t+k_4}$$

Theorem 2. *Let* (*E*): $\dfrac{\mathrm{d}y}{\mathrm{d}t} = a(t)y^2 + b(t)y + c(t)$ *be the Riccati equation. The equation* (*E*) *is invariant under the rational transformation*:

$$y = \frac{\gamma_1(t)w+\gamma_2(t)}{\gamma_3(t)w+\gamma_4(t)}$$

that is w is a solution of the Riccati equation:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = l_1(t)w^2 + l_2(t)w + l_3(t)$$

Proof.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(\gamma_1'w+\gamma_1w'+\gamma_2')(\gamma_3+\gamma_1)-(\gamma_1w+\gamma_2)(\gamma_3'w+\gamma_3w'+\gamma_4'w)}{(\gamma_3w+\gamma_4)^2} \quad \Rightarrow$$

$$= a\left(\frac{\gamma_1w+\gamma_2}{\gamma_3w+\gamma_4}\right)^2 + b\left(\frac{\gamma_1w+\gamma_2}{\gamma_3w+\gamma_4}\right) + c$$

$$\frac{\gamma_1'\gamma_3w^2+\gamma_1'\gamma_4w+\gamma_2'\gamma_3w+\gamma_2'\gamma_4+\gamma_1\gamma_4w'-\gamma_1\gamma_3'w^2-\gamma_1\gamma_4'w-\gamma_2\gamma_3'w-\gamma_2\gamma_3w'-\gamma_2\gamma_4'}{(\gamma_3w+\gamma_4)^2} \quad \Rightarrow$$

$$= a\frac{\gamma_1^2w^2+2\gamma_1\gamma_2+\gamma_2^2}{(\gamma_3w+\gamma_4)^2} + b\frac{(\gamma_1w+\gamma_2)(\gamma_3w+\gamma_4)}{(\gamma_3w+\gamma_4)^2} + c\frac{(\gamma_3w+\gamma_4)^2}{(\gamma_3+\gamma_4)^2}$$

$$\begin{aligned}&(\gamma_1'\gamma_3-\gamma_1\gamma_3')w^2+(\gamma_1'\gamma_4-\gamma_1\gamma_4'+\gamma_2'\gamma_3-\gamma_2\gamma_3')w+(\gamma_1\gamma_4-\gamma_2\gamma_3)w'+\gamma_2'\gamma_4-\gamma_2\gamma_4\\ &= a(\gamma_1^2w^2+2\gamma_1\gamma_2w+\gamma_2^2)+b(\gamma_1\gamma_3w^2+\gamma_1\gamma_4w+\gamma_2\gamma_3w+\gamma_2\gamma_4) \quad \Rightarrow\\ &\quad +c(\gamma_3^2w^2+2\gamma_3\gamma_4w+\gamma_4^2)\end{aligned}$$

$$\begin{aligned}&\left(\gamma_1'\gamma_3-\gamma_1\gamma_3'\right)w^2+\left(\gamma_1'\gamma_4-\gamma_1'\gamma_4+\gamma_2'\gamma_3-\gamma_2\gamma_3'\right)w+\left(\gamma_1\gamma_4-\gamma_2\gamma_3\right)w'+\left(\gamma_2'\gamma_4-\gamma_2\gamma_4'\right)\\&=\left(a\gamma_1^2+b\gamma_1\gamma_3+c\gamma_3^2\right)w^2+\left(2a\gamma_1\gamma_2+b\gamma_1\gamma_4+b\gamma_2\gamma_3+2c\gamma_3\gamma_4\right)w \quad\Rightarrow\\&\quad+\left(a\gamma_2^2+b\gamma_2\gamma_4+c\gamma_4^2\right)\end{aligned}$$

$$\begin{aligned}w'&=\frac{a\gamma_1^2+b\gamma_1\gamma_3+c\gamma_3^3-\gamma_1'\gamma_3+\gamma_1\gamma_3'}{\gamma_1\gamma_4-\gamma_2\gamma_3}w^2\\&\quad+\frac{2a\gamma_1\gamma_2+b\gamma_1\gamma_4+b\gamma_2\gamma_3+2c\gamma_3\gamma_4-\gamma_1'\gamma_4+\gamma_1'\gamma_4-\gamma_2'\gamma_3+\gamma_2\gamma_3'}{\gamma_1\gamma_4-\gamma_2\gamma_3}w\\&\quad+\frac{a\gamma_2^2+b\gamma_2\gamma_4+c\gamma_4^2-\gamma_2'\gamma_4+\gamma_2\gamma_4'}{\gamma_1\gamma_4-\gamma_2\gamma_3}\end{aligned}$$

