mathematics

MDPI

Article

Queueing-Inventory with One Essential and m Optional Items with Environment Change Process Forming Correlated Renewal Process (MEP) †

Jaison Jacob [1], Dhanya Shajin [2], Achyutha Krishnamoorthy [3,*], Vladimir Vishnevsky [4,*] and Dmitry Kozyrev [4,5]

1 Department of Mathematics, St. Aloysius College, Elthuruth, Thrissur 680611, India; jaisjacobt@gmail.com
2 Department of Mathematics, Sree Narayana College, Chempazhanthy, Thiruvanathapuram 695587, India; dhanya.shajin@gmail.com
3 Centre for Research in Mathematics, CMS College, Kottayam 686001, India
4 V.A. Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 65 Profsoyuznaya Street, 117997 Moscow, Russia; kozyrev-dv@rudn.ru
5 Applied Probability and Informatics Department, Peoples' Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya Street, 117198 Moscow, Russia
* Correspondence: achyuthacusat@gmail.com or krishnamoorthy@cmscollege.ac.in (A.K.); vishn@inbox.ru (V.V.)

† This paper is an extended version of our paper published in *Ann. Oper. Res.* (February 2021). https://doi.org/10.1007/s10479-021-03975-8. Accepted on 30 January 2021, Published on 22 February 2021.

Abstract: We consider a queueing inventory with one essential and m optional items for sale. The system evolves in environments that change randomly. There are n environments that appear in a random fashion governed by a Marked Markovian Environment change process. Customers demand the main item plus none, one, or more of the optional items, but were restricted to at most one unit of each optional item. Service time of the main item is phase type distributed and that of optional items have exponential distributions with parameters that depend on the type of the item, as well as the environment under consideration. If the essential item is not available, service will not be provided. The lead times of optional and main items have exponential distributions having parameters that depend on the type of the item. The condition for stability of the system is analyzed by considering a multi-dimensional continuous time Markov chain that represent the evolution of the system. Under this condition, various performance characteristics of the system are derived. In terms of these, a cost function is constructed and optimal control policies of the different types of commodities are investigated. Numerical results are provided to give a glimpse of the system performance.

Keywords: essential item; optional item(s); random environment; Markovian arrival process; phase-type distribution; marked Markovian environment change process

Citation: Jacob, J.; Shajin, D.; Krishnamoorthy, A.; Vishnevsky, V.; Kozyrev, D. Queueing-Inventory with One Essential and m Optional Items with Environment Change Process Forming Correlated Renewal Process (MEP). *Mathematics* **2022**, *10*, 104. https://doi.org/10.3390/math10010104

Academic Editor: Alexander Zeifman

Received: 12 November 2021
Accepted: 23 December 2021
Published: 29 December 2021

1. Introduction

Queueing inventory models have been extensively analyzed since 1992. Very few of these discuss multi-commodity systems in randomly changing environments. Queueing systems which evolve under influences from external sources have, for a long time, inspired interest. In real life situations, inventory systems are often subject to randomly changing exogenous environment conditions that affect the demand for the product, the supply, and the cost structure. The area of queues in random environments is today a field of active research in applied probability. Queueing systems with correlated arrival flow of customers give adequate mathematical models for different real world systems including computer and telecommunication systems, and network protocols [1]. The following papers are relevant to the present paper only in that the authors consider multi-commodity inventory systems without any specified main and optional items. FaizAl-Khayyal et al. [2] consider a multi-commodity network model in maritime routing and scheduling. They tried to optimize the cost and the quantity of each commodity with constrained production rates, consumption rates, and storage capacities in each port. This paper addresses the common problem faced in the maritime transportation of petrochemical products. The problem is formulated as a mixed-integer non-linear programming problem.

Two-echelon, multi-commodity supply chain network design problem is considered by Hannan et al. [3]. The authors formulate the problem as a mixed-integer programming model in deterministic, single-period, and multi-commodity contexts. They develop a heuristic solution procedure based on Lagrangian relaxation due to the complexity of the problem. The problem deals with developing an optimum strategy in locating and sizing factories and warehouses and minimizing the total cost of the system which includes the costs associated with production, storage, transportation, and lead-times of commodities.

Jin QIN et al. [4] discuss an optimization problem in multi-commodity logistic network design. The problem is formulated as a non-linear mixed-integer programming model based on assumed normally distributed stochastic demands of the retailers. In this work, the strategic decisions regarding inventory controls and facility locations are incorporated simultaneously. The authors also developed a combined simulated annealing (CSA) algorithm to solve the problem. Optimization of the cost function involving costs associated with location, inventory, and transportation is also done.

Ronald et al. [5] examine a multi-commodity logistics network design problem with simultaneous emphasis on establishing ideal location facilities and distribution of commodities to ensure minimization of costs together with improved services. Factors such as the location of facilities and warehouses, storage capacity of the warehouses, and transportation routes involving all these are considered in the optimization problem. They later designed, tested, and compared a genetic algorithm and a specific problem heuristic on several realistic scenarios.

Claudio et al. [6] discuss multi-commodity inventory location models where inventory control policies are reviewed periodically and continuously under modular stochastic capacity constraints. The model is formulated as a mixed-integer non-linear programming model. The logistic problem of supplying certain commodities to the warehouses, which acts as customer service centers, from a single factory is under consideration. An objective function with factors associated with the selection of warehouses, type, and quantity of commodities to be assigned, type of customers to be served, etc., is optimized. They have developed a Lagrangian heuristic to obtain a feasible integer solution at each iteration of the subgradient method.

Ali et al. [7] examine a dynamic multi-commodity inventory and facility location problem in steel supply chain networks. Demand is assumed to be stochastic with normal distribution. In this model, the authors suggested a potential production capacity with emergency and shared safety stocks. The authors have presented a mixed-integer non-linear programming model and a mixed-integer linear programming model in this paper. The paper focuses on the strategic and tactical design of steel supply chain networks.

In Shajin et al. [8], the authors discuss a single server multi-commodity queueing inventory system with one essential and m optional items. They were the first to introduce the concept of optional items for sales/service. Customer arrival follows the Markovian arrival process, service completion with respect to the essential inventory follows Phase-type distribution and that with respect to optional inventories follows an exponential distribution. In this model, immediately after the service of an essential item, the customer either leaves the system with probability p or with probability $1 - p$ the customer goes for optional item(s). The system is assumed to be idle either in the absence of an essential item or when there is no customer in the system. Each customer is allowed to purchase only one unit of the essential item, whereas more than one type of optional items can be purchased with an imposed restriction of, at most, one unit per item. The stability condition is obtained by using the well-known fact that the left drift rate should be less than that of the right drift. Optimization of the control variables with respect to the cost function is also done numerically.

The following papers deal with queueing inventory systems influenced by randomly changing environments. In Song et al. [9], the authors consider an inventory model where the rate of demand is dependent on the environment variables. These variables can be anything, such as different stages in the life cycle of the particular inventory or changes in various factors linked with the economy, etc. They not only derived basic characteristics of the optimal policies but also observed the influence of various patterns in problem data on optimal policies and developed algorithms for computing optimal policies

Özekici et al. [10] describe inventory models with unreliable suppliers in randomly changing environments. The environment change follows a Markov chain. The dependence of the stock-flow equations of the system on random environments is represented by a two-dimensional stochastic process. Under specified conditions, they have derived an optimality condition for the base-stock policy and (s, S) policy. Computational issues and some extensions are also determined.

A single item inventory model which is observed periodically in a randomly changing environment is considered in Erdem et al. [11]. All the model parameters are dependent on a time-homogenous Markov chain environment. The replenishment quantity is minimum{Order quantity, Vendors capacity}. The problem is analyzed in single, multiple, and infinite periods. In all these cases, the authors prove that the optimal base-stock level depends on the state of the environment. Comparisons of the results with the case when the replenishment quantity equals the quantity ordered is also done.

Perry et al. [12] discuss production-inventory models with an unreliable facility operating in a two-state random environment. The system is characterized by a production machine. The production can even be stopped purposefully when there is a limited stocking capacity. When the machine is in ON period, the input into the buffer is assumed to be continuous and uniform until the threshold is reached, whereas the output from the buffer follows a compound Poisson process during OFF periods. Two different models are discussed and the factors controlling OFF periods are determined.

A continuous review (s, S) inventory system in a randomly changing environment is discussed in Feldman et al. [13] and its steady-state distribution obtained. The demand process is an environment dependent compound Poisson process when the environment is in a fixed state during an interval of time. The environmental process follows a continuous-time Markov process.

Kalpakam et al. [14] consider a lost sales (s, S) inventory system in a random environment. No backlog is allowed. The demand and supply rates are influenced by the environment process which is a finite irreducible Markov chain in continuous time. They have obtained the transform solution of the inventory level distribution and also an efficient algorithm to evaluate the long run system state is provided. Moreover, transient and limiting values of the mean reorder and shortage rates are also obtained. Goh et al. [15] discuss price-dependent inventory models with discount offers at random times. The offer is accepted when the inventory position is lower than a threshold level. Three different pricing policies are considered in which demand is induced by the retailer's price variation. They have obtained expressions for optimal order quantities, prices, and profits under the assumptions of constant demand rates.

Highlights of this paper are:

- It considers multi-commodity inventory with positive service time [16] in finite number of randomly changing environments;
- The first paper to introduce optional items for service in random environments;
- Except for one item (essential), all others are optional;
- The customer demand process follows Markovian arrival process (MAP);
- The environment change process follows marked Markovian environment arrival process (MMEAP)[n] of order n;
- Service time of customers, being served with the essential inventory follows phase type distribution and that w.r.t optional item(s) follows exponential distribution (depending

on the environment). The latter has a parameter, depending on the specific item(s) demanded by the customer.

The rest of the paper is organized as follows. The mathematical formulation of the model including the stability condition and the steady-state probability vector is described in detail in Section 2. Section 3 deals with some system performance measures and in Section 4, the construction of the cost function for optimizing the system control variables is discussed. Numerical illustrations and the numerical analysis of the cost function are discussed in Section 5. Section 6 gives the conclusion followed by references.

Notations and abbreviations used:

- (s, S) ordering policy: An inventory policy which says that when the inventory level falls below a certain minimum number s, the order for replenishment is made to restore the inventory to a maximum number S;
- e = Column vector of appropriate order with all its entries as $1'$s;
- $\bar{0}$ = Matrix of appropriate order with all its entries as 0;
- I_k = Identity matrix of order k;
- $[G]_{pq} = (p, q)th$ element of the matrix G;
- $CTMC$: Continuous time Markov chain;
- $LIQBD$: Level Independent Quasi-Birth and Death process;
- MAP = Markovian arrival process;
- $(MMEAP)[n]$ = Marked Markovian environment arrival process (with n distinct environments);
- $C \otimes D$ = The Kronecker product of two given matrices $C_{m\times n}$ and $D_{p\times q}$, given by $([C]_{pq}D)$ of order $mp \times nq$;
- $C \oplus D$ = The Kronecker sum of two square matrices B and C of orders m and n, respectively, given by $C \otimes I_n + I_m \otimes D$;
- C_{cor} = The Correlation coefficient;
- MAP^p = Markovian arrival process with positive C_{cor};
- MAP^n = Markovian arrival process with negative C_{cor};
- $PH(\gamma, T)$: Phase type distribution with the initial probability vector γ and the transition generator matrix T.

2. Mathematical Formulation

Consider a single server multi-commodity queueing inventory system with one essential and m optional inventories in n random environments. Only one environment will be in operation at any given time. The arrival of customers follows Markovian arrival process (MAP) with representation (H_0, H_1), where each H_i for $0 \le i \le 1$ is of order m_3. The generator matrix of the underlying $CTMC$ $(\delta(t), t \ge 0)$ on the state space $\{1, 2, 3, \ldots, m_3\}$ is given by $H = H_0 + H_1$. These matrices, H_0 and H_1, are of the form

$$H_0 = \begin{bmatrix} h_{11}^{(0)} & h_{12}^{(0)} & \dots & h_{1m_3}^{(0)} \\ h_{21}^{(0)} & h_{22}^{(0)} & \dots & h_{2m_3}^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m_31}^{(0)} & h_{m_32}^{(0)} & \dots & h_{m_3m_3}^{(0)} \end{bmatrix}, H_1 = \begin{bmatrix} h_{11}^{(1)} & h_{12}^{(1)} & \dots & h_{1m_3}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)} & \dots & h_{2m_3}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m_31}^{(1)} & h_{m_32}^{(1)} & \dots & h_{m_3m_3}^{(1)} \end{bmatrix}$$

where, $h_{ii}^{(0)} = -\left(\sum_{j=1,j\neq i}^{m_3} h_{ij}^{(0)} + \sum_{j=1}^{m_3} h_{ij}^{(1)}\right)$ for $1 \le i \le m_3$. Thus, $h_{ij}^{(1)}, 1 \le i, j \le m_3$ gives the transition rate from ith state to jth state through an arrival, while $h_{ij}^{(0)}, 1 \le i, j \le m_3$, gives the transition from ith state to jth state without an arrival. Note that the transition rate between the ith states, given by $h_{ii}^{(1)}, 1 \le i \le m_3$ occurs only with an arrival. Let η be the steady-state probability vector of H. Then, η satisfy ηH = 0 and ηe = 1. The fundamental rate λ^A of this MAP is given by $\lambda^A = \eta H_1 e$ which gives the expected number of arrivals per unit of time. The coefficient of variation C_{var} of intervals between arrivals is calculated as $C_{var} = 2\lambda^A \eta(-H_0)^{-1}e - 1$ and coefficient of correlation C_{cor} of intervals between successive arrivals is given as $C_{cor} = (\lambda^A \eta(-H_0)^{-1}H_1(-H_0)^{-1}e - 1)/C_{var}$.

