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Article

Hyers Stability and Multi-Fuzzy Banach Algebra

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Abstract: In this paper, we define multi-fuzzy Banach algebra and then prove the stability of involution on multi-fuzzy Banach algebra by fixed point method. That is, if $f : A \to A$ is an approximately involution on multi-fuzzy Banach algebra A, then there exists an involution $H : A \to A$ which is near to f. In addition, under some conditions on f, the algebra A has multi C^*-algebra structure with involution H.

Keywords: stability; multi fuzzy Banach algebra; fixed point theory

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1. Introduction

H. G. Dales and M. E. Polyakov introduced the concept of multi-normed space in their article [1]. Multi normed space has a relation with ordered vector spaces and operator spaces. Furthermore, this concept is somewhat similar to that of the operator sequence space. We have collected some properties of multi-normed spaces which will be used in this article. We refer readers to [1–4] for more details.

Functional equations and their stability are some of the classical and practical issues in the area of mathematical analysis. About half a century ago, the stability of functional equations was raised with the important question of Ulam [5]. It is said that a functional equation G is stable if each function g satisfying the equation $G-$approximately is near to the true solution of G. D. H. Hyers developed Ulam's question and theorem [6]. He posed the following theorem:

Suppose that U and V be Banach spaces and let ρ be a function from U to V such that the following inequality satisfies for some $\delta > 0$ and for every $u, v \in U$,

$$\|\rho(u+v) - \rho(u) - \rho(v)\| \leq \delta.$$

Then there is only one additive function $T : U \to V$ so that

$$\|T(u) - \rho(u)\| \leq \delta$$

for any $u \in U$.

Mathematicians developed the results of the Hyers theorem. By changing the space, the norm, the control function, and functional equation, they could prove more interesting theorems [7–14]. For example, the Jenson functional equation or the integral and differential equations were used instead of the functional equation (in the theorem) and the validity of the theorem was proved. Now, we change the functional equation to a different lattice functional equation and various control functions in the above theorem are replaced.

Definition 1. *Let X be a set. A function $d : X^2 \to [0, \infty]$ is a called a generalized metric on X if and only if d satisfies*

(M_1) $d(x,y) = 0$ *if and only if* $x = y$;
(M_2) $d(x,y) = d(y,x)$, *for all* $x, y \in X$;
(M_3) $d(x,z) \le d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

We now introduce one of the fundamental results of the fixed point theory.

Theorem 1 ([15,16])**.** *Let (X, d) be a generalized complete metric space. Assume that $G : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a non-negative integer n_0 such that $d(G^{n_0+1}x, G^{n_0}x) < \infty$ for some $x \in X$, then the following statements are held:*

(i) *The sequence $\{G^n x\}$ converges to a fixed point x_0 of G;*
(ii) *x_0 is the unique fixed point of G in $Y = \{y \in X | d(G^{n_0}x, y) < \infty\}$;*
(iii) *If $y \in Y$, then*

$$d(y, x_0) \le \frac{1}{1-L} d(Gy, y).$$

Now, recall the notion of a multi-normed space from [1,4]. Let $(E, ||.||)$ be a complex normed space and let $k \in \mathbb{N}$. We denote by E^k, the linear space $E \oplus ... \oplus E$ consisting of k-tuples $(x_1, ..., x_k)$, where $x_1, ..., x_k \in E$. The linear operations on E^k are defined coordinat-wise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, ..., k\}$ and by $\mathcal{G}_k$ the group of permutations on k symbols.