Theorem 3. *If the rational function* $y(t)=\frac{k_1t+k_2}{k_3t+k_4}$ *is a non-constant solution of the Riccati equation* $\frac{\mathrm{d}y}{\mathrm{d}t}=a(t)y^2+b(t)y+c(t)$ *where* k_1,k_2,k_3 *and* k_4 *are constant, then the coeffcient* $a(t)$, $b(t)$, *and* $c(t)$ *are constant functions.*

Proof. If we choose $w=t$ and k_1,k_2,k_3 and k_4 are constant, then $y(t)=\frac{k_1t+k_2}{k_3t+k_4}$ in the previous theorem. Now replacing w by t in the Riccati equation in the previous theorem, we obtain:

$$\begin{aligned}1&=\frac{a(t)k_1^2+b(t)k_1k_3+c(t)k_3^2}{k_1k_4-k_2k_3}t^2\\&\quad+\frac{2a(t)k_1k_2+b(t)(k_1k_4+k_2k_3)+2c(t)k_3k_4}{k_1k_4-k_2k_3}t\\&\quad+\frac{a(t)k_2^2+b(t)k_2k_4+c(t)k_4^2}{k_1k_4-k_2k_3}\end{aligned}$$

for all t.

$$a(t)k_1^2+b(t)k_1k_3+c(t)k_3^2=0$$

$$2a(t)k_1k_2+b(t)(k_1k_4+k_2k_3)+2c(t)k_3k_4=0$$

$$a(t)k_2^2+b(t)k_2k_4+c(t)k_4^2=k_1k_4-k_2k_3$$

The Matrix of the system is given by:

$$\boldsymbol{M}=\begin{pmatrix}k_1^2 & k_1k_3 & k_3^2\\ 2k_1k_2 & k_1k_4+k_2k_3 & 2k_3k_4\\ k_2^2 & k_2k_4 & k_4^2\end{pmatrix}$$

The determinant of M is given by:

$$|\boldsymbol{M}|=k_3^2\begin{vmatrix}k_1k_4+k_2k_3 & 2k_3k_4\\ k_2k_4 & k_4^2\end{vmatrix}-k_1k_3\begin{vmatrix}2k_1k_2 & 2k_3k_4\\ k_2^2 & k_4^2\end{vmatrix}+k_3^2\begin{vmatrix}2k_1k_2 & k_1k_4+k_2k_3\\ k_2^2 & k_2k_4\end{vmatrix}\quad\Rightarrow$$

$$k_1^2\left(k_1k_4k_4^2+k_2k_3k_4^2-2k_2k_3k_4^2\right)-k_1k_3\left(2k_1k_2k_4^2-2k_2^2k_3k_4\right)$$
$$+k_3^2\left(2k_1k_2^2k_4-k_1k_2^2k_4-k_2^3k_3\right)$$
$$=k_1^3k_4^3+k_1^2k_2k_3k_4^2-2k_1^2k_2k_3k_4^2-2k_1^2k_2k_3k_4^2+2k_1k_2^2k_3^2k_4$$
$$+2k_1k_2^2k_3^2k_4-k_1k_2^2k_3^2k_4-k_2^3k_3^3$$
$$=k_1^3k_4^3-3k_1^2k_2k_3k_4^2+3k_1k_2^2k_3^2k_4-k_2^3k_3^3=\left(k_1k_4-k_2k_3\right)^3$$
$$=\left|M\right|=\left(k_1k_4-k_2k_3\right)^3$$

Since $y(t)$ is a non-constant solution, then $k_1k_4-k_2k_3\neq 0$. Therefore the system has a unique solution $\left(A(t),B(t),C(t)\right)$ expressed only in terms of k_1,k_2,k_3 and k_4. Since k_1,k_2,k_3, and k_4 are constant, then $A(t),B(t)$ and $C(t)$ are constant functions.