There are n environments that occur randomly and the occurrence of the environments follows marked Markovian environment arrival process ($MMEAP[n]$) with representation (D_0,D_1,D_2,...,D_n), where each D_i for $0 \leq i \leq n$ is of order m_2. As stated earlier, only one environment will be in operation at any given time. The change in environment is directed by the stochastic process $\{\mathcal{V}(t); t \geq 0\}$ which is an irreducible continuous time Markov chain with the state space $\{1, 2, \ldots, m_2\}$. The sojourn time of this chain in the state $v, 1 \leq v \leq m_2$, is exponentially distributed with parameter $\lambda^{(v)}$. When the sojourn time in the state v expires,the process $\{\mathcal{V}(t); t \geq 0\}$ jumps to the state v' without any change in the environment with probability $p^{(0)}_{(v,v')}$ where $v, v' \in \{1, 2, \ldots, m_2\}, v \neq v'$. On the other hand, the process $\{\mathcal{V}(t); t \geq 0\}$ jumps to the state v' with the arrival of lth environment with probability $p^{(l)}_{(v,v')}$ where $v, v' \in \{1, 2, \ldots, m_2\}, v \neq v'$ and $l \in \{1, 2, \ldots, n\}$.

The behavior of the $MMEAP$ is completely characterized by the matrices D_l, $l = 0, 1, 2, \ldots n$ defined by

$$D_0 = \begin{bmatrix} -\lambda^{(1)} & \lambda^{(1)} p^{(0)}_{1,2} & \cdots & \lambda^{(1)} p^{(0)}_{1,m_2} \\ \lambda^{(2)} p^{(0)}_{2,1} & -\lambda^{(2)} & \cdots & \lambda^{(2)} p^{(0)}_{2,m_2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{(m_2)} p^{(0)}_{m_2,1} & \lambda^{(m_2)} p^{(0)}_{m_2,2} & \cdots & -\lambda^{(m_2)} \end{bmatrix},$$

$$D_l = \begin{bmatrix} \lambda^{(1)} p^{(l)}_{1,1} & \lambda^{(1)} p^{(l)}_{1,2} & \cdots & \lambda^{(1)} p^{(l)}_{1,m_2} \\ \lambda^{(2)} p^{(l)}_{2,1} & \lambda^{(2)} p^{(l)}_{2,2} & \cdots & \lambda^{(2)} p^{(l)}_{2,m_2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{(m_2)} p^{(l)}_{m_2,1} & \lambda^{(m_2)} p^{(l)}_{m_2,2} & \cdots & \lambda^{(m_2)} p^{(l)}_{m_2,m_2} \end{bmatrix},$$

$l = \{1, 2, \ldots, n\}$ The matrix $D = \sum_{l=0}^{n} D_l$ represents the generator of the process $\{\mathcal{V}(t); t \geq 0\}$.

Service time of those customers who are served with the essential item is phase type distributed with representation $PH(\gamma, T)$ of order m_1. This service time is the time until the undergoing Markov chain $(\zeta(t), t \geq 0)$ with a finite state space $\{1, 2, 3, \ldots, m_1 + 1\}$ reaches the absorbing state $m_1 + 1$. $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{m_1})$ gives the initial probability of starting in any of the m_1 states. T is the generator matrix that gives transition rates within the states $\{1, 2, 3, \ldots, m_1\}$. The absorption rates from the individual transient states $\{1, 2, 3, \ldots, m_1\}$ to the absorption state $m_1 + 1$ is given by $T^0 = -Te$. $\mu' = -\gamma T^{-1}e$ gives the mean service of the customer.

Service time of those customers who are served with the optional items are environment dependent and they are exponentially distributed with parameter μ^k_i, where $1 \leq k \leq n$ and $i \in \{i_1, (i_1 i_2), (i_1 i_2 i_3), \ldots, (i_1 i_2 i_3 \ldots i_m)\}$. It is important to note that no order preference has been given to any element, i.e., $(i_j i_l) = (i_l i_j)$, and so on where each $i_l \in \{1, 2, 3, \ldots, m\}$ with $1 \leq j \neq l \leq m$.

In this model, a customer is allowed to demand exactly one unit of the essential inventory where as more than one type of optional inventory can be demanded by a customer with an imposed restriction of, at most, one item from each optional inventories. Service rates of the optional items are assumed to be environment dependent. The ith optional item is served in the kth environment with probability p^k_i, similarly the lth and rth optional inventories are served in the kth environment with probability p^k_{lr}, and so on. If the demanded optional inventory is not available, the customer is expected to quit the system after acquiring the essential item together with those available optional inventories. The server is assumed to be in the idle state in the absence of customers, as well as essential inventories. Essential and optional inventories have exponentially distributed positive lead time with parameters β and β_j for $1 \leq j \leq m$, respectively. The essential inventories are under the (s, S) control policy whereas the environment dependent optional inventories are under the (s_i^k, S_i) for $i = 1, 2, 3, \ldots, m$ and $k = 1, 2, 3, \ldots, n$ control policies in the kth environment.

At any given time t, let $N(t)$, $S(t)$, $E(t)$, $O_l(t)$, $J_1(t)$, $J_2(t)$, and $J_3(t)$ denote, respectively, number of customers in the system, status of environment, number of essential inventory items, number of lth optional inventory items for $1 \leq l \leq m$, service phase, environment phase and arrival phase of the customers. The status of the server at any given time t is defined as,

$$C(t) = \begin{cases} 0^* & \text{Idle server} \\ 0 & \text{Essential inventory service} \\ l & lth \text{ optional inventory service} \\ \vdots & \\ lr & \text{combined service of } lth \text{ and } rth \text{ optional inventories} \\ \vdots & \\ lru & \text{combined service of } lth, rth \text{ and } uth \text{ optional inventories} \\ \vdots & \\ 123, \ldots, m & \text{combined service of all optional inventories} \end{cases}$$

Let Δ be the collection of all the permitted combinations of different optional inventories and let C_u denotes the server status, in general, for the combined service of u optional items, for $1 \leq u \leq m$. Thus, the process $\Gamma = \{(N(t), S(t), E(t), C(t), O_1(t), O_2(t), \ldots, O_m(t), J_1(t), J_2(t), J_3(t)), t \geq 0\}$ is a $CTMC$ which is a level independent quasi-birth and death process($LIQBD$) with state space as follows

$\{(0, j, i, 0^*, i_1, i_2, \ldots, i_m, j_2, j_3), 0 \leq i \leq S, 1 \leq j \leq n, 0 \leq i_r \leq S_r^j$ for $1 \leq r \leq m$, $1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\}$
$\bigcup \{(\bar{n}, j, 0, 0^*, i_1, i_2, \ldots, i_m, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq n, 0 \leq i_r \leq S_r^j$ for $1 \leq r \leq m, 1 \leq j_2 \leq m_2$, $1 \leq j_3 \leq m_3\}$
$\bigcup \{(\bar{n}, j, i, 0, i_1, i_2, \ldots, i_m, j_1, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq n, 1 \leq i \leq S, 0 \leq i_r \leq S_r^j$ for $1 \leq r \leq m$, $1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\}$
$\bigcup \{(\bar{n}, j, i, C_1, i_1, i_2, \ldots, i_m, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq n, 1 \leq i \leq S, C_1 \in \Delta, 1 \leq j_2 \leq m_2$, $1 \leq j_3 \leq m_3$ for if $C_1 = l$ where $1 \leq l \leq m$ then $0 \leq i_k \leq S_k^j$ for $k \in \{1, 2, \ldots, m\} - \{l\}$ and $1 \leq i_l \leq S_l^j\}$
$\bigcup \{(\bar{n}, j, i, C_2, i_1, i_2, \ldots, i_m, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq n, 1 \leq i \leq S, C_2 \in \Delta, 1 \leq j_2 \leq m_2$, $1 \leq j_3 \leq m_3$ for if $C_2 = lq$ where $l \neq j, 1 \leq l, q \leq m$ then $0 \leq i_k \leq S_k^j$ for $k \in \{1, 2, \ldots, m\} - \{l, q\}$ and $1 \leq i_k \leq S_k^j$ for $k \in \{l, q\}\}$
$\bigcup \{(\bar{n}, j, i, C_3, i_1, i_2, \ldots, i_m, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq n, 1 \leq i \leq S, C_3 \in \Delta, 1 \leq j_2 \leq m_2$, $1 \leq j_3 \leq m_3$ for if $C_3 = hql$ where $h \neq q \neq l, 1 \leq h, q, l \leq m$ then $0 \leq i_k \leq S_k^j$ for $k \in \{1, 2, \ldots, m\} - \{h, q, l\}$ and $1 \leq i_k \leq S_k^j$ for $k \in \{h, q, l\}\} \bigcup \ldots \bigcup \{(\bar{n}, j, i, 12\ldots m, i_1, i_2, \ldots, i_m, j_2), \bar{n} \geq 1, 1 \leq j \leq n, 1 \leq i \leq S, 12\ldots m \in \Delta, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3, 1 \leq i_k \leq S_k^j$ for $k \in \{1, 2, \ldots, m\}\}$.

The infinitesimal generator $\mathcal{Q}$ of the system is of the form

$$\mathcal{Q} = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Matrices A_{01} and A_0 are of order $a \times b$ and $a \times a$, respectively, their entries are due to the arrival of customers following MAP with representation (H_0, H_1). Matrices A_{10} and A_2 are of order $c \times a$ and $c \times c$, respectively, their entries are due to the service of essential inventories following phase type distribution with representation $PH(\gamma, T)$ and also due to the environment dependent, exponentially distributed service of optional inventories. Matrices A_{00} and A_1 are square matrices of order a and c, respectively, their entries includes the replenishment rates of the inventories in addition to the negative sign of sum of other entries of the same row found in A_{01}, A_0, A_{10}, and A_2, where $a = n(S+1)\Pi_{k=1}^{m}(S_k+1)m_2m_3$, $b = n(S+1)\Pi_{k=1}^{m}(S_k+1)m_1m_2m_3$ and $c = n(\Pi_{k=1}^{m}(S_k+1)m_2m_3 + S\sum_{u\in\Delta} l_u)$.

When $u = 0$, then $l_0 = \pi_{k=1}^{m}(S_k+1)m_1m_2m_3$.

When $u = i$, then $l_i = \pi_{k=1,k\neq i}^{m}(S_k+1)S_im_2m_3$ for $1 \leq i \leq m$.

When $u = ij$, then $l_{ij} = \pi_{k=1,k\neq i,j}^{m}(S_k+1)\Pi_{k\in\{i,j\}}S_km_2m_3$ for $1 \leq i,j \leq m$

When $u = hij$, then $l_{hij} = \pi_{k=1,k\neq h,i,j}^{m}(S_k+1)\Pi_{k\in\{h,i,j\}}S_km_2m_3$ for $1 \leq h,i,j \leq m$ and so on.

When $u = 12\dots m$, then $l_{12\dots m} = \pi_{k=1}^{m}S_km_2m_3$.

In order to have a better understanding of the system, a detailed illustration of the model has been provided in Appendix A by fixing the number of optional items $m = 2$ and the number of environments $n = 2$. All the transitions and resultant component matrices are shown clearly in the Appendix A.

2.1. Stability Condition

Let $\pi = (\pi_0, \pi_1, \pi_2, \dots, \pi_S)$ be the steady-state probability vector of $A = A_0 + A_1 + A_2$, where $\pi_k = (\pi_k^{(1)}, \pi_k^{(2)}, \dots, \pi_k^{(n)})$ for $0 \leq k \leq S$.