Definition 2. *Let $(E, ||.||)$ be a complex (real) normed space. A multi-normed on $\{E^k, K \in \mathbb{N}\}$ is a sequence $\{||.||_k\}_{k\in\mathbb{N}}$ of norms on $E^k (k \in N)$ such that $||x||_1 = ||x||$, for each $x \in E$ and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \ge 2$:*

(N_1) $||(x_{\sigma(1)}, ..., x_{\sigma(k)})||_k = ||(x_1, ..., x_k)||_k \qquad (\sigma \in \mathcal{G}_k; x_1, ..., x_k \in E)$;
(N_2) $||(\alpha_1 x_1, ..., \alpha_k x_k)||_k \le (\max_{i\in\mathbb{N}_k} |\alpha_i|)||(x_1, ..., x_k)||_k$
$(\alpha_1, ..., \alpha_k \in \mathbb{C}; x_1, ..., x_k \in E)$;
(N_3) $||(x_1, ..., x_{k-1}, 0)||_k = ||(x_1, ..., x_{k-1})||_{k-1} \qquad (x_1, ..., x_{k-1} \in E)$;
(N4) $||(x_1, ..., x_{k-1}, x_{k-1})||_k = ||(x_1, ..., x_{k-1})||_{k-1} \qquad (x_1, ..., x_{k-1} \in E)$.

In this case, we say that $\{(E^k, ||.||_k), k \in \mathbb{N}\}$ is a multi-normed space.

Suppose that $\{(E^k, ||.||_k), k \in \mathbb{N}\}$ is a multi-normed space. The following properties are almost immediate consequences of the axioms:

(i) $||(x, ..., x)||_k = ||x|| \qquad (x \in E)$;
(ii) $max_{i\in\mathbb{N}_k} ||x_i|| \le ||(x_1, ..., x_k)||_k \le \sum_{i=1}^k ||x_i|| \le k \max_{i\in\mathbb{N}_k} ||x_i||$
$(x_1, ..., x_k \in E)$.

Applying (ii) one concludes that if $(E, ||.||)$ is a Banach space, then $(E^K, ||.||_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case, $\{(E^k, ||.||_k), k \in \mathbb{N}\}$ is called a multi-Banach space.

Definition 3. *Let $(E, ||.||)$ be a normed algebra such that $\{(E^K, ||.||) : k \in \mathbb{N}\}$ is a multi-normed space. Then $\{(E^K, ||.||) : k \in \mathbb{N}\}$ is a multi-normed algebra if*

$$||(x_1y_1, ..., x_ky_k)|| \le ||(x_1, ..., x_k)||_k||(y_1, ..., y_k)||_k \qquad (\forall k \in \mathbb{N} \quad x_1, ..., x_k, y_1, ..., y_k \in E).$$

Furthermore, the multi-normed algebra $\{(E^K, ||.||) : k \in \mathbb{N}\}$ is a multi-Banach algebra if $\{(E^K, ||.||) : k \in \mathbb{N}\}$ is a multi-Banach space.

Definition 4 ([17])**.** *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$*

(N_1) $N(x, t) = 0$, *for all* $t \le 0$;
(N_2) $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ *if* $c \ne 0$;
(N_4) $N(x + y, s + t) \ge \min\{N(x, s), N(y, t)\}$;

(N_5) *$N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and* $\lim_{t\to\infty} N(x,t) = 1$;

(N_6) *For $x \neq 0$, $N(x,.)$ is continuous on $\mathbb{R}$.*

The pair (X, N) is called a fuzzy normed vector space.

Definition 5. *Let (X, N) be a fuzzy normed vector space.*

(1) *A sequence $\{x_n\}$ in X is said to be convergent if there exists an $x \in X$ such that* $\lim_{n\to\infty} N(x_n - x, t) = 1, \quad \forall t > 0$. *In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by* $N - \lim_{n\to\infty} x_n = x$.

(2) *A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space. We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x_0 \in X$, then $f : X \to Y$ is said to be continuous on X.

Definition 6 ([18])**.** *Let X be an algebra and (X, N) a fuzzy normed space. The fuzzy normed space (X, N) is called a fuzzy normed algebra if*

$$N(xx', st) \geq N(x, s)N(x', t) \qquad \forall x, x' \in X, \quad s, t \in \mathbb{R}^+.$$

Complete fuzzy normed algebra is called a fuzzy Banach algebra.