4. Differential Transform

The Differential Transform Method (DTM) is a numerical and analytical method for solving a wide variety of differential equations, such as, but not limited to, ordinary and partial differential equations. This method provides the solution of the given differential equation in terms of convergent series with easily computable components. The concept of DTM was first introduced by Zhou [4] and its main application are in linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the nth derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the Taylor series method, which requires symbolic computation of derivatives and involves more computations especially for higher order differential equations. Different applications of DTM can be found in [18]-[26], to name a few.

In this chapter, the Differential Transform is used to find explicit solution of some Riccati equations. The solutions are expressed in terms of a Taylor series expansion.

4.1. The Essentials of Differential Transform Method

For convenience, we present a review of the DTM. The differential transform of the kth derivative of a function $f(x)$ is defined as

$$F(k)=\frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}f(x)\right]_{x=x_0}$$

where $f(x)$ is the original function and $F(x)$ is the transformed function. The differential inverse transform of $f(x)$ is defined as,

$$f(x)=\sum_{k=0}^{\infty}F(k)\left(x-x_0\right)^k$$

From the differential transform and the differential inverse transform we get,

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left[\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right]_{x=x_0}$$

showing that the concept of differential transform derives from the Taylor series expansion. This method, however, does not evaluate the derivatives symbolically. Instead, relative derivatives are calculated by an iterative way which are described by the transformed equation of the original function.

The fundamental operations performed by differential transform can be readily obtained and are listed below in section 4.2. The main steps of DTM is first to apply the differential transform to the given equation, which will result in a recurrence relation. Then solve the relation and use the differential inverse transform to obtain the solution to the problem.

4.2. Properties

The following properties can be deduced from the differential transformation and the differential inverse transformation and are given below:

1) If $f(x)=g(x)\pm h(x)$, then $F(k)=G(k)\pm H(k)$

2) If $f(x)=\lambda g(x)$, then $F(k)=\lambda G(k)$, where λ is a constant

3)If $f(x)=\frac{\partial f}{\partial x}$, then $F(k)=(k+1)G(k+1)$

4) If $f(x)=\frac{\partial^m g(x)}{\partial x^m}$, then $F(k)=(k+1)(k+2)\cdots(k+m)G(k+m)$

5) If $f(x)=x^m$, then $F(k)=\delta(k-m)=\begin{cases}1 & \text{if } k=m\\ 0 & \text{otherwise}\end{cases}$

6) If $f(x)=g(x)h(x)$, then $F(k)=\sum_{r=0}^{k}G(r)H(k-r)$

7) If $f(x)=f_1(x)f_2(x)\cdots f_m(x)$ then

$$F(k)=\sum_{k_{m-1}=0}^{k}\cdots\sum_{k_1=0}^{k_2}F_1(k_1)F_2(k_2-k_1)\cdots F_m(k-k_{m-1})$$

8) If $f(x)=\mathrm{e}^{Bx}$, then $F(k)=\frac{B^k}{k!}$

9) If $f(x)=\int_0^x g(t)\mathrm{d}t$ then $F(k)=\frac{G(k-1)}{k}$, $k\geq 1$, $F(0)=0$

10) If $f(x)=\sin(\omega x+\alpha)$, then $F(k)=\frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2}+\alpha\right)$

11) If $f(x)=\cos(\omega x+\alpha)$, then $F(k)=\frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2}+\alpha\right)$

Example 2

We first start by considering the following equation (E):

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-y^2+1,$$

with initial condition

$$y(0)=0$$

Let Y be the differential transform of y.