Then

$$\pi A = 0, \pi e = 1. \tag{1}$$

Refer to Appendix A for the component matrix representations From (1),

$$\pi_0 N_0 + \pi_1 M_0 = 0 \tag{2}$$

$$\pi_i N_1 + \pi_{i+1} M = 0, 1 \leq i \leq s \tag{3}$$

$$\pi_i N_2 + \pi_{i+1} M = 0, s+1 \leq i \leq S-1 \tag{4}$$

$$\pi_0 Z^0 + \sum_{i=1}^{s} \pi_i Z + \pi_S N_2 = 0 \tag{5}$$

where,

$$\begin{gathered} N_0 = [Z_0^{ij}]; M_0 = [M_0^{ij}] \text{ for } 1 \leq i,j \leq n \\ N_k = [L^{ij} + Z_k^{ij} + \bar{M}^{ij}], 1 \leq k \leq 2 \text{ for } 1 \leq i,j \leq n \\ M = [M^{ij}]; Z^0 = [Z^{0ij}]; Z = [Z^{ij}] \text{ for } 1 \leq i,j \leq n \end{gathered}$$

Solving Equations (2)–(5), we get

$$\pi_i = \begin{cases} \pi_S \mathcal{W}_0 & i = 0 \\ \pi_S \mathcal{W}_i & 1 \leq i \leq s \\ \pi_S \hat{\mathcal{W}}_i & s+1 \leq i \leq S \end{cases} \tag{6}$$

where,

$$\mathcal{W}_0 = (-1)^S (MN_2^{-1})^{S-s-1} (MN_1^{-1})^s (M_0 N_0^{-1}) \tag{7}$$

$$\mathcal{W}_i = (-1)^{S-i} (MN_2^{-1})^{S-s-1} (MN_1^{-1})^{s+1-i} \tag{8}$$

$$\hat{\mathcal{W}}_i = (-1)^{S-i} (MN_2^{-1})^{S-i} \tag{9}$$

The only unknown probability vector π_S is obtained from the normalizing condition

$$\pi_S \left(\mathcal{W}_0 + \sum_{i=1}^{s} \mathcal{W}_i + \sum_{i=s+1}^{S} \hat{\mathcal{W}}_i \right) \boldsymbol{e} = 1.$$

Theorem 2.1. *The necessary and sufficient condition for the stability of queuing inventory system under study is*

$$\pi_S \mathcal{H}_0 \mathcal{V}_0 < \pi_S(\mathcal{H}_1 \mathcal{V}_1 + \mathcal{H}_2 \mathcal{V}_2)$$

Proof. The queueing system with the generator $\mathcal{Q}$ under study is stable if, and only if,

$$\pi A_0 e < \pi A_2 e \tag{10}$$

Refer to Appendix A for the component matrix representations.

Using Equations (6)–(9) together with the matrices A_0 and A_2 we get

$$\pi A_0 e = \pi_S \left(\sum_{i=1}^{s} \mathcal{W}_i + \sum_{i=s+1}^{S} \hat{\mathcal{W}}_i \right) L.e$$

where $L = [L^{ij}]$ for $1 \le i, j \le n$,

$$\pi A_2 e = \pi_S \left[\mathcal{W}_1 (M_0 + \bar{M}) + \left(\sum_{i=2}^{s} \mathcal{W}_i + \sum_{i=s+1}^{S} \hat{\mathcal{W}}_i \right) (\bar{M} + M) \right] .e$$

where $\bar{M} = [\bar{M}^{ij}]$ for $1 \le i, j \le n$. Let

$$\mathcal{H}_0 = \sum_{i=1}^{s} \mathcal{W}_i + \sum_{i=s+1}^{S} \hat{\mathcal{W}}_i, \quad \mathcal{V}_0 = L.e, \quad \mathcal{H}_1 = \mathcal{W}_1, \quad \mathcal{V}_1 = (M_0 + \bar{M}).e,$$

$$\mathcal{H}_2 = \sum_{i=2}^{s} \mathcal{W}_i + \sum_{i=s+1}^{S} \hat{\mathcal{W}}_i \ \ \& \ \ \mathcal{V}_2 = (\bar{M} + M).e$$

Then, by (10) we get the stated result. □

2.2. Steady State Probability Vector

Let $\mathbf{x}$ denote the steady state probability vector of the generator $\mathcal{Q}$. Then, we have

$$\mathbf{x}\mathcal{Q} = 0, \qquad \mathbf{xe} = 1. \tag{11}$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$, from (23) we get

$$\begin{gathered} \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} = 0 \\ \mathbf{x}_0 A_{01} + \mathbf{x}_1 A_1 + \mathbf{x}_2 A_2 = 0 \\ \mathbf{x}_{n-1} A_0 + \mathbf{x}_n A_1 + \mathbf{x}_{n+1} A_2 = 0; n \ge 2 \end{gathered} \tag{12}$$

By assuming the stability condition, we see that $\mathbf{x}$ is obtained as (see [17])

$$\mathbf{x}_n = \mathbf{x}_1 R^{n-1}; n \ge 2, \tag{13}$$

where R is the minimal non-negative solution of the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0 \tag{14}$$

The boundary conditions are given by

$$\begin{gathered} \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} = 0 \\ \mathbf{x}_0 A_{01} + \mathbf{x}_1 [A_1 + R A_2] = 0 \end{gathered} \tag{15}$$

From Equation (23) we get,

$$\mathbf{x}_1 = \mathbf{x}_0 \mathcal{K} \tag{16}$$

and by the normalizing condition in (23), we get

$$[\mathbf{x}_0 + \mathbf{x}_0\mathcal{K}(I-R)^{-1}]\mathbf{e} = 1 \tag{17}$$

where

$$\mathcal{K} = (-A_{01})(A_1 + RA_2)^{-1} \tag{18}$$

3. System Performance Measures

1. Expected re-ordering rate of the essential inventory

$$E_{RE} = \mu' \sum_{r=1}^{\infty}\sum_{t=1}^{n}\sum_{k=1}^{m}\sum_{i_k=1}^{S_k}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} \mathbf{x}_r(t, s+1, 0, i_1, \ldots, i_m, j_1, j_2, j_3)$$

2. Expected re-ordering rate of lth optional item in the ath environment, $1 \leq l \leq m$, $1 \leq a \leq n$

$$E_{RO}(l)^a = \mu_l^a \sum_{r=1}^{\infty}\sum_{i=1}^{S}\Big(\sum_{\substack{u\in\Delta\\ l\in u}}\sum_{\substack{i_k=0,\\ k\notin u,\\ 1\leq k\leq m}}^{S_k}\sum_{\substack{i_k=1,\\ k\in u,\\ 1\leq k\leq m}}^{S_k}\Big)\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} \mathbf{x}_r(a, i, u, i_1, \ldots, i_{h-1}, s_l^a+1, i_{h+1}, \ldots, j_2, j_3)$$

3. Expected number of customers in the system $E_C = \sum_{i=1}^{\infty} i.\mathbf{x}_i.e.$
4. Expected number of essential inventories in the system

$$\begin{aligned}
E_{EI} &= \sum_{t=1}^{n}\sum_{i=1}^{S}\sum_{k=1}^{m}\sum_{i_k=1}^{S_k}\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} i.\mathbf{x}_0(t, i, 0^*, i_1, \ldots, i_m, j_2, j_3) \\
&+ \sum_{r=1}^{\infty}\sum_{t=1}^{n}\sum_{i=1}^{S}\sum_{k=1}^{m}\sum_{i_k=1}^{S_k}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} i.\mathbf{x}_r(t, i, 0, i_1, \ldots, i_m, j_1, j_2, j_3) \\
&+ \sum_{r=1}^{\infty}\sum_{t=1}^{n}\sum_{i=1}^{S}\Big(\sum_{u\in\Lambda}\sum_{\substack{i_k=0,\\ k\notin u,\\ 1\leq k\leq m}}^{S_k}\sum_{\substack{i_k=1,\\ k\in u,\\ 1\leq k\leq m}}^{S_k}\Big)\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} i.\mathbf{x}_r(t, i, u, i_1, \ldots, i_m, j_2, j_3)
\end{aligned}$$

5. Expected number of lth optional inventories in the system for $1 \leq l \leq m$.

$$\begin{aligned}
E_{OI}(l) &= \sum_{t=1}^{n}\sum_{i=1}^{S}\sum_{k=1}^{m}\sum_{i_k=1}^{S_k}\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} i_l.\mathbf{x}_0(t, i, 0^*, i_1, \ldots, i_m, j_2, j_3) \\
&+ \sum_{r=1}^{\infty}\sum_{t=1}^{n}\sum_{i=1}^{S}\sum_{k=1}^{m}\sum_{i_k=1}^{S_k}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} i_l.\mathbf{x}_r(t, i, 0, i_1, \ldots, i_m, j_1, j_2, j_3) \\
&+ \sum_{r=1}^{\infty}\sum_{t=1}^{n}\sum_{i=1}^{S}\Big(\sum_{u\in\Lambda}\sum_{\substack{i_k=0,\\ k\neq l, k\notin u,\\ 1\leq k\leq m}}^{S_k}\sum_{\substack{i_k=1,\\ k=l, k\in u,\\ 1\leq k\leq m}}^{S_k}\Big)\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} i_l.\mathbf{x}_r(t, i, u, i_1, \ldots, i_m, j_2, j_3)
\end{aligned}$$

6. Expected loss rate of customers in the absence of essential item

$$E_L = \lambda^A \sum_{r=1}^{\infty}\sum_{t=1}^{n}\sum_{k=1}^{m}\sum_{i_k=1}^{S_k}\sum_{j_2=1}^{m_2}\sum_{j_3=1}^{m_3} \mathbf{x}_r(t, 0, 0^*, i_1, \ldots, i_m, j_2, j_3)$$

4. Cost Function

In-order to optimize the inventory levels s, S, s_i^k, and S_i for $1 \leq i \leq m, 1 \leq k \leq n$, we construct the following cost function,

$$K(s, s_1^1, \ldots, s_1^n, \ldots, s_m^1, \ldots, s_m^n, S, S_1, \ldots, S_m) = C^0 E_{R_E} + \sum_{i=1}^{m} \sum_{j=1}^{n} C_i^j E_{RO}(i)^j$$
$$+ C_{EI} E_{EI} + \sum_{i=1}^{m} C_{OI}(i) E_{OI}(i) + C_1 E_C + C_2 E_L,$$

where

1. C^0 = Fixed set up cost per unit of the essential item;
2. C_i^j = Fixed set up cost per unit of the ith optional item in the jth environment;
3. C_{EI} = Carrying cost per unit of the essential item;
4. $C_{OI}(i)$ = Carrying cost per unit of the ith optional item;
5. C_1 = Customer holding cost per unit time;
6. C_2 = Cost due to loss of goodwill per unit time, in the absence of the essential item.

5. Numerical Illustration

In this section, we provide the numerical illustration of the system performance measures with varied values of the underlying parameters. The model with one essential and two optional inventories is considered here.

The phase-type service process of the customer is characterized by

$$T = \begin{pmatrix} -8 & 4 \\ 1 & -4 \end{pmatrix}, T^0 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \gamma = \begin{pmatrix} 0.3 & 0.7 \end{pmatrix}$$

where the mean service time is obtained as $\mu^{'} = 0.3107$.

To exhibit the correlation effect, we introduce two MAP arrival of customers coded ass MAP^p and MAP^n for the customer arrivals with positive correlation coefficient and negative correlation coefficient, respectively.

- MAP^p is defined by

  $$H_0 = \begin{pmatrix} -3.45 & 0.7 \\ 0.85 & -5 \end{pmatrix}, H_1 = \begin{pmatrix} 2.3 & 0.45 \\ 0.5 & 3.65 \end{pmatrix}$$

  The coefficient of correlation $C_{cor} = +0.13$.
- MAP^n is defined by

  $$H_0 = \begin{pmatrix} -3.35 & 0.6 \\ 0.85 & -4.105 \end{pmatrix}, H_1 = \begin{pmatrix} 2.3 & 0.45 \\ 0.005 & 3.25 \end{pmatrix}$$

  The coefficient of correlation $C_{cor} = -0.23$.

Tables 1 and 2 show the effect of μ_1^1 and μ_1^2, respectively, the environment dependent and exponentially distributed service completion rates of the first optional inventory on different performance measures. From the table, it is clear that the values of the performance measures show similar behavior in both the cases, with MAP^p and MAP^n, respectively. Values of $E_{ROI_1^1}, E_{ROI_1^2}, E_{ROI_2^1}, E_{ROI_2^2}, E_{EI}, E_{OI(1)}, E_{OI(2)}$, and E_L seems to increase, respectively, with increased values of μ_1^1 and μ_1^2 whereas, the values of E_{RE} and E_C seems to decrease, respectively, with the increased values of μ_1^1 and μ_1^2.