Example 1. *Every normed algebra $(X, ||.||)$ defines a fuzzy normed algebra (X, N), where N is defined by*

$$N(x, t) = \frac{t}{t + ||x||} \qquad \forall x \in X, \quad \forall t > 0.$$

This space is called the induced fuzzy normed algebra.

Now, we recall the notion of a multi-fuzzy normed space. The readers can consider [19] for more details about the features of this space.

Definition 7. *Let (E, N) be a fuzzy normed space. A multi-fuzzy norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence $\{N_k\}$ such that N_k is a fuzzy norm on $E^k \quad k \in \mathbb{N}$, $N_1(x, t) = N(x, t)$ for each $x \in E$ and $t \in \mathbb{R}$ and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:*

(F_1) $N_k((x_{\sigma(1)}, ..., x_{\sigma(k)}), t) = N_k((x_1, ..., x_k), t)$
$(\forall \sigma \in \mathcal{G}_k, \quad \forall x_1, ..., x_k \in E, \quad \forall t \in \mathbb{R})$;

(F_2) $N_k((\alpha_1 x_1, ..., \alpha_k x_k), t) \geq N_k(\max_{i\in\mathbb{N}_k} |\alpha_i|(x_1, ..., x_k), t)$
$(\forall \alpha_1, ..., \alpha_k \in \mathbb{C}, \quad \forall x_1, ..., x_k \in E, \forall t \in \mathbb{R})$;

(F_3) $N_k((x_1, ..., x_{k-1}, 0), t) = N_{k-1}((x_1, ..., x_{k-1}), t)$
$(\forall x_1, ..., x_{k-1} \in E, \quad \forall t \in \mathbb{R})$;

(F4) $N_k((x_1, ..., x_{k-1}, x_{k-1}), t) = N_{k-1}((x_1, ..., x_{k-1}), t)$
$(\forall x_1, ..., x_{k-1} \in E, \quad \forall t \in \mathbb{R})$.

In this case, we say that $\{(E^k, N_k), k \in N\}$ is a multi-fuzzy normed space.

If (E, N_1) is a fuzzy Banach space, then $\{(E^k, N_k), k \in N\}$ is a multi-fuzzy Banach space (see [19]).

2. Main Result

We begin this section by introducing the notion of multi-fuzzy normed algebra. Then we develop the subject of the article [20] in multi-fuzzy Banach algebras.

Definition 8. *Let (E, N) be a fuzzy normed algebra, and let $\{(E^k, N_k), k \in N\}$ be a multi-fuzzy normed space. Then $\{(E^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy normed algebra if*

$$N_k((x_1y_1, ..., x_ky_k), st) \geq N_k((x_1, ..., x_k), s)N_k((y_1, ..., y_k), t),$$

for all $k \in \mathbb{N}$, $x_1, ..., x_k, y_1, ..., y_k \in E$ and $s, t \in \mathbb{R}^+$. Furthermore, the multi-fuzzy normed algebra $\{(E^k, N_k), k \in N\}$ is a multi-fuzzy Banach algebra if $\{(E^k, N_k), k \in N\}$ is a multi-fuzzy Banach space.

Example 2. *Every multi-Banach algebra $\{(E^k, ||.||_k), k \in N\}$ defines a multi-fuzzy Banach algebra $\{(E^k, N_k), k \in N\}$, where*

$$N_k((x_1, ..., x_k), t) = \frac{t}{t + ||(x_1, ..., x_k)||_k} \qquad t \in \mathbb{R}^+, \quad x_1, ..., x_k \in E.$$

In this article, we assume that m_0 is a natural number. We also assume that $\mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{T}^1_{\frac{1}{m_0}} := \{e^{i\theta}; 0 \leq \theta \leq \frac{2\pi}{m_0}\}$. Moreover, we suppose that (E, N) is fuzzy Banach algebra. For a given mapping $f : E \to E$, we define

$$D_{\lambda\gamma}f(x, y) = \bar{\lambda}f(\frac{x + \gamma y}{2}) + \bar{\lambda}f(\frac{x - \gamma y}{2}) - f(\lambda x) \qquad \forall x, y \in E \quad \text{and} \quad \forall \lambda, \gamma \in \mathbb{C}. \tag{1}$$