By taking the differential transform of (E) using properties 1 - 5, the following recurrence relation is obtained:

$$(k+1)Y(k+1) = -\sum_{r=0}^{k} Y(r)Y(k-r) + \delta(k)$$

Since $y(0)=0$ then $Y(0)=\frac{1}{0!}y(0)=0$

If $k=0$ then $Y(1)=-\sum_{r=0}^{0} Y(0)Y(0-r)+\delta(0)$

$$Y(1) = -Y(0)Y(0)+1$$

$$Y(1)=1$$

Following the same recursive procedure to find $Y(2),Y(3),Y(4),Y(5)$ when $k=1$, $k=2$, and $k=3$ we have:

If $k=1$ then $2Y(2)=-\sum_{r=0}^{1} Y(r)Y(1-r)+\delta(1)$

$$Y(2)=0$$

We find in the same way $Y(2)=\frac{1}{3}$, $Y(4)=0$, $Y(5)=\frac{2}{15}$.

Since $x_0=0$, the differential inverse becomes:

$$y=\sum_{k=0}^{\infty} Y(k)x^k$$

then

$$y(t)=t-\frac{1}{3}t^3+\frac{2}{15}t^5-\frac{17}{315}t^7+\frac{62}{2835}t^9-\cdots$$

where the explicit solution is:

$$y(t)=\frac{e^{2t}-1}{e^{2t}+1}$$

Example 3

Solve (E)

$$\frac{dy}{dt}=-e^t y^2+2e^{2t}y+e^t-e^{-3t}$$

with initial condition

$$y(0)=1$$

Taking the differential in both sides gives:

$$(k+1)Y(k+1)=-\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} F_1(k_1)Y(k_2-k_1)Y(k-k_2)$$
$$+2\sum_{r=0}^{k} F_2(r)Y(k-r)+\frac{1}{k!}-\frac{(-3)^k}{k!}$$

$$(k+1)Y(k+1)=-\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{1}{k_1!}Y(k_2-k_1)Y(k-k_2)$$
$$+2\sum_{r=0}^{k}\frac{2^r}{r!}Y(k-r)+\frac{1}{k!}-\frac{(-3)^k}{k!}$$

The initial condition at $x_0 = 0$ is:

$$Y(0) = \frac{1}{0!} y(0) = 1$$

If $k = 0$ then

$$Y(1) = -\sum_{k_2=0}^{0} \sum_{k_1=0}^{k_2} \frac{1}{k_1!} Y(k_2 - k_1) Y(-k_2) + 2\sum_{r=0}^{0} \frac{2^r}{r!} Y(-r) + 1 - \frac{1}{1}$$

$$Y(1) = 2$$

If $k = 1$ then

$$2Y(2) = -\sum_{k_2=0}^{1} \sum_{k_1=0}^{k_2} \frac{1}{k_1!} Y(k_2 - k_1) Y(k - k_2) + 2\sum_{r=0}^{1} \frac{2^r}{r!} Y(1-r) + \frac{1}{1!} - \frac{-3}{1!}$$

$$Y(2) = \frac{7}{2}$$

If $k = 2$ then using the same procedure, find $Y(3) = -\frac{1}{6}$

Substituting $Y(0), Y(1), Y(2), Y(3), \cdots$ into the differential inverse transformation, we get the following solution:

$$y(t) = 1 + 2x + \frac{7}{2}x^2 - \frac{1}{6}x^3 + \cdots$$

5. Applications

In this section, we will see that many equations that arise in physics, cosmology and quantum mechanics can also be changed into a Riccati equation.

5.1. Quantum Mechanics and Schrodinger Equation

In Classical Mechanics, we describe the motion of an object using Newton's second law which states that the sum of the force (F) acting on an object of Mass (m) equals the mass of the object times the acceleration (a) of the object:

$$F = ma$$

In quantum mechanics that represents the analogy of classical mechanics, but at the smallest scale. For example, for electrons orbiting the nucleus of an atom Newton's Mechanics does not apply. Also, in classical mechanics, we describe a state of a physical system using the position and the momentum of the objects that compose the system. However, in Quantum mechanics, a particle is described using a wave function $\psi(x, y, z)$, where the square of the magnitude $|\psi|^2$ represents the probability that the particle is located at the position (x, y, z, t), for a single particle moving around a box in three dimension.