Table 1. Effects of μ_1^1 (fix $\mu_1^2 = 4$) and μ_1^2 (fix $\mu_1^1 = 5$): Fix $S = 4, S_1 = 3, S_2 = 3, s = 2, s_1^1 = 1, s_1^2 = 1, s_2^1 = 1, s_2^2 = 2, m1 = m2 = m3 = 2, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3$.

MAP^p						
3	4	5	$\leftarrow \mu_1^2$: $\mu_1^1 \rightarrow$	5	4	3
4.5×10^{-3}	2.9×10^{-3}	2.5×10^{-3}	E_{RE}	2.9×10^{-3}	3.8×10^{-3}	6.5×10^{-3}
4.4×10^{-3}	5.2×10^{-3}	5.4×10^{-3}	$E_{ROI_1^1}$	5.2×10^{-3}	4.2×10^{-3}	2.4×10^{-3}
3.8×10^{-3}	6.0×10^{-3}	7.9×10^{-3}	$E_{ROI_1^2}$	6.0×10^{-3}	5.7×10^{-3}	4.1×10^{-3}
1.3×10^{-3}	1.58×10^{-3}	1.65×10^{-3}	$E_{ROI_2^1}$	1.6×10^{-3}	1.4×10^{-3}	9.4×10^{-4}
1.9×10^{-3}	2.2×10^{-3}	2.248×10^{-3}	$E_{ROI_2^2}$	2.2×10^{-3}	2.1×10^{-3}	1.5×10^{-3}
0.8010	0.9202	0.9568	E_{EI}	0.9202	0.8524	0.6213
0.6550	0.5306	0.5220	E_C	0.5306	0.5648	1.0821
0.2643	0.2790	0.2837	$E_{OI(1)}$	0.2790	0.2691	0.2358
0.3040	0.3261	0.3331	$E_{OI(2)}$	0.3261	0.3122	0.2649
3.11×10^{-5}	3.76×10^{-5}	4.01×10^{-5}	E_L	3.76×10^{-5}	3.45×10^{-5}	2.35×10^{-5}

Table 2. Effects of μ_1^1 (fix $\mu_1^2 = 4$) and μ_1^2 (fix $\mu_1^1 = 5$): Fix $S = 4, S_1 = 3, S_2 = 3, s = 2, s_1^1 = 1, s_1^2 = 1, s_2^1 = 1, s_2^2 = 2, m1 = m2 = m3 = 2, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3$.

MAP^n						
3	4	5	$\leftarrow \mu_1^2$: $\mu_1^1 \rightarrow$	5	4	3
3.8×10^{-3}	3.0×10^{-3}	2.7×10^{-3}	E_{RE}	3.0×10^{-3}	3.5×10^{-3}	4.7×10^{-3}
5.2×10^{-3}	5.6×10^{-3}	5.7×10^{-3}	$E_{ROI_1^1}$	5.6×10^{-3}	4.6×10^{-3}	3.5×10^{-3}
4.9×10^{-3}	7.1×10^{-3}	9.2×10^{-3}	$E_{ROI_1^2}$	7.1×10^{-3}	6.9×10^{-3}	6.4×10^{-3}
1.6×10^{-3}	1.7×10^{-3}	1.8×10^{-3}	$E_{ROI_2^1}$	1.8×10^{-3}	1.7×10^{-3}	1.4×10^{-3}
2.3×10^{-3}	2.4×10^{-3}	2.5×10^{-3}	$E_{ROI_2^2}$	2.5×10^{-3}	2.4×10^{-3}	2.29×10^{-2}
1.0923	1.1719	1.2028	E_{EI}	1.1719	1.1107	0.9824
0.3668	0.3316	0.3264	E_C	0.3316	0.3455	0.4189
0.3078	0.3173	0.3213	$E_{OI(1)}$	0.3173	0.3071	0.2886
0.3468	0.3596	0.3649	$E_{OI(2)}$	0.3596	0.3470	0.3232
1.58×10^{-5}	1.74×10^{-5}	1.82×10^{-5}	E_L	1.74×10^{-5}	1.65×10^{-5}	1.43×10^{-5}

Tables 3 and 4 show the effect of μ_2^1 and μ_2^2, respectively, the environment dependent and exponentially distributed service completion rates of the second optional inventory on different performance measures. From the table, it is clear that the values of the performance measures show similar behavior in both the cases, with MAP^p and MAP^n, respectively. The values of E_{RE}, $E_{ROI_1^1}$, $E_{ROI_1^2}$, $E_{ROI_2^1}$, $E_{ROI_2^2}$, E_{EI}, $E_{OI(1)}$, $E_{OI(2)}$, and E_L seems to increase, respectively, with increased values of μ_2^1 and μ_2^2 whereas the value of E_C seems a decrease, respectively, with the increased values of μ_2^1 and μ_2^2.

Table 3. Effects of μ_2^1 (fix $\mu_2^2 = 4$) and μ_2^2 (fix $\mu_2^1 = 5$): Fix $S = 4, S_1 = 3, S_2 = 3, s = 2, s_1^1 = 1, s_1^2 = 1, s_2^1 = 1, s_2^2 = 2, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3$.

MAP^p						
3	4	5	$\leftarrow \mu_2^2$: $\mu_2^1 \rightarrow$	5	4	3
2.0×10^{-3}	2.7×10^{-3}	2.9×10^{-3}	E_{RE}	2.7×10^{-3}	2.3×10^{-3}	1.3×10^{-3}
3.5×10^{-3}	4.6×10^{-3}	5.1×10^{-3}	$E_{ROI_1^1}$	4.6×10^{-3}	3.8×10^{-3}	2.0×10^{-3}
4.2×10^{-3}	5.7×10^{-3}	6.3×10^{-3}	$E_{ROI_1^2}$	5.7×10^{-3}	5.1×10^{-3}	3.0×10^{-3}
8.98×10^{-4}	1.2×10^{-3}	1.3×10^{-3}	$E_{ROI_2^1}$	1.2×10^{-3}	8.28×10^{-4}	3.50×10^{-4}
1.2×10^{-3}	2.0×10^{-3}	2.7×10^{-3}	$E_{ROI_2^2}$	2.0×10^{-3}	1.7×10^{-3}	9.71×10^{-4}
0.7109	0.8643	0.9238	E_{EI}	0.8643	0.7737	0.5353
0.8829	0.5570	0.5190	E_C	0.5570	0.6612	1.6951
0.2731	0.2943	0.3001	$E_{OI(1)}$	0.2898	0.2818	0.2784
0.2993	0.3156	0.3229	$E_{OI(2)}$	0.3156	0.3018	0.2681
2.50×10^{-5}	3.47×10^{-5}	3.91×10^{-5}	E_L	3.47×10^{-5}	2.98×10^{-5}	1.66×10^{-5}

Table 4. Effects of μ_2^1 (fix $\mu_2^2 = 4$) and μ_2^2 (fix $\mu_2^1 = 5$): Fix $S = 4, S_1 = 3, S_2 = 3, s = 2, s_1^1 = 1, s_1^2 = 1, s_2^1 = 1, s_2^2 = 2, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3$.

MAP^n						
3	4	5	$\leftarrow \mu_2^2$: $\mu_2^1 \rightarrow$	5	4	3
2.4×10^{-3}	2.8×10^{-3}	2.9×10^{-3}	E_{RE}	2.8×10^{-3}	2.5×10^{-3}	1.9×10^{-3}
4.4×10^{-3}	5.1×10^{-3}	5.4×10^{-3}	$E_{ROI_1^1}$	5.1×10^{-3}	4.4×10^{-3}	3.3×10^{-3}
5.9×10^{-3}	7.0×10^{-3}	7.4×10^{-3}	$E_{ROI_1^2}$	7.0×10^{-3}	6.5×10^{-3}	5.4×10^{-3}
1.2×10^{-3}	1.4×10^{-3}	1.5×10^{-3}	$E_{ROI_2^1}$	1.4×10^{-3}	1.0×10^{-3}	6.13×10^{-4}
1.6×10^{-3}	2.4×10^{-3}	3.1×10^{-3}	$E_{ROI_2^2}$	2.4×10^{-3}	2.1×10^{-3}	1.7×10^{-3}
0.9929	1.1144	1.1705	E_{EI}	1.1144	1.0193	0.8552
0.4277	0.3423	0.3291	E_C	0.3423	0.3857	0.5572
0.3112	0.3104	0.3130	$E_{OI(1)}$	0.3104	0.3043	0.3003
0.3319	0.3477	0.3571	$E_{OI(2)}$	0.3477	0.3305	0.3064
1.37×10^{-5}	1.64×10^{-5}	1.78×10^{-5}	E_L	1.64×10^{-5}	1.47×10^{-5}	1.17×10^{-5}

Tables 5 and 6 show the effect of μ_{12}^1 and μ_{12}^2, respectively, the environment dependent and exponentially distributed service completion rates of the combined optional inventories on different performance measures. From the table it is clear that the values of the performance measures show similar behavior in both the cases with MAP^p and MAP^n, respectively. The values of $E_{RE}, E_{ROI_1^1}, E_{ROI_1^2}, E_{ROI_2^1}, E_{ROI_2^2}, E_{EI}, E_{OI(1)}, E_{OI(2)}$, and E_L seems to increase, respectively, with increased values of μ_{12}^1 and μ_{12}^2 whereas the value of E_C seems to decrease, respectively, with the increased values of μ_{12}^1 and μ_{12}^2.

Table 5. Effects of μ_{12}^1 (fix $\mu_{12}^2 = 5$) and μ_{12}^2 (fix $\mu_{12}^1 = 4$): Fix $S = 4, S_1 = 3, S_2 = 3, s = 2, s_1^1 = 1,$ $s_1^2 = 1, s_2^1 = 1, s_2^2 = 2, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6, \mu_2^2 = 4, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5,$ $\beta_2^1 = 4, \beta_2^2 = 3.$

MAP^p						
3	4	5	$\leftarrow \mu_{12}^2 :$ $\mu_{12}^1 \rightarrow$	5	4	3
6.4×10^{-4}	2.9×10^{-3}	3.5×10^{-3}	E_{RE}	3.9×10^{-3}	3.5×10^{-3}	2.9×10^{-3}
1.1×10^{-3}	5.1×10^{-3}	6.1×10^{-3}	$E_{ROI_1^1}$	6.6×10^{-3}	6.1×10^{-3}	5.2×10^{-3}
1.4×10^{-3}	6.0×10^{-3}	7.3×10^{-3}	$E_{ROI_1^2}$	8.2×10^{-3}	7.3×10^{-3}	6.0×10^{-3}
3.4×10^{-4}	1.5×10^{-3}	1.8×10^{-3}	$E_{ROI_2^1}$	1.8×10^{-3}	1.78×10^{-3}	1.6×10^{-3}
4.93×10^{-4}	2.2×10^{-3}	2.6×10^{-3}	$E_{ROI_2^2}$	2.9×10^{-3}	2.6×10^{-3}	2.2×10^{-3}
0.4330	0.9077	1.027	E_{EI}	1.097	1.027	0.9202
3.2931	0.5618	0.4106	E_C	0.3745	0.4106	0.5306
0.1393	0.278	0.3115	$E_{OI(1)}$	0.3337	0.3115	0.279
0.2273	0.3280	0.3461	$E_{OI(2)}$	0.3615	0.3461	0.3261
6.77×10^{-6}	3.28×10^{-5}	4.18×10^{-5}	E_L	4.36×10^{-5}	4.18×10^{-5}	3.76×10^{-5}

Table 6. Effects of μ_{12}^1 (fix $\mu_{12}^2 = 5$) and μ_{12}^2 (fix $\mu_{12}^1 = 4$) with MAP^p and MAP^n: Fix $S = 4,$ $S_1 = 3, S_2 = 3, s = 2, s_1^1 = 1, s_1^2 = 1, s_2^1 = 1, s_2^2 = 2, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6,$ $\mu_2^2 = 4, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3.$

3	4	5	$\leftarrow \mu_{12}^2 :$ $\mu_{12}^1 \rightarrow$	5	4	3
1.9×10^{-3}	2.9×10^{-3}	3.3×10^{-3}	E_{RE}	3.5×10^{-3}	3.3×10^{-3}	3.0×10^{-3}
3.7×10^{-3}	5.5×10^{-3}	6.1×10^{-3}	$E_{ROI_1^1}$	6.4×10^{-3}	6.1×10^{-3}	5.6×10^{-3}
4.8×10^{-3}	7.2×10^{-3}	8.1×10^{-3}	$E_{ROI_1^2}$	8.8×10^{-3}	8.1×10^{-3}	7.1×10^{-3}
1.1×10^{-3}	1.7×10^{-3}	1.8×10^{-3}	$E_{ROI_2^1}$	1.8×10^{-3}	1.78×10^{-3}	1.75×10^{-3}
1.7×10^{-3}	2.5×10^{-3}	2.8×10^{-3}	$E_{ROI_2^2}$	3.1×10^{-3}	2.8×10^{-3}	2.5×10^{-3}
0.888	1.1598	1.2702	E_{EI}	1.3336	1.2702	1.1719
0.7408	0.3387	0.2854	E_C	0.2708	0.2854	0.3316
0.2511	0.3174	0.3444	$E_{OI(1)}$	0.3624	0.3444	0.3173
0.3187	0.3608	0.3800	$E_{OI(2)}$	0.3949	0.3800	0.3596
9.72×10^{-6}	1.56×10^{-5}	1.82×10^{-5}	E_L	1.84×10^{-5}	1.82×10^{-5}	1.74×10^{-5}

Numerical Optimization of the Cost Function

Numerical optimization of the cost function constructed in section 4 is shown in this section. Tables 7 and 8 show the effect of values of the (s, S) control policy with respect to the essential inventory in the cost incurred on the system with MAP^p and MAP^n, respectively. Both the tables give a concrete idea of the optimality of the variables S and s of the cost function. As expected, it can be observed that the cost incurred increases with the increased value of S, the maximum level of the main inventory. The factors, such as holding cost, contribute to this behavior. The cost is seen minimum when the values S and s are close enough, i.e., when the re-order level is close to the maximum inventory level. Additionally, the cost incurred on the system is seen to increase with the increase in the difference between the values S and s, i.e., when the re-order level is not closer to the maximum inventory level. This shows that it is not profitable to hold more essential inventories in the system at any given time and that the re-ordering level of the essential inventory is to be kept very close to the value of S.