Let us recall some of the necessary definitions.
Let A be an algebra over $\mathbb{C}$. An involution on A is a mapping

$$\begin{aligned} \star : & A \to A \\ & a \longmapsto a^\star \end{aligned}$$

such that

(i) $(\alpha a + \beta b)^\star = \bar{\alpha}a^\star + \bar{\beta}b^\star \qquad \forall a, b \in A, \quad \forall \alpha, \beta \in \mathbb{C};$
(ii) $(ab)^\star = b^\star a^\star \qquad \forall a, b \in A;$
(iii) $a^{\star\star} = a \qquad \forall a \in A.$

1. A complex algebra with an involution is a $\star$-algebra.
2. A $C^\star$-algebras is a (non-zero) Banach algebra with an involution, such that:

$$||a^\star a|| = ||a||^2.$$

Definition 9. *Let A be an $\star$-algebra and (A, N) a fuzzy normed algebra. The fuzzy normed algebra (A, N) is called a fuzzy normed $\star$-algebra if*

$$N(a^\star, t) = N(a, t) \qquad \forall a \in A, \quad \forall t \in \mathbb{R}^+.$$

A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.

Definition 10. *Let (A, N) be a fuzzy Banach $*$-algebra. Then (A, N) is called a fuzzy C^*-algebra if*

$$N(a^\star a, st) = N(a^\star, s)N(a, t) \qquad \forall a \in A, \quad \forall s, t \in \mathbb{R}^+.$$

Theorem 2. *Let (E, N) be a fuzzy Banach algebra and $\{(E^K, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach algebra. In addition, suppose that $\psi : E^{2k} \to [0, \infty)$ is a given function and there exists a constant L, $0 < L < 1$, such that:*

$$\psi(x_1, y_1, ..., x_k, y_k) \leq 2L\psi(\frac{x_1}{2}, \frac{y_1}{2}, ..., \frac{x_k}{2}, \frac{y_k}{2}), \tag{2}$$

$$N_k((D_{\lambda\gamma}f(x_1, y_1), D_{\lambda\gamma}f(x_2, y_2), ..., D_{\lambda\gamma}f(x_k, y_k)), t) \geq \frac{t}{t + \psi(x_1, y_1, ..., x_k, y_k)}, \tag{3}$$

$$N_k((f(x_1y_1) - f(y_1)f(x_1), ..., f(x_ky_k) - f(y_k)f(x_k)), t) \geq \frac{t}{t + \psi(x_1, y_1, ..., x_k, y_k)}, \tag{4}$$

$$N - \lim_{n\to\infty} 2^{-n} f(2^n(N - \lim_{n\to\infty} 2^{-n} f(2^n x))) = x, \tag{5}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$, *all* $t > 0$ *and all* $\lambda, \gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$, *then there exists a unique involution* $H : E \to E$ *such that*

$$H(x) := N - \lim_{n\to\infty} 2^{-n} f(2^n x)$$

and

$$N_k((f(x_1) - H(x_1), ..., f(x_k) - H(x_k)), t) \geq \frac{(1-L)t}{(1-L)t + L\psi(x_1, 0, ..., x_k, 0)}. \tag{6}$$

Also, if for all $x_1, ..., x_k \in E$ *and for all* $t > 0$

$$N_k(([N(f(x_1), t) - N(x_1, t)]x_1, ..., [N(f(x_k), t) - N(x_k, t)]x_k), t) \geq \frac{t}{t + \psi(x_1, x_1, ..., x_k, x_k)}, \tag{7}$$

then (E, N) *is a fuzzy Banach* $*$*-algebra.*
Moreover, if for all $x_1, ..., x_k \in E$ *and for all* $s, t > 0$

$$\begin{aligned} &N_k(([N(f(x_1)x_1, st) - N(f(x_1), s)N(x_1, t)]x_1, ..., \\ &[N(f(x_k)x_k, st) - N(f(x_k), s)N(x_k, t)]x_k), t) \geq \frac{t}{t + \psi(x_1, x_1, ..., x_k, x_k)}, \end{aligned} \tag{8}$$

then (E, N) *is a fuzzy* C^**-algebra with involution* $x^* = H(x)$ *for all* $x \in E$.