$\psi(x, y, z, t)$ is a solution of the Schrodinger equation:

$$\frac{i\hbar}{2\pi} \frac{\partial \psi}{\partial t} = -\frac{h^2}{8\pi^2 m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + V\psi$$

where h is a constant called then Planck's constant

$h = 6.626068 \times 10^{-34}\ \mathrm{m}^2 \cdot \mathrm{kg/s}$ and $\hbar = \dfrac{h}{2\pi}$ is the reduced Planck's constant.

i is an imaginary number $i = \sqrt{-1}$

m is the mass of the particle. V is the potential energy of the particle.

5.1.1. Relationship between Riccati Equation and Schrodinger Equation

Let's consider the one-dimensional time-independent Schrodinger equation (one particle)

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\psi(x,t)$$

If $V(x) = 0$, we deal with a free motion of the particle. If $V(x) = \frac{1}{2}mw^2x^2$, then we deal with the Harmonic oscillator with constant frequency $w = w_0$ or time-independent frequency w(t).

In either cases, the solution of the one dimensional time independent Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi$$

or

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2\hbar}w^2x^2\right)\psi$$

can be written as:

$$\psi(x,t) = A(t)\mathrm{e}^{i\alpha(x,y,t)} = A(t)\mathrm{e}^{i\left(\alpha(t)x^2 + \beta xy + y(t)y^2\right)}$$

The real-value functions of time $\alpha(t)$, $\beta(t)$, $y(t)$ satisfy the following system of ordinary differential equation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{m}{2\hbar}w^2 + \frac{2\hbar}{m}\alpha^2 = 0 \tag{2}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} + \frac{2\hbar}{m}\alpha\beta = 0 \tag{3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\hbar}{2m}\beta^2 = 0 \tag{4}$$

(2) is the Riccati equation that we can solve using differential transform.

Example 4 *Choose* $\frac{2\hbar}{m} = 1$, $w = \mathrm{e}^{-t}$.

Solve:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \mathrm{e}^{-2t} + \alpha^2 = 0 \tag{5}$$

Taking the differential transform in both sides of the equation

$$(k+1)\alpha(k+1) + \sum_{r=0}^{k}\alpha(r)\alpha(k-r) + \frac{(-2)^k}{k!} = 0, \quad \Rightarrow \alpha(0) = 0$$

If $k=0$ then $\alpha(1)+\sum_{r=0}^{0}\alpha(0-r)+1=0$, $\alpha(1)=-1$

If $k=1$ then $2\alpha(2)+\sum_{r=0}^{1}\alpha(r)\alpha(1-r)+\frac{-2}{1}=0$, $\alpha(2)=\frac{2}{2}=1$ using the same process, $\alpha(3)=-1$, $\alpha(4)=\frac{5}{6}$, $\alpha(5)=-\frac{11}{5}$, $\alpha(6)=\frac{59}{90}$, $\cdots$

The solution of (5) is given by:

$$\alpha(t)=-t+t^2-t^3+\frac{5}{6}t^4-\frac{11}{15}t^5+\frac{59}{90}t^6-\cdots$$

5.1.2. Time-Independent Schrodinger Equation and Riccati Equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi+(V-E)\psi=0 \quad \text{(Harmonic Oscillator)}$$

Suppose that the particle of mass m is confined in a box that lies in the interval $0\le x\le L$ Let's consider the case where the potential energy is

$$V(x)=\frac{mW_0^2}{2}x^2$$

Let $q=mW_0^2$ so that $V(x)=\frac{q}{2}x^2$ The Schrodinger equation then becomes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+\left(\frac{q}{2}x^2-E\right)\psi=0$$

Making the change of variable $y=\frac{\psi'}{\psi}$ leads to the equation

$$y'+y^2-\frac{2m}{\hbar^2}\left(\frac{q}{2}x^2-E\right)=0$$

$$y'+y^2-\frac{m}{\hbar^2}qx^2+\frac{2m}{\hbar^2}E=0$$

Solving

$$y'+y^2-\frac{mq}{\hbar^2}x^2+\frac{2m}{\hbar^2}E=0$$

We apply the Differential Transform in both sides of the equations

$$(k+1)Y(k+1)+\sum_{r=0}^{k}Y(r)y(k-r)-\frac{mq}{\hbar^2}\delta(k-2)+\frac{2m}{t^2}E\delta(k)=0$$