Table 7. Effect of (s, S) with MAP^p on the Cost function: Fix $S_1 = 2$, $S_2 = 2$, $s_1^1 = 1$, $s_1^2 = 1$, $s_2^1 = 1$, $s_2^2 = 1, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6,$ $\beta_1^1 = 4$, $\beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3, C^0 = 120\$, C_1^1 = 35\$, C_1^2 = 40\$, C_2^1 = 45\$, C_2^2 = 50\$, C_{EI} = 110\$,$ $C_{OI}(1) = 500\$, C_{OI}(2) = 450\$, C_1 = 70\$, C_2 = 50\$.$

MAP^p				
$S \downarrow s \rightarrow$	4	3	2	1
8	545.2348	631.7666	905.0076	1003.001
7	403.4487	711.095	985.8775	1181.701
6	292.7348	435.4059	677.8474	914.2366
5	295.0661	297.7624	612.2427	933.1728

Table 8. Effect of (s, S) with MAP^non the Cost function: Fix $S_1 = 2, S_2 = 2, s_1^1 = 1, s_1^2 = 1,$ $s_2^1 = 1, s_2^2 = 1, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6,$ $\beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3, C^0 = 120\$, C_1^1 = 35\$, C_1^2 = 40\$, C_2^1 = 45\$, C_2^2 = 50\$, C_{EI} = 110\$,$ $C_{OI}(1) = 500\$, C_{OI}(2) = 450\$, C_1 = 70\$, C_2 = 50\$.$

MAP^n				
$S \downarrow s \rightarrow$	4	3	2	1
8	633.9178	667.4360	841.9043	879.9268
7	547.5167	709.7668	899.4244	995.1082
6	406.4411	492.9354	647.1244	786.6579
5	398.9954	397.9574	586.8929	781.3571

Tables 9 and 10 show the effect of the pairs (s_1^1, S_1) and (s_1^2, S_1) concerning the first optional item in the cost incurred on the system under consideration with MAP^p and MAP^n, respectively. The table shows that the value of the cost incurred tends to vary in proportional to the value of S_1. The optimum minimum cost value is seen to be obtained with the decrease in the values of S_1, s_1^1, and s_1^2, respectively.

Table 9. Effect of (s_1^1, S_1) (fix $s_1^2 = 1$) and (s_1^2, S_1) (fix $s_1^1 = 1$) with MAP^p in the cost incurred on the system: Fix $S = 3, s = 1, S_2 = 2, s_2^1 = 1, s_2^2 = 1, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6,$ $\mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3, C^0 = 120\$, C_1^1 = 35\$, C_1^2 = 40\$,$ $C_2^1 = 45\$, C_2^2 = 50\$, C_{EI} = 110\$, C_{OI}(1) = 500\$, C_{OI}(2) = 450\$, C_1 = 70\$, C_2 = 50\$.$

MAP^p						
1	2	3	$\leftarrow s_1^2$: $S_1 \downarrow$: $s_1^1 \rightarrow$	3	2	1
935.5729	894.1549	863.9671	6	922.4228	935.3796	935.5729
865.7215	872.3495	874.6748	5	883.7009	877.5753	865.7215
809.9514	824.6832	809.5636	4	796.8894	827.6646	809.9514

Table 10. Effect of (s_1^1, S_1) (fix $s_1^2 = 1$) and (s_1^2, S_1) (fix $s_1^1 = 1$) with MAP^n in the cost incurred on the system: Fix $S = 4, S_2 = 2, s_2^1 = 1, s_2^2 = 1, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3, C^0 = 120\$, C_1^1 = 35\$, C_1^2 = 40\$, C_2^1 = 45\$, C_2^2 = 50\$, C_{EI} = 110\$, C_{OI}(1) = 500\$, C_{OI}(2) = 450\$, C_1 = 70\$, C_2 = 50\$$.

MAP^n						
1	2	3	$\leftarrow s_1^2$: $S_1 \downarrow$: $s_1^1 \rightarrow$	3	2	1
509.8432	494.74	524.1961	6	545.1854	510.3812	509.8432
524.6889	520.1184	501.4385	5	562.9512	543.2073	524.6889
467.4061	459.2036	467.5296	4	480.5882	469.6073	467.4061

Tables 11 and 12 show the effect of the pairs (s_2^1, S_2) and (s_2^2, S_2) concerning the second optional inventory in the cost incurred on the system with MAP^p and MAP^n, respectively. Both the tables gives an idea on the condition for optimal values of the variables S_2, s_2^1, and s_2^2. It can be seen that the cost function show a decreasing tendency with decreased values of S_2, s_2^1, and s_2^2.

Table 11. Effect of (s_2^1, S_2) (fix $s_2^2 = 1$) and (s_2^2, S_2) (fix $s_2^1 = 1$) with MAP^p in the cost incurred on the system: Fix $S = 3, s = 1, S_1 = 2, s_1^1 = 1, s_1^2 = 1, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3, C^0 = 120\$, C_1^1 = 35\$, C_1^2 = 40\$, C_2^1 = 45\$, C_2^2 = 50\$, C_{EI} = 110\$, C_{OI}(1) = 500\$, C_{OI}(2) = 450\$, C_1 = 70\$, C_2 = 50\$$.

MAP^p						
1	2	3	$\leftarrow s_2^2$: $S_2 \downarrow$: $s_2^1 \rightarrow$	3	2	1
979.5430	986.6257	975.6525	6	970.0710	986.1526	979.5430
885.1778	854.4908	851.0899	5	840.7033	844.198	885.1778
692.9451	696.7333	693.5159	4	699.5264	701.7346	692.9451

Table 12. Effect of (s_2^1, S_2) (fix $s_2^2 = 1$) and (s_2^2, S_2) (fix $s_2^1 = 1$) with MAP^n in the cost incurred on the system: Fix $S = 3, S_1 = 2, s_1^1 = 1, s_1^2 = 1, m1 = m2 = m3 = 2, \mu_1^1 = 5, \mu_1^2 = 4, \mu_2^1 = 6, \mu_2^2 = 4, \mu_{12}^1 = 3, \mu_{12}^2 = 5, \beta = 6, \beta_1^1 = 4, \beta_1^2 = 5, \beta_2^1 = 4, \beta_2^2 = 3, C^0 = 120\$, C_1^1 = 35\$, C_1^2 = 40\$, C_2^1 = 45\$, C_2^2 = 50\$, C_{EI} = 110\$, C_{OI}(1) = 500\$, C_{OI}(2) = 450\$, C_1 = 70\$, C_2 = 50\$$.

MAP^n						
1	2	3	$\leftarrow s_2^2$: $S_2 \downarrow$: $s_2^1 \rightarrow$	3	2	1
850.5189	853.9868	843.1651	6	842.5212	854.2780	850.5189
776.4571	755.0671	753.3682	5	747.1692	747.7253	776.4571
631.4084	634.4203	634.0225	4	640.1808	639.8542	631.4084

6. Conclusions

We studied a single server multi-commodity queueing inventory system with one essential and m optional items in n random environments. The condition for stability of the system is obtained. Under this condition, different performance measures of the system are derived. A cost function involving these measures and inventory control variables is constructed. Optimization of the cost function along with the control variables is also done numerically. The obtained numerical results showed huge resemblances with what we see and experience around us. A very familiar example is a car/truck (Heavy automobiles) showroom, where one can see only very few items (main item) displayed. In this example, one can think of the additional accessories as the optional inventories. Here, booking of the essential inventory with other optional inventories has to be done as per the requirement of the customer.

Author Contributions: Conceptualization, J.J. and A.K.; methodology, J.J. and D.S.; software, D.S.; validation, J.J., D.S., and A.K.; formal analysis, A.K., V.V., and D.K.; writing—original draft preparation, J.J.; writing—review and editing, J.J., D.S., V.V., and D.K.; supervision, A.K.; project administration, A.K.; funding acquisition, V.V. and D.K. All authors have read and agreed to the published version of the manuscript.

Funding: This paper was supported by the RUDN University Strategic Academic Leadership Program (recipient D.K., formal analysis) and funded by Russian Foundation for Basic Research according to the research project number 19-29-06043 (recipients V.V. and D.K., writing—review and editing).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors express their gratitude to the referees for valuable suggestions that improved the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Appendix A

$\boldsymbol{Illustration}$ **: (1)**: Fix the number of optional items $m = 2$ and the number of environments $n = 2$, the state space of the system is precisely,

$\{(0, j, i, 0^*, i_1, i_2, j_2, j_3), 0 \leq i \leq S, 1 \leq j \leq 2, 0 \leq i_1 \leq S_1, 0 \leq i_2 \leq S_2, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\}$
$\bigcup \{(\bar{n}, j, 0, 0^*, i_1, i_2, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq 2, 0 \leq i_r \leq S_r^j$ for $1 \leq r \leq 2, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\} \bigcup \{(n, j, i, 0, i_1, i_2, j_1, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq 2, 1 \leq i \leq S, 0 \leq i_r \leq S_r^j$ for $1 \leq r \leq 2,\ 1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\} \bigcup \{(n, j, i, 1, i_1, i_2, j_2, j_3),\ \bar{n} \geq 1, 1 \leq j \leq 2, 1 \leq i \leq S, 1 \leq i_1 \leq S_1^j; 0 \leq i_2 \leq S_2^j, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\} \bigcup \{(n, j, i, 2, i_1, i_2, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq 2, 1 \leq i \leq S, 0 \leq i_1 \leq S_1^j; 1 \leq i_2 \leq S_2^j, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3\} \bigcup \{(n, j, i, 12, i_1, i_2, j_2, j_3), \bar{n} \geq 1, 1 \leq j \leq 2, 1 \leq i \leq S, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3; 1 \leq i_k \leq S_k^j$ for $k \in \{1, 2\}\}$.