Proof. Consider the set $S := \{g : E \to E\}$ and introduce the generalized metric on S.

$$d(g, h) = \inf\{\delta \in [0, \infty] : N_k((g(x_1) - h(x_1), ..., g(x_k) - h(x_k)), \delta t) \geq \frac{t}{t + \psi(x_1, 0, ..., x_k, 0)}\},$$

for all $x_1, ..., x_k \in E$ and $t > 0$. Where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [21]). Now, we define mappings $J : S \to S$ by

$$Jg(x) := \frac{1}{2} g(2x) \qquad \forall x \in E.$$

First, we prove that J is strictly contractive on S. Let $g, h \in S$ be given such that $d(g, h) \neq +\infty$. Then for some $\epsilon > 0$

$$N_k((g(x_1) - h(x_k), ..., g(x_k) - h(x_k)), \epsilon t) \geq \frac{t}{t + \psi(x_1, 0, ..., x_k, 0)} \qquad \forall x_1, ..., x_k \in E, \forall t > 0.$$

If we replace x_k in the above inequality with $2x_k$, for $k = 1, ..., n$, and make use of (2), then we have

$$\begin{aligned} &N((Jg(x_1) - Jh(x_1), ..., Jg(x_k) - Jh(x_k)), L\epsilon t) \\ &= N((\frac{1}{2}g(2x_1) - \frac{1}{2}h(2x_1), ..., \frac{1}{2}g(2x_k) - \frac{1}{2}h(2x_k)), L\epsilon t) \\ &= N((g(2x_1) - h(2x_1), ..., g(2x_k) - h(2x_k)), 2L\epsilon t) \\ &\geq \frac{2Lt}{2Lt + \psi(2x_1, 0, ..., 2x_k, 0)} \\ &\geq \frac{2Lt}{2Lt + 2L\psi(x_1, 0, ..., x_k, 0)} = \frac{t}{t + \psi(x_1, 0, ..., x_k, 0)}, \end{aligned} \tag{9}$$

for all $x_1, ..., x_k \in E$ and all $t > 0$. Therefore, using the definition of d metric, we can conclude that $d(Jg, Jh) \leq L\epsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h) \qquad \forall g, h \in S.$$

Next, we assert that $d(Jf, f) < \infty$. Putting $\lambda = 1$ and $y_1 = ... = y_k = 0$ in (3), we get

$$\begin{aligned}
&N_k((\frac{1}{2}f(2x_1) - f(x_1), ..., \frac{1}{2}f(2x_k) - f(x_k)), Lt) \\
&= N_k((f(2x_1) - 2f(x_1), ..., f(2x_k) - 2f(x_k)), 2Lt) \\
&\geq \frac{2Lt}{2Lt + \psi(2x_1, 0, ..., 2x_k, 0)} \\
&\geq \frac{2Lt}{2Lt + 2L\psi(x_1, 0, ..., x_k, 0)} = \frac{t}{t + \psi(x_1, 0, ..., x_k, 0)},
\end{aligned}$$

for any $x_1, ..., x_k \in E$, that is

$$d(Jf, f) \leq L < \infty. \tag{10}$$

Now, it follows Theorem 1 that there exists a function $H : E \to E$ which is a fixed point of J, i.e,

$$H(x) = \frac{1}{2}H(2x),$$

such that $\lim_{n\to\infty} d(J^n f, H) = 0$. Therefore, it can be concluded that

$$N - \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = H(x) \qquad \forall x \in E.$$