Then we obtain: $Y(0)=0$, $Y(1)=\frac{-2m}{\hbar^2}E$, $Y(2)=0$, $Y(3)=\frac{-m^2}{\hbar^4}E^2$, $Y(4)=0$,

$$Y(5)=\frac{-4}{5}\frac{m^3}{\hbar^6}E^3,\ Y(6)=0,\ Y(7)=\frac{-3}{5}\frac{m^4}{\hbar^8}E^4.$$

Choosing $w_0=\frac{E}{\hbar}$ so $q=m\frac{E^2}{\hbar^2}$.

We then have:

$$y=\frac{-2m}{\hbar^2}Ex-\frac{m^2}{\hbar^4}E^2x^3-\frac{4}{5}\frac{m^3}{\hbar^6}E^3x^5-\frac{3}{5}\frac{m^4}{\hbar^8}E^4x^7-\frac{4}{9}\frac{m^5}{\hbar^10}E^5x^9-\cdots$$

$$y = \frac{\psi'}{\psi}$$

$$\int \frac{\psi'}{\psi} \mathrm{d}x = \int y \mathrm{d}x = \frac{-m}{\hbar^2} E x^2 - \frac{m^2}{4\hbar^4} E^2 x^4 - \frac{4}{30} \frac{m^3}{\hbar^6} E^3 x^6$$
$$- \frac{3m^4}{40\hbar^8} E^4 x^8 - \frac{4}{90} \frac{m^5}{\hbar^{10}} E^5 x^{10} - \cdots$$

$$\psi(x) = c\mathrm{e}^{-\left(\frac{m^2}{\hbar} E x^2\right) - \frac{1}{4}\left(\frac{m}{\hbar^2} E x^2\right)^2 - \frac{2}{15}\left(\frac{m}{\hbar^2} E x^2\right)^3 - \frac{3}{40}\left(\frac{m}{\hbar^2} E x^2\right)^4 - \frac{4}{90}\left(\frac{m}{\hbar^2} E x^2\right)^5 - \cdots}$$

We show that

$$\psi(x) = A\cos\left(\sqrt{\frac{2mE}{\hbar^2}} x\right)\cosh\left(\frac{mE}{2\hbar^2} x^2\right)$$

We can find A by Normalizing $\psi(x)$. We obtain

$$A = \pm\frac{1}{2} \frac{1}{\int_0^L \cos^2\left(\sqrt{\frac{2mE}{\hbar^2}} x\right)\cosh^2\left(\frac{mE}{2\hbar^2} x^2\right)\mathrm{d}x}$$

5.2. Applications in Physics—Motion of a Particle in Central Force Field

In a central force field, the force acting on the particle of mass m has the properties that:

The force is always directed from a fixed point towards, or away from the particle.

The magnitude of the force only depends on the distance r from 0.

$\boldsymbol{F}$ is called a central force and the particle is said to move in a central force field. The point 0 is reference as the center of the force.

We have the following properties for a particle moving under the influence of a central force.

1) The path of the particle lies on a plane (planar)

2) The angular momentum of the particle is conserved

3) The time rate of change in an area swept by the position vector (from the point 0) is constant.

Using Figure 1, $\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$, and $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_r$ are unit vectors. That is, $\|\boldsymbol{e}_\theta\| = 1$ and $\|\boldsymbol{e}_r\| = 1$. Since $\boldsymbol{F}$ is always directed from a fixed point 0 towards or away from the particle, then:

$$\boldsymbol{F} = f(r)\boldsymbol{e}_r$$

If $f(r) < 0$, then the force is attracted towards 0.

If $f(r) > 0$, then $\boldsymbol{F}$ is repulsive away from 0.