The transitions rates are:

1. Transition rates due to customer arrival.
   (a) $(0, l_1, i, 0^*, i_1, i_2, j_2, j_3) \to (1, l_2, i, 0, i_1, i_2, j_1, j_2, j_3^{'})$
   at the rate $\gamma_{j_1} [H_1]_{j_3, j_3^{'}}$ when $l_1 = l_2$
   for $1 \leq l_1, l_2 \leq 2, 1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2$, and $1 \leq j_3, j_3^{'} \leq m_3$
   where $1 \leq i \leq S; 0 \leq i_r \leq S_r$ for$1 \leq r \leq 2$.
   (b) $(\bar{n}, l_1, i, 0, i_1, i_2, j_1, j_2, j_3) \to (\bar{n} + 1, l_2, i, 0, i_1, i_2, j_1, j_2, j_3^{'})$ at the rate $[H_1]_{j_3, j_3^{'}}$ when $l_1 = l_2$
   for $1 \leq l_1, l_2 \leq 2$ and $1 \leq j_1 \leq m_1$, $1 \leq j_2 \leq m_2$ and $1 \leq j_3, j_3^{'} \leq m_3$ where $\bar{n} \geq 1; 1 \leq i \leq S; 0 \leq i_r \leq S_r$ for $1 \leq r \leq 2$.
   (c) $(\bar{n}, l_1, i, C_u, i_1, i_2, j_2, j_3) \to (\bar{n} + 1, l_2, i, C_u, i_1, i_2, j_2, j_3^{'})$
   at the rate $[H_1]_{j_3, j_3^{'}}$ when $l_1 = l_2$
   for $1 \leq l_1, l_2 \leq 2$, $1 \leq j_2 \leq m_2$, $1 \leq j_3, j_3^{'} \leq m_3$ where $\bar{n} \geq 1; 1 \leq i \leq S$ and, when
   $C_u = 1 \longrightarrow 1 \leq i_1 \leq S_1; 0 \leq i_2 \leq S_2;$
   $C_u = 2 \longrightarrow 0 \leq i_1 \leq S_1; 1 \leq i_2 \leq S_2;$
   $C_u = 12 \longrightarrow 1 \leq i_1 \leq S_1; 1 \leq i_2 \leq S_2$

2. Transition rates due to the service completion of essential and optional items.
   (a) Transition rates from level 1 to level 0 due to the service completion of optional inventory
      i. $(1, l_1, i, 1, i_1, i_2, j_2, j_3) \rightarrow (0, l_2, i, 0^*, i_1 - 1, i_2, j_2, j_3)$
      at the rate $\mu_1^{l_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$, $1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 1 \leq i_1 \leq S_1; 0 \leq i_2 \leq S_2$.
      ii. $(1, l_1, i, 2, i_1, i_2, j_2, j_3) \rightarrow (0, l_2, i, 0^*, i_1, i_2 - 2, j_2, j_3)$
      at the rate $\mu_2^{l_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$, $1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 0 \leq i_1 \leq S_1; 1 \leq i_2 \leq S_2$.
      iii. $(1, l_1, i, 12, i_1, i_2, j_2, j_3) \rightarrow (0, l_2, i, 0^*, i_1 - 1, i_2 - 1, j_2, j_3)$
      at the rate $\mu_{12}^{l_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$, $1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 1 \leq i_1 \leq S_1; 1 \leq i_2 \leq S_2$.
   (b) Transition rates from level 1 to level 0 due to the service completion of essential inventory
      i. $(1, l_1, i, 0, 0, 0, j_1, j_2, j_3) \rightarrow (0, l_2, i - 1, 0^*, 0, 0, j_2, j_3)$
      at the rate T_{j_1} when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2, 1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S$
      ii. $(1, l_1, i, 0, 0, i_2, j_1, j_2, j_3) \rightarrow (0, l_2, i - 1, 0^*, 0, i_2, j_2', j_3)$
      at the rate $\eta_1 T_{j_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$, $1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$ where $1 \leq i \leq S; 1 \leq i_2 \leq S_2$ and $\eta_1 = p + (1 - p)(p_1 + p_{12})$.
      iii. $(1, l_1, i, 0, i_1, 0, j_1, j_2, j_3) \rightarrow (0, l_2, i - 1, 0^*, i_1, 0, j_2', j_3)$
      at the rate $\eta_2 T_{j_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$, $1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$ where $1 \leq i \leq S; 1 \leq i_1 \leq S_1$ and $\eta_2 = p + (1 - p)(p_2 + p_{12})$.
      iv. $(1, l_1, i, 0, i_1, i_2, j_1, j_2, j_3) \rightarrow (0, l_2, i - 1, 0^*, i_1^*, i_2^*, j_2, j_3)$
      at the rate pT_{j_1} when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2, 1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 1 \leq i_k = i_k^* \leq S_k$ for $k \in \{1, 2\}$.
   (c) Transition rates from level $\bar{n}$ to level $\bar{n} - 1$ for $\bar{n} \geq 2$ due to the service completion of optional inventory
      i. $(\bar{n}, l_1, i, 1, i_1, i_2, j_2, j_3) \rightarrow (\bar{n} - 1, l_2, i, 1, i_1 - 1, i_2, j_2, j_3)$
      at the rate $\mu_1^{l_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$,$\bar{n} \geq 2, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 1 \leq i_1 \leq S_1; 0 \leq i_2 \leq S_2$.
      ii. $(\bar{n}, l_1, i, 2, i_1, i_2, j_2, j_3) \rightarrow (\bar{n} - 1, l_2, i, 2, i_1, i_2 - 1, j_2, j_3)$
      at the rate $\mu_2^{l_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$,$\bar{n} \geq 2, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 0 \leq i_1 \leq S_1; 1 \leq i_2 \leq S_2$.
      iii. $(\bar{n}, l_1, i, 12, i_1, i_2, j_2, j_3) \rightarrow (\bar{n} - 1, l_2, i, 12, i_1 - 1, i_2 - 1, j_2, j_3)$
      at the rate $\mu_{12}^{l_1}$ when $l_1 = l_2$
      for $1 \leq l_1, l_2 \leq 2$,$\bar{n} \geq 2, 1 \leq j_2 \leq m_2, 1 \leq j_3 \leq m_3$
      where $1 \leq i \leq S; 1 \leq i_1 \leq S_1; 1 \leq i_2 \leq S_2$.
   (d) Transition rates from level $\bar{n}$ to level $\bar{n} - 1$ for $\bar{n} \geq 2$ due to the service completion of essential inventory
      i. $(\bar{n}, l_1, 1, 0, 0, 0, j_1, j_2, j_3) \rightarrow (\bar{n} - 1, l_2, 0, 0^*, 0, 0, j_2, j_3)$
      at the rate T_{j_1} when $l_1 = l_2$

for $1 \leq l_1, l_2 \leq 2, 1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq i \leq S$.

ii. $(\bar{n}, l_1, 1, 0, 0, i_2, j_1, j_2, j_3) \to (\bar{n}-1, l_2, 0, 0^*, 0, i_2, j_2, j_3)$
at the rate $\eta_1 T_{j_1}$ when $l_1 = l_2$
for $1 \leq l_1, l_2 \leq 2, 1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq i_2 \leq S_2$
and $\eta_1 = p + (1-p)(p_1 + p_{12})$.

iii. $(\bar{n}, l_1, 1, 0, i_1, 0, j_1, j_2, j_3) \to (\bar{n}-1, l_2, 0, 0^*, i_1, 0, j_2, j_3)$
at the rate $\eta_2 T_{j_1}$ when $l_1 = l_2$
for $1 \leq l_1, l_2 \leq 2, 1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq i_1 \leq S_1$
and $\eta_2 = p + (1-p)(p_2 + p_{12})$.

iv. $(\bar{n}, l_1, 1, 0, i_1, i_2, j_1, j_2, j_3) \to (\bar{n}-1, l_2, 0, 0^*, i_1, i_2, j_2, j_3)$
at the rate pT_{j_1} when $l_1 = l_2$
for $1 \leq l_1, l_2 \leq 2; 1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq i_k \leq S_k$ for $k \in \{1, 2\}$.

v. $(\bar{n}, l_1, i, 0, 0, 0, j_1, j_2, j_3) \to (\bar{n}-1, l_2, i-1, 0, 0, 0, j_1, j_2, j_3)$
at the rate T_{j_1} when $l_1 = l_2$
for $1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq l_1, l_2 \leq 2; 1 \leq i \leq S$.

vi. $(\bar{n}, l_1, i, 0, 0, i_2, j_1, j_2, j_3) \to (\bar{n}-1, l_2, i-1, 0, 0, i_2, j_1, j_2, j_3)$
at the rate $\eta_1 T_{j_1}$ when $l_1 = l_2$
for $1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq l_1, l_2 \leq 2, 2 \leq i \leq S; 1 \leq i_2 \leq S_2$ and $\eta_1 = p + (1-p)(p_1 + p_{12})$.

vii. $(\bar{n}, l_1, i, 0, i_1, 0, j_1, j_2, j_3) \to (\bar{n}-1, l_2, i-1, 0, i_1, 0, j_1, j_2, j_3)$
at the rate $\eta_2 T_{j_1}$ when $l_1 = l_2$
for $1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$
where $\bar{n} \geq 2; 1 \leq l_1, l_2 \leq 2, 2 \leq i \leq S; 1 \leq i_1 \leq S_1$ and $\eta_2 = p + (1-p)(p_2 + p_{12})$.

viii. $(\bar{n}, l_1, i, 0, i_1, i_2, j_1, j_2, j_3) \to (\bar{n}-1, l_2, i-1, 0, i_1, i_2, j_1, j_2, j_3)$ at the rate pT_{j_1} when $l_1 = l_2$
for $1 \leq j_1 \leq m_1; 1 \leq j_2 \leq m_2; 1 \leq j_3 \leq m_3$ where $1 \leq l_1, l_2 \leq 2, \bar{n} \geq 2$; $2 \leq i \leq S; 1 \leq i_k \leq S_k$ for $k \in \{1, 2\}$.

The infinitesimal generator $\mathcal{Q}$ of the system is obtained to be

$$\mathcal{Q} = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

A_{00} is a square matrix of order a. A_{01} is a matrix of order $a \times b$. A_{10} is a matrix of order $c \times a$. A_0, A_1, and A_2 are square matrices of order c, where $a = 2(S+1)(S_1+1)(S_2+1)m_2m_3$, $b = 2(S+1)(S_1+1)(S_2+1)m_1m_2m_3$ and $c = 2((S_1+1)(S_2+1)m_2m_3 + S(b_1+b_2+b_3+b_4))$ where $b_1 = (S_1+1)(S_2+1)m_1m_2m_3$, $b_2 = S_1(S_2+1)m_2m_3$, $b_3 = (S_1+1)S_2m_2m_3$ and $b_4 = S_1S_2m_2m_3$.

The structure of A_{00}, A_{01}, A_{10}, A_1, A_0, A_2 and their corresponding ijth sub matrices A_{00}^{ij}, A_{01}^{ij}, A_{10}^{ij}, A_1^{ij}, A_0^{ij}, A_2^{ij} for $1 \leq i, j \leq 2$ are obtained as given in Equations (A1)–(A12).

The structure of A_{00}, A_{01}, A_{10}, A_0, A_1 andA_2 are obtained as

$$A_{00} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} A_{00}^{11} & A_{00}^{12} \\ A_{00}^{21} & A_{00}^{22} \end{pmatrix} \end{array} \tag{A1}$$

where each A_{00}^{ij} for $1 \le i, j \le 2$ has the structure

$$A_{00}^{ij} = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} Z_0^{ij} & & & & & & \hat{Z}^{ij} \\ & \hat{Z_1^{ij}} & & & & & \hat{Z}^{ij} \\ & & \ddots & & & & \vdots \\ & & & \hat{Z_1}^{ij} & & & \hat{Z}^{ij} \\ & & & & \hat{Z_2}^{ij} & & \\ & & & & & \ddots & \\ & & & & & & \hat{Z_2}^{ij} \end{pmatrix} \end{array} \tag{A2}$$

$$A_{01} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} A_{01}^{11} & A_{01}^{12} \\ A_{01}^{21} & A_{01}^{22} \end{pmatrix} \end{array} \tag{A3}$$

where each A_{01}^{ij} for $1 \le i, j \le 2$ has the structure

$$A_{01}^{ij} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & \cdots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{0} & & & \\ & L^{*ij} & & \\ & & \ddots & \\ & & & L^{*ij} \end{pmatrix} \end{array} \tag{A4}$$

$$A_{10} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} A_{10}^{11} & A_{10}^{12} \\ A_{10}^{21} & A_{10}^{22} \end{pmatrix} \end{array} \tag{A5}$$

where each A_{10}^{ij} for $1 \le i, j \le 2$ has the structure

$$A_{10}^{ij} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & \cdots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{0} & & & & \\ M_0^{ij} & \hat{M}^{ij} & & & \\ & M_0^{ij} & \hat{M}^{ij} & & \\ & & \ddots & \ddots & \\ & & & M_0^{ij} & \hat{M}^{ij} \end{pmatrix} \end{array} \tag{A6}$$

$$A_1 = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{pmatrix} \end{array} \tag{A7}$$

where each A_1^{ij} for $1 \leq i, j \leq 2$ has the structure

$$A_1^{ij} = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} & \begin{pmatrix} Z_0^{ij} & & & & & & Z^{0ij} \\ & Z_1^{ij} & & & & & Z^{ij} \\ & & \ddots & & & & \vdots \\ & & & Z_1^{ij} & & & Z^{ij} \\ & & & & Z_2^{ij} & & \\ & & & & & \ddots & \\ & & & & & & Z_2^{ij} \end{pmatrix} \end{array} \tag{A8}$$

$$A_0 = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{pmatrix} \end{array} \tag{A9}$$

where each A_0^{ij} for $1 \leq i, j \leq 2$ has the structure

$$A_0^{ij} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & \cdots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{0} & & & \\ & L^{ij} & & \\ & & \ddots & \\ & & & L^{ij} \end{pmatrix} \end{array} \tag{A10}$$

$$A_2 = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} A_2^{11} & A_2^{12} \\ A_2^{21} & A_2^{22} \end{pmatrix} \end{array} \tag{A11}$$

where each A_2^{ii} for $1 \leq i, j \leq 2$ has the structure

$$A_2^{ij} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & \cdots & S \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ S \end{array} & \begin{pmatrix} \bar{0} & & & & \\ M_0^{ij} & \bar{M}^{ij} & & & \\ & M^{ij} & \bar{M}^{ij} & & \\ & & \ddots & \ddots & \\ & & & M^{ij} & \bar{M}^{ij} \end{pmatrix} \end{array} \tag{A12}$$