Then $H \in X^*$, which:

$$X^* = \{g \in S : d(f, g) < \infty\}.$$

Again, by Theorem 1 and (10), we obtain

$$d(f, H) \leq \frac{1}{1-L} d(Jf, f) \leq \frac{L}{1-L},$$

i.e, the inequality (6) is true for all $x \in E$. Suppose $\lambda = \gamma = 1$ in (2), we have

$$\begin{aligned}
&N_k\Big((2^{-n}f(2^n(\frac{x_1+y_1}{2})) + 2^{-n}f(2^n(\frac{x_1-y_1}{2})) - 2^{-n}f(2^n x_1), ..., \\
&2^{-n}f(2^n(\frac{x_k+y_k}{2})) + 2^{-n}f(2^n(\frac{x_k-y_k}{2})) - 2^{-n}f(2^n x_k)), 2^{-n}t\Big) \\
&\geq \frac{t}{t + \psi(2^n x_1, 2^n y_1, ..., 2^n x_k, ..., 2^n y_k)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&N_k\Big((2^{-n}f(2^n(\frac{x_1+y_1}{2})) + 2^{-n}f(2^n(\frac{x_1-y_1}{2})) - 2^{-n}f(2^n x_1), ..., \\
&2^{-n}f(2^n(\frac{x_k+y_k}{2})) + 2^{-n}f(2^n(\frac{x_k-y_k}{2})) - 2^{-n}f(2^n x_k), t\Big) \\
&\geq \frac{2^n t}{2^n t + 2^n L^n \psi(x_1, y_1, ..., x_k, ..., y_k)} \to 1 \text{ as } n \to \infty,
\end{aligned}$$

for all $x_1, ..x_k, y_1, ..., y_k$ in E and for all $t > 0$. Therefore

$$N_k((H(\frac{x_1+y_1}{2}) + H(\frac{x_1-y_1}{2}) - H(x_1), ..., H(\frac{x_k+y_k}{2}) + H(\frac{x_k-y_k}{2}) - H(x_k)), t) = 1.$$

By replacing $x_1, ..., x_k$ with x and $y_1, ..., y_k$ with y in the last inequality, we conclude

$$N(H(\frac{x+y}{2}) + H(\frac{x-y}{2}) - H(x), t) = 1.$$

We get

$$H(x) = H(\frac{x+y}{2}) + H(\frac{x-y}{2}),$$

for all $x, y \in E$. If $y_1 = ... = y_k = 0$ in (3), then we have

$$\begin{aligned}
&N_k\Big((2^{-n}\bar{\lambda}f(2^{n-1}x_1) + 2^{-n}\bar{\lambda}f(2^{n-1}x_1) - 2^{-n}f(\lambda 2^n x_1), ...,\\
&2^{-n}\bar{\lambda}f(2^{n-1}x_k) + 2^{-n}\bar{\lambda}f(2^{n-1}x_k) - 2^{-n}f(\lambda 2^n x_k)), 2^{-n}t\Big)\\
&\geq \frac{t}{t + \psi(2^n x_1, 0, ..., 2^n x_k, ..., 0)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&N_k\Big((2^{-n}\bar{\lambda}f(2^{n-1}x_1) + 2^{-n}\bar{\lambda}f(2^{n-1}x_1) - 2^{-n}f(\lambda 2^n x_1), ...,\\
&2^{-n}\bar{\lambda}f(2^{n-1}x_k) + 2^{-n}\bar{\lambda}f(2^{n-1}x_k) - 2^{-n}f(\lambda 2^n x_k)), t\Big)\\
&\geq \frac{2^n t}{2^n t + 2^n L^n \psi(x_1, 0, ..., x_k, ..., 0)} \to 1 \text{ as } n \to \infty,
\end{aligned}$$

for all $x_1, ..., x_k \in E$ and $t > 0$, then

$$N_k((2\bar{\lambda}H(\frac{x_1}{2}) - H(\lambda x_1), ..., 2\bar{\lambda}H(\frac{x_k}{2}) - H(\lambda x_k)), t) = 1,$$
$$\forall x_1, .., x_k \in E, \quad \forall \lambda \in \mathbb{T}^1_{\frac{1}{m_0}}, \quad \forall t > 0.$$