5.2.1. Equation of Motion for a Particle under the Influence of a Central Force

The position of the particle is: $\boldsymbol{r} = r\boldsymbol{e}_r$

The velocity of the particle is: $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$

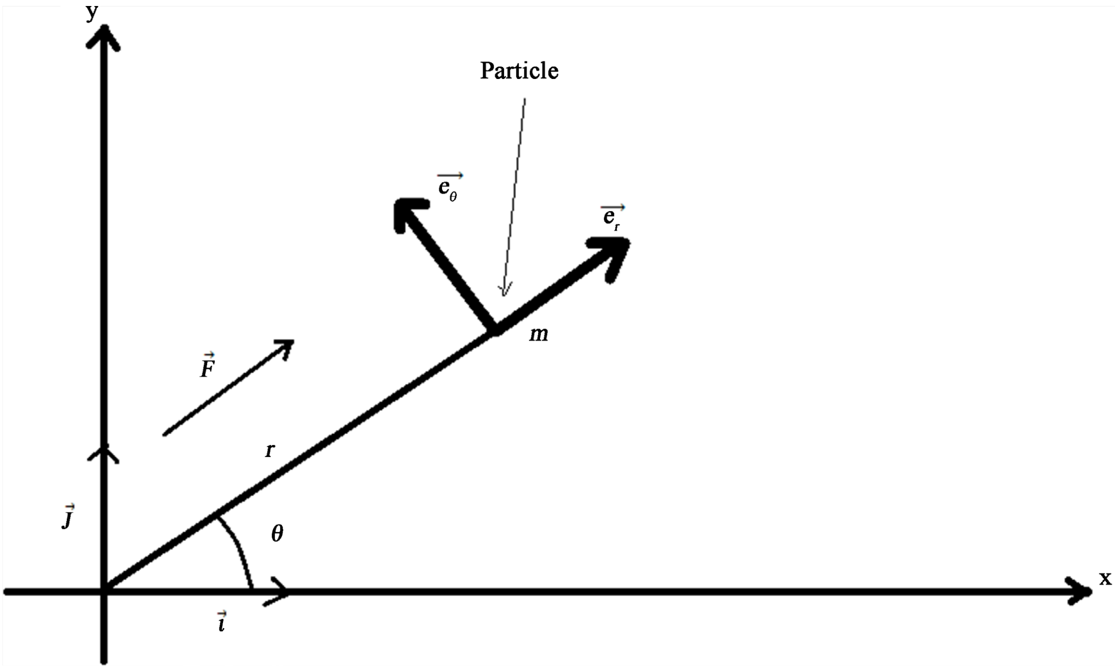

Figure 1. Illustration.

The acceleration is: $\boldsymbol{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\boldsymbol{e}_\theta$

Using the Newton's second law: $F = ma$

We obtain

$$\ddot{r} - \frac{c^2}{r^3} = \frac{f(r)}{m}$$

This is a nonlinear equation that we can turn to a Riccati equation using the change of variable $u = \frac{r'}{r}$ where $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$.

If $f(r) = \frac{k}{r^3}$ we, therefore, deal with the Riccati equation

$$u' - u^2 - 1 = \frac{k}{m}$$

that can be easily solved.

6. Conclusion

In this article, a number of integrability conditions of the Riccati equation have been discussed in Chapter 2. Method 5 has been added to the list of methods that eventually lead to an analytical solution. As the previous methods, the main idea is to turn the Riccati equation into a separable Differential equation. For each of these methods, the complexity of the method depends on the coefficients of the Riccati equation $\left(A(t), B(t), C(t)\right)$.

The notion of differential transform is also presented in this article. The small size of computation and the rapid convergence show that this method is very different and could be an alternative for solving non-linear differential equations over existing methods. The usefulness of the Riccati equation has been pointed out in the study of the Schrodinger equation in quantum mechanics. Some change of variable in the Schrodinger equation leads to the Riccati equation and the differential transform is used to write analytic solutions to the Schrodinger equation. The motion of a particle under the influence of a central force was explored, and again the equation of the motion found was turned into a Riccati equation using a change of variable.

A new result has also been presented in chapter 3 about solutions of the Riccati equations in the form:

$$y(t)=\frac{at+b}{ct+d}$$

where a, b, c and d are constants. It is shown that solutions of this form to the Riccati equation cannot exist unless it is a constant solution. A natural extension of this article would be to explore the general rational functions of the form):

$$y(t)=\frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomial of arbitrary degree.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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