1. Submatrices of A_{00}^{ij}

   For $i = j$,
   $Z_0^{ij} = \begin{pmatrix} I_{s_1^i+1} \otimes C_1 & e_{s_1^i+1} \otimes [I_{(S_2+1)} \otimes \beta_1^i I_{m_2 m_3}] \\ \bar{0} & I_{S_1-s_1^i} \otimes C_2 \end{pmatrix}$,
   where
   $C_1 = \begin{pmatrix} I_{s_2^i+1} \otimes B_1 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes B_2 \end{pmatrix}$,
   $C_2 = \begin{pmatrix} I_{s_2^i+1} \otimes B_3 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes B_4 \end{pmatrix}$,
   $B_1 = (D_0 + D_i - (\beta + \beta_1^i + \beta_2^i) I_{m_2}) \oplus H$,
   $B_2 = (D_0 + D_i - (\beta + \beta_1^i) I_{m_2}) \oplus H$,
   $B_3 = (D_0 + D_i - (\beta + \beta_2^i) I_{m_2}) \oplus H$,
   $B_4 = (D_0 + D_i - \beta I_{m_2}) \oplus H$.
   For $i \neq j$, $Z_0^{ij} = I_{(S_1+1)(S_2+1)} \otimes [D_j \otimes I_{m_3}]$
   For $i = j$,
   $\hat{Z}_1^{ij} = \begin{pmatrix} I_{s_1^i+1} \otimes \hat{C}_1 & e_{s_1^i+1} \otimes \beta_1^i I_{(S_2+1)m_2 m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes \hat{C}_2 \end{pmatrix}$, where
   $\hat{C}_1 = \begin{pmatrix} I_{s_2^i+1} \otimes \hat{B}_1 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_2 \end{pmatrix}$,
   $\hat{C}_2 = \begin{pmatrix} I_{s_2^i+1} \otimes \hat{B}_3 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_4 \end{pmatrix}$,
   $\hat{B}_1 = (D_0 + D_i - (\beta + \beta_1^i + \beta_2^i) I_{m_2}) \oplus H_0$,
   $\hat{B}_2 = (D_0 + D_i - (\beta + \beta_1^i) I_{m_2}) \oplus H_0$,
   $\hat{B}_3 = (D_0 + D_i - (\beta + \beta_2^i) I_{m_2}) \oplus H_0$, $\hat{B}_4 = (D_0 + D_i - \beta I_{m_2}) \oplus H_0$.
   For $i \neq j$,
   $\hat{Z}_1^{ij} = I_{(S_1+1)(S_2+1)} \otimes [D_j \otimes I_{m_3}]$

   For $i = j$,
   $\hat{Z}_2^{ij} = \begin{pmatrix} I_{s_1^i+1} \otimes \bar{C}_1 & e_{s_1^i+1} \otimes \beta_1^i I_{(S_2+1)m_2 m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes \bar{C}_2 \end{pmatrix}$, where
   $\bar{C}_1 = \begin{pmatrix} I_{s_2^i+1} \otimes \bar{B}_1 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \bar{B}_2 \end{pmatrix}$,
   $\bar{C}_2 = \begin{pmatrix} I_{s_2^i+1} \otimes \bar{B}_3 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \bar{B}_4 \end{pmatrix}$,
   $\bar{B}_1 = (D_0 + D_i - (\beta_1^i + \beta_2^i) I_{m_2}) \oplus H_0$,
   $\bar{B}_2 = (D_0 + D_i - (\beta_1^i) I_{m_2}) \oplus H_0$,
   $\bar{B}_3 = (D_0 + D_i - (\beta_2^i) I_{m_2}) \oplus H_0$, $\bar{B}_4 = D_0 + D_i \oplus H_0$.
   For $i \neq j$,
   $\hat{Z}_2^{ij} = I_{(S_1+1)(S_2+1)} \otimes [D_j \otimes I_{m_3}]$
   For $i = j$,
   $\hat{Z}^{ij} = \beta I_{(S_1+1)(S_2+1)m_2 m_3}$.
   For $i \neq j$, $\hat{Z}^{ij} = \bar{0}$
2. Submatrices of A_{01}^{ij}

   For $i = j$, $L^{*ij} = I_{(S_1+1)(S_2+1)} \otimes [\gamma \otimes (I_{m_2} \otimes H_1)]$
   For $i \neq j$, $L^{*ij} = \bar{0}$

3. Submatrices of A_{10}^{ij}

$$\text{For } i=j,\ M_0^{ij} = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & \cdots & S_1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ S_1 \end{array} & \begin{pmatrix} m_0 & & & & \\ & m_1 & & & \\ & & m_1 & & \\ & & & \ddots & \\ & & & & m_1 \end{pmatrix} \end{array}, \quad \text{where}$$

$$m_0 = \begin{pmatrix} T^0 \otimes I_{m_2m_3} & \bar{0} \\ \bar{0} & I_{S_2} \otimes \eta_1 T^0 \otimes I_{m_2m_3} \end{pmatrix}, \text{ where } \eta_1 = p + (1-p)(p_1 + p_{12})$$

$$m_1 = \begin{pmatrix} \eta_2 T^0 \otimes I_{m_1m_2} & \\ \bar{0} & I_{S_2} \otimes pT^0 \otimes I_{m_1m_2} \end{pmatrix}, \text{ where } \eta_2 = p + (1-p)(p_2 + p_{12})$$

For $i \neq j$, $M_0^{ij} = \bar{0}$

$$\text{For } i=j,\ \hat{M}^{ij} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 12 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 12 \end{array} & \begin{pmatrix} \bar{0} & & & \\ \mu_1^i I_{S_1(S_2+1)m_2m_3} & \bar{0} & & \\ \mu_2^i I_{(S_1+1)S_2m_2m_3} & & \bar{0} & \\ \mu_{12}^i I_{S_1S_2m_2m_3} & & & \bar{0} \end{pmatrix} \end{array}.$$

For $i \neq j$, $\hat{M}^{ij} = \bar{0}$

4. Submatrices of A_1^{ij}
For $i = j$, $Z^{0ij} = I_{(S_1+1)(S_2+1)} \otimes [\gamma \otimes \beta I_{m_2m_3}]$.
For $i \neq j$, $Z^{0ij} = \bar{0}$
For $i = j$, $Z^{ij} = \beta I_{[(S_1+1)(S_2+1)m_1m_2m_3 + S_1(S_2+1)m_2m_3 + (S_1+1)S_2m_2m_3 + S_1S_2m_2m_3]}$
For $i \neq j$, $Z^{ij} = \bar{0}$

$$Z_1^{ij} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 12 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 12 \end{array} & \begin{pmatrix} G_1 & G^{12} & G^{13} & G^{14} \\ & G_2 & & \\ & & G_3 & \\ & & & G_4 \end{pmatrix} \end{array} \quad \text{where}$$

$$\text{For } i=j,\ G_1 = \begin{pmatrix} I_{s_1^i+1} \otimes C_3 & e_{s_1^i+1} \otimes \beta_1^i I_{(S_2+1)m_1m_2m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes C_4 \end{pmatrix}.$$

$$C_3 = \begin{pmatrix} I_{s_2^i+1} \otimes B_5 & e_{s_2^i+1} \otimes \beta_2^i I_{m_1m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes B_6 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} I_{s_2^i+1} \otimes B_7 & e_{s_2^i+1} \otimes \beta_2^i I_{m_1m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes B_8 \end{pmatrix},$$

$$B_5 = T \oplus [(D_0 + D_i) - (\beta + \beta_1^i + \beta_2^i) I_{m_2} \oplus H_0],$$

$$B_6 = T \oplus [(D_0 + D_i) - (\beta + \beta_1^i) I_{m_2} \oplus H_0],$$

$$B_7 = T \oplus [(D_0 + D_i) - (\beta + \beta_2^i) I_{m_2} \oplus H_0],$$
$$B_8 = T \oplus [(D_0 + D_i) - \beta I_{m_2} \oplus H_0].$$

For $i \neq j$, $G_1 = I_{(S_1+1)(S_2+1)m_1} \otimes [D_j \otimes I_{m_3}]$

$$\text{For } i=j,\ G^{12} = \begin{pmatrix} \bar{0} \\ H_1 \end{pmatrix},\ H_1 = I_{S_1(S_2+1)} \otimes [(1-p)p_1 T^0 \otimes I_{m_2m_3}]$$

For $i \neq j$, $G^{12} = \bar{0}$

For $i = j$, $G^{13} = I_{S_1+1} \otimes Z^*$, where

$$Z^* = \begin{pmatrix} \bar{0} \\ H_2 \end{pmatrix}, H_2 = \quad I_{S_2} \otimes [(1-p)p_2 T^0 \otimes I_{m_2 m_3}].$$

For $i \neq j$, $G^{13} = \bar{0}$

For $i = j$, $G^{14} = \begin{pmatrix} \bar{0} \\ I_{S_1} \otimes \bar{Z} \end{pmatrix}$, where

$$\bar{Z} = \begin{pmatrix} \bar{0} \\ H_3 \end{pmatrix}, H_3 = \quad I_{S_2} \otimes [(1-p)p_{12} T^0 \otimes I_{m_2 m_3}],$$

For $i \neq j$, $G^{14} = \bar{0}$

For $i = j$, $G_2 = \begin{pmatrix} I_{s_1^i} \otimes C_5 & e_{s_1^i} \otimes \beta_1^i I_{(S_2+1)m_2 m_3} \\ \bar{0} & I_{S_1 - s_1^i} \otimes C_6 \end{pmatrix}$, where

$$C_5 = \begin{pmatrix} I_{s_2^i+1} \otimes B_9 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2 - s_2^i} \otimes B_{10} \end{pmatrix},$$

$$C_6 = \begin{pmatrix} I_{s_2^i+1} \otimes B_{11} & e_{s_2^i+1} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2 - s_2^i} \otimes B_{12} \end{pmatrix}$$

$$\begin{aligned}
B_9 &= \quad [(D_0 + D_i) - (\beta + \beta_1^i + \beta_2^i + \mu_1^i) I_{m_2}] \oplus H_0, \\
B_{10} &= \quad [(D_0 + D_i) - (\beta + \beta_1^i + \mu_1^i) I_{m_2}] \oplus H_0, \\
B_{11} &= \quad [(D_0 + D_i) - (\beta + \beta_2^i + \mu_1^i) I_{m_2}] \oplus H_0, \\
B_{12} &= \quad [(D_0 + D_i) - (\beta + \mu_1^i) I_{m_2}] \oplus H_0.
\end{aligned}$$

For $i \neq j$, $G_2 = I_{S_1(S_2+1)} \otimes [D_j \otimes I_{m_3}]$

For $i = j$, $G_3 = \begin{pmatrix} I_{s_1^i+1} \otimes C_7 & e_{s_1^i+1} \otimes \beta_1^i I_{S_2 m_2 m_3} \\ \bar{0} & I_{S_1 - s_1^i} \otimes C_8 \end{pmatrix}$, where

$$C_7 = \begin{pmatrix} I_{s_2^i} \otimes B_{13} & e_{s_2^i} \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2 - s_2^i} \otimes B_{14} \end{pmatrix}, C_8 = \begin{pmatrix} I_{s_2^i} \otimes B_{15} & e_{s_2^i} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2 - s_2^i} \otimes B_{16} \end{pmatrix}$$

$$\begin{aligned}
B_{13} &= \quad [(D_0 + D_i) - (\beta + \beta_1^i + \beta_2^i + \mu_2^i) I_{m_2}] \oplus H_0, \\
B_{14} &= \quad [(D_0 + D_i) - (\beta + \beta_1^i + \mu_2^i) I_{m_2}] \oplus H_0, \\
B_{15} &= \quad [(D_0 + D_i) - (\beta + \beta_2^i + \mu_2^i) I_{m_2}] \oplus H_0, \\
B_{16} &= \quad [(D_0 + D_i) - (\beta + \mu_2^i) I_{m_2}] \oplus H_0.
\end{aligned}$$