By replacing $x_1, ..., x_k$ with x in the last inequality, we conclude

$$N((2\bar{\lambda}H(\frac{x}{2}) - H(\lambda x), t) = 1 \qquad \forall x_1, .., x_k \in E, \quad \forall \lambda \in \mathbb{T}^1_{\frac{1}{m_0}}, \quad \forall t > 0.$$

It follows by the last equation and additivity of H that $H(\lambda x) = \bar{\lambda}H(x)$, for all $x \in E$ and all $\lambda \in \mathbb{T}^1_{\frac{1}{m_0}}$.

We will use techniques [22] to continue proving. Now, we show that H is conjugate linear. We have to show that $H(\alpha x) = \bar{\alpha}H(x)$ for all $\alpha \in \mathbb{C}$, $x \in E$. To this end, let $\alpha \in \mathbb{C}$. If α belongs to $\mathbb{T}^1$, then there exists $\theta \in [0, 2\pi]$ such that $\alpha = e^{i\theta}$. We set $\alpha_1 = e^{\frac{i\theta}{m_0}}$, thus α_1 belongs to $\mathbb{T}^1_{\frac{1}{m_0}}$ and $H(\alpha x) = H(\alpha_1^{m_0} x) = \bar{\alpha}_1^{m_0} H(x) = \bar{\alpha}H(x)$.

If α belong to $n\mathbb{T}^1 = \{nz; \quad z \in \mathbb{T}^1\}$ for some $n \in \mathbb{N}$, then by additivity of H, $H(\alpha x) = \bar{\alpha}H(x)$ for all $x \in E$.
Let $t \in (0, \infty)$ then by Archimedes property of $\mathbb{C}$, there exists a positive real number n such that the point $(t, 0)$ lies in the interior of a circle with centre at origin and radius n. Putting $t_1 := t + \sqrt{n^2 - t^2}i$, $t_2 := t - \sqrt{n^2 - t^2}i$. Then we have $t = \frac{t_1 + t_2}{2}$ and $t_1, t_2 \in n\mathbb{T}^1$. It follows that

$$H(tx) = H(\frac{t_1 + t_2}{2}x) = \frac{\bar{t}_1}{2}H(x) + \frac{\bar{t}_2}{2}H(x) = \bar{t}H(x) = tH(x) \qquad \forall x \in E.$$

On the other hand, there exists $\theta \in [0, 2\pi]$ such that $\alpha = |\alpha|e^{i\theta}$. It follows that

$$H(\alpha x) = H(|\alpha|e^{i\theta}x) = |\alpha|e^{-i\theta}H(x) = \bar{\alpha}H(x) \qquad \forall x \in E.$$

Hence, $H : E \to E$ is conjugate $\mathbb{C}$-linear mapping. By (4)

$$\begin{aligned}
&N_k\Big((4^{-n}f(4^n x_1 y_1) - 2^{-n}f(2^n y_1).2^{-n}f(2^n x_1), ..., \\
&4^{-n}f(4^n x_k y_k) - 2^{-n}f(2^n y_k).2^{-n}f(2^n x_k)), 4^{-n}t\Big) \\
&\geq \frac{t}{t + \psi(2^n x_1, 2^n y_1, ..., 2^n x_k, 2^n y_k)} x \\
&\geq \frac{t}{t + 2^n L^n \psi(x_1, y_1, ..., x_k, y_k)},
\end{aligned}$$

thus

$$\begin{aligned}
&N_k\Big((4^{-n}f(4^n x_1 y_1) - 2^{-n}f(2^n y_1).2^{-n}f(2^n x_1), ..., \\
&4^{-n}f(4^n x_k y_k) - 2^{-n}f(2^n y_k).2^{-n}f(2^n x_k)), t\Big) \\
&\geq \frac{4^n t}{4^n t + 2^n L^n \psi(x_1, y_1, ..., x_k, y_k)} \to 1 \text{ as } n \to \infty,
\end{aligned}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$ and all $t > 0$, so we have

$$N_k((H(x_1 y_1) - H(y_1)H(x_1), ..., H(x_k y_k) - H(y_k)H(x_k)), t) = 1 \qquad \forall x_1, ..., x_k, y_1, ..., y_k \in E, \forall t > 0$$

By replacing $x_1, ..., x_k$ with x and $y_1, ..., y_k$ with y in the last inequality, we conclude

$$N(H(xy) - H(y)H(x), t) = 1 \qquad \forall x, y \in E, \quad \forall t > 0$$

and therefore,

$$H(xy) = H(y)H(x) \qquad \forall x, y \in E.$$

On the other hand, by (5)

$$H(H(x)) = N - \lim_{n\to\infty} 2^{-n}f(2^n(N - \lim_{k\to\infty} 2^{-n}f(2^n x))) = x,$$

for all x in E. Hence $H : E \to E$ is an involution satisfying (6).

In addition, we must prove the uniqueness of H. In fact, assume the existence of another involution H' satisfies (6), hence $H'(\frac{x}{2^k}) = \frac{1}{2^k}H'(x), \forall x \in E$, so we have

$$\begin{aligned}
&N_k((2^{-n}f(2^n x_1) - 2^{-n}H'(2^n x_1), ..., 2^{-n}f(2^n x_k) - 2^{-n}H'(2^n x_k)), t) \\
&= N((f(2^n x_1) - H'(2^n x_1), ..., f(2^n x_k) - H'(2^n x_k)), 2^n t) \\
&\geq \frac{(1-L)2^n t}{(1-L)2^n t + L\psi(2^n x_1, 0, ..., 2^n x_k, 0)} \\
&= \frac{(1-L)2^n t}{(1-L)2^n t + 2^n L^{n+1}\psi(x_1, 0, ..., x_k, 0)} \to 1 \text{ as } n \to \infty,
\end{aligned}$$

for all $x_1, ..., x_k \in E$, $t > 0$, then

$$N_k((H(x_1) - H'(x_1), ..., H(x_k) - H'(x_k)), t) = 1 \qquad \forall x_1, ..., x_k \in E, \quad \forall t > 0$$

By replacing $x_1, ..., x_k$ with x in the last inequality, we conclude

$$N(H(x) - H'(x), t) = 1 \qquad \forall x \in E, \quad \forall t > 0$$

Therefore, $H(x) = H'(x)$ for all $x \in E$. Now, suppose that H satisfies (7), then we have

$$\begin{aligned} N_k\Big(& ([N(2^{-n} f(2^n x_1), 2^{-n} t) - N(x_1, 2^{-n} t)]x_1, ..., \\ & [N(2^{-n} f(2^n x_k), 2^{-n} t) - N(x_k, 2^{-n} t)]x_k), 2^{-n} t\Big) \\ & \geq \frac{t}{t + \psi(2^n x_1, 2^n x_1, ..., 2^n x_k, 2^n x_k)}, \end{aligned}$$

thus,

$$\begin{aligned} & N_k(([N(2^{-n} f(2^n x_1), t) - N(x_1, t)]x_1, ..., \\ & [N(2^{-n} f(2^n x_k), t) - N(x_k, t)]x_k), t) \\ & \geq \frac{2^n t}{2^n t + 2^n L^n \psi(x_1, x_1, ..., x_k, x_k)} \to 1 \text{ as } n \to \infty, \end{aligned}$$

for all $x_1, ..., x_k \in E$ and $t > 0$, therefore

$$N_k(([N(H(x_1), t) - N(x_1, t)]x_1, ..., [N(H(x_k), t) - N(x_k, t)]x_k), t) = 1$$

for all $x_1, ..., x_k \in E$ and $t > 0$. Putting $x_1 = ... = x_k := x$ in the above equality, we get

$$\begin{aligned} & N([N(H(x), t) - N(x, t)]x, t) = 1 && \forall x \in E, \quad t > 0 \\ \Longrightarrow \quad & [N(H(x), t) - N