For $i \neq j$, $G_3 = I_{(S_1+1)S_2} \otimes [D_j \otimes I_{m_3}]$

For $i = j$, $G_4 = \begin{pmatrix} I_{s_1^i} \otimes C_9 & e_{s_1^i} \otimes \beta_1^i I_{S_2 m_2 m_3} \\ \bar{0} & I_{S_1 - s_1^i} \otimes C_{10} \end{pmatrix}$, where

$$C_9 = \begin{pmatrix} I_{s_2^i} \otimes B_{17} & e_{s_2^i} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2 - s_2^i} \otimes B_{18} \end{pmatrix}, C_{10} = \begin{pmatrix} I_{s_2^i} \otimes B_{19} & e_{s_2^i} \otimes \beta_2^i I_{m_2 m_3} \\ \bar{0} & I_{S_2 - s_2^i} \otimes B_{20} \end{pmatrix}$$

$$\begin{aligned}
B_{17} &= \quad [(D_0 + D_i) - (\beta + \beta_1^i + \beta_2^i + \mu_{12}^i) I_{m_2}] \oplus H_0, \\
B_{18} &= \quad [(D_0 + D_i) - (\beta + \beta_1^i + \mu_{12}^i) I_{m_2}] \oplus H_0, \\
B_{19} &= \quad [(D_0 + D_i) - (\beta + \beta_2^i + \mu_{12}^i) I_{m_2}] \oplus H_0, \\
B_{20} &= \quad [(D_0 + D_i) - (\beta + \mu_{12}^i) I_{m_2}] \oplus H_0.
\end{aligned}$$

For $i \neq j$, $G_4 = I_{S_1 S_2} \otimes [D_j \otimes I_{m_3}]$

$$Z_2^{ij} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} \hat{G}_1 & G^{12} & G^{13} & G^{14} \\ & \hat{G}_2 & & \\ & & \hat{G}_3 & \\ & & & \hat{G}_4 \end{pmatrix} \end{array},$$

where,

For $i = j$, $\hat{G}_1 = \begin{pmatrix} I_{s_1^i+1} \otimes \hat{C}_3 & e_{s_1^i+1} \otimes \beta_1^i I_{(S_2+1)m_1m_2m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes \hat{C}_4 \end{pmatrix}$, where

$$\hat{C}_3 = \begin{pmatrix} I_{s_2^i+1} \otimes \hat{B}_5 & e_{s_2^i+1}\beta_2^i I_{m_1m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_6 \end{pmatrix},$$
$$\hat{C}_4 = \begin{pmatrix} I_{s_2^i+1} \otimes \hat{B}_7 & e_{s_2^i+1} \otimes \beta_2^i I_{m_1m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_8 \end{pmatrix},$$
$$\begin{aligned}
\hat{B}_5 &= T \oplus [(D_0 + D_i) - (\beta_1^i + \beta_2^i) I_{m_2} \oplus H_0], \\
\hat{B}_6 &= T \oplus [(D_0 + D_i) - (\beta_1^i) I_{m_2} \oplus H_0], \\
\hat{B}_7 &= T \oplus [(D_0 + D_i) - (\beta_2^i) I_{m_2} \oplus H_0], \\
\hat{B}_8 &= T \oplus [(D_0 + D_i) \oplus H_0].
\end{aligned}$$

For $i \neq j$, $\hat{G}_1 = G_1$

For $i = j$, $\hat{G}_2 = \begin{pmatrix} I_{s_1^i} \otimes \hat{C}_5 & e_{s_1^i} \otimes \beta_1^i I_{(S_2+1)m_2m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes \hat{C}_6 \end{pmatrix}$, where

$$\hat{C}_5 = \begin{pmatrix} I_{s_2^i+1} \otimes \hat{B}_9 & e_{s_2^i+1} \otimes \beta_2^i I_{m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_{10} \end{pmatrix},$$
$$\hat{C}_6 = \begin{pmatrix} I_{s_2^i+1} \otimes \hat{B}_{11} & e_{s_2^i+1} \otimes \beta_2^i I_{m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_{12} \end{pmatrix}$$
$$\begin{aligned}
\hat{B}_9 &= [(D_0 + D_i) - (\beta_1^i + \beta_2^i + \mu_1^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{10} &= [(D_0 + D_i) - (\beta_1^i + \mu_1^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{11} &= [(D_0 + D_i) - (\beta_2^i + \mu_1^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{12} &= [(D_0 + D_i) - (\mu_1^i) I_{m_2}] \oplus H_0.
\end{aligned}$$

For $i \neq j$, $\hat{G}_2 = G_2$.

For $i = j$, $\hat{G}_3 = \begin{pmatrix} I_{s_1^i+1} \otimes \hat{C}_7 & e_{s_1^i+1} \otimes \beta_1^i I_{S_2m_2m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes \hat{C}_8 \end{pmatrix}$, where

$$\hat{C}_7 = \begin{pmatrix} I_{s_2^i} \otimes \hat{B}_{13} & e_{s_2^i} \otimes \beta_2^i I_{m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_{14} \end{pmatrix}, \hat{C}_8 = \begin{pmatrix} I_{s_2^i} \otimes \hat{B}_{15} & e_{s_2^i} \otimes \beta_2^i I_{m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_{16} \end{pmatrix}$$
$$\begin{aligned}
\hat{B}_{13} &= [(D_0 + D_i) - (\beta_1^i + \beta_2^i + \mu_2^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{14} &= [(D_0 + D_i) - (\beta_1^i + \mu_2^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{15} &= [(D_0 + D_i) - (\beta_2^i + \mu_2^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{16} &= [(D_0 + D_i) - (\mu_2^i) I_{m_2}] \oplus H_0.
\end{aligned}$$

For $i \neq j$, $\hat{G}_3 = G_3$

For $i = j$, $\hat{G}_4 = \begin{pmatrix} I_{s_1^i} \otimes \hat{C}_9 & e_{s_1^i} \otimes \beta_1^i I_{S_2m_2m_3} \\ \bar{0} & I_{S_1-s_1^i} \otimes \hat{C}_{10} \end{pmatrix}$, where

$$\hat{C}_9 = \begin{pmatrix} I_{s_2^i} \otimes \hat{B}_{17} & e_{s_2^i} \otimes \beta_2^i I_{m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_{18} \end{pmatrix}, \hat{C}_{10} = \begin{pmatrix} I_{s_2^i} \otimes \hat{B}_{19} & e_{s_2^i} \otimes \beta_2^i I_{m_2m_3} \\ \bar{0} & I_{S_2-s_2^i} \otimes \hat{B}_{20} \end{pmatrix}$$
$$\begin{aligned}
\hat{B}_{17} &= [(D_0 + D_i) - (\beta_1^i + \beta_2^i + \mu_{12}^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{18} &= [(D_0 + D_i) - (\beta_1^i + \mu_{12}^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{19} &= [(D_0 + D_i) - (\beta_2^i + \mu_{12}^i) I_{m_2}] \oplus H_0, \\
\hat{B}_{20} &= [(D_0 + D_i) - (\mu_{12}^i) I_{m_2}] \oplus H_0.
\end{aligned}$$

For $i \neq j$, $\hat{G}_4 = G_4$

5. Submatrices of A_0^{ij}
   For $i = j$,

$$L^{ij} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} I_{(S_1+1)(S_2+1)m_1m_2} \otimes H_1 & & & \\ & I_{S_1(S_2+1)m_2} \otimes H_1 & & \\ & & I_{(S_1+1)S_2m_2} \otimes H_1 & \\ & & & I_{S_1S_2m_2} \otimes H_1 \end{pmatrix} \end{array}.$$

For $i \neq j, L^{ij} = \bar{0}$

6. Submatrices of A_2^{ij}

For $i = j$, $M^{ij} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S_1 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & S_1 \end{array} \\ \begin{pmatrix} \hat{m}_0 & & & & \\ & \hat{m}_1 & & & \\ & & \hat{m}_1 & & \\ & & & \ddots & \\ & & & & \hat{m}_1 \end{pmatrix} \end{array}$, where

$$\hat{m}_0 = \begin{pmatrix} diagT^0 \otimes I_{m_2m_3} & \bar{0} \\ \bar{0} & I_{S_2} \otimes [\eta_1 diagT^0 \otimes I_{m_2m_3}] \end{pmatrix},$$

where $\eta_1 = p + (1-p)(p_1 + p_{12})$

$$\hat{m}_1 = \begin{pmatrix} \eta_2 diagT^0 \otimes I_{m_1m_2} & \bar{0} \\ \bar{0} & I_{S_2} \otimes p diagT^0 \otimes I_{m_1m_2} \end{pmatrix},$$

where $\eta_2 = p + (1-p)(p_2 + p_{12})$

For $i \neq j, M^{ij} = \bar{0}$

$$\bar{M}^{ij} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} \bar{0} & & & \\ & \mu_1^i I_{S_1(S_2+1)m_2m_3} & & \\ & & \mu_2^i I_{(S_1+1)S_2m_2m_3} & \\ & & & \mu_{12}^i I_{S_1S_2m_2m_3} \end{pmatrix} \end{array}.$$

For $i \neq j, \bar{M}^{ij} = \bar{0}$

References

1. Dudin, A.N.; Klimenok, V.I.; Vishnevsky, V.M. *The Theory of Queuing Systems with Correlated Flows*; Springer: Cham, Switzerland, 2020; pp. 1–410. [CrossRef]
2. Al-Khayyal, F.; Hwang, S.-J. Inventory constrained maritime routing and scheduling for multi-commodity liquid bulk, Part I: Applications and model. *Eur. J. Oper. Res.* **2007**, *176*, 106–130. [CrossRef]
3. Sadjady, H.; Davoudpour, H. Two-echelon, multi-commodity supply chain network design with mode selection, lead-times and inventory costs. *Comput. Oper. Res.* **2012**, *39*, 1345–1354 [CrossRef]
4. Qin, J.; Shi, F.; Miao, L.; Tan, G. Optimal Model and Algorithm for Multi-Commodity Logistics Network Design Considering Stochastic Demand and Inventory Control. *Syst. Eng. Theory Pract.* **2009**, *29*, 176–183. [CrossRef]
5. Askin, R.G.; Baffo, I.; Xia, M. Multi-commodity warehouse location and distribution planning with inventory consideration. *Int. J. Prod. Res.* **2014**, *52*, 1897–1910 [CrossRef]
6. Araya-Sassi, C.; Paredes-Belmar, G.; Gutiérrez-Jarpa, G. Multi-commodity inventory-location problem with two different review inventory control policies and modular stochastic capacity constraints. *Comput. Ind. Eng.* **2020**, *143*. [CrossRef]
7. Sabzevari Zadeh, A.; Sahraeian, R.; Homayouni, S.M. A dynamic multi-commodity inventory and facility location problem in steel supply chain network design. *Int. J. Adv. Manuf. Technol.* **2014**, *70*, 1267–1282. [CrossRef]
8. Shajin, D.; Jacob, J.; Krishnamoorthy, A. On a queueing inventory problem with necessary and optional inventories. *Ann. Oper. Res.* **2021**, 1–12. [CrossRef]
9. Song, J.S.; Zipkin, P. Inventory control in a fluctuating demand environment. *Oper. Res.* **1993**, *41*, 351–370. [CrossRef]
10. Özekici, S.; Parlar, M. Inventory models with unreliable suppliers in a random environment. *Ann. Oper. Res.* **1999**, *91*, 123–136. [CrossRef]

11. Erdem, A.S.; Özekici, S. Inventory models with random yield in a random environment. *Int. J. Prod. Econ.* **2002**, *78*, 239–253. [CrossRef]
12. Perry, D.; Posner, M.J.M. Production-inventory models with an unreliable facility operating in a two-state random environment. *Probab. Eng. Inf. Sci.* **2002**, *16*, 325–338. [CrossRef]
13. Feldman, R.M. A continuous review (s, S) inventory system in a random environment. *J. Appl. Probab.* **1978**, *15*, 654–659. [CrossRef]
14. Kalpakam, S.; Arivariqnan, G. A lost sales inventory system in a random environment. *Stoch. Anal. Appl.* **1989**, *7*, 367–385. [CrossRef]
15. Goh, M.; Sharafali, M. Price-dependent inventory models with discount offers at random times. *Prod. Oper. Manag.* **2002**, *11*, 139–156. [CrossRef]
16. Krishnamoorthy, A.; Shajin, D.; Viswanath, C.N. Inventory with Positive Service Time:A Survey. In *Advanced Trends in Queueing Theory: Series of Books "Mathematics and Statistics", Sciences*; Anisimov, V., Limnios, N., Eds.; ISTE & J. Wiley: London, UK, 2019.
17. Neuts, M.F. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*; The Johns Hopkins University Press: Baltimore, MD, USA, 1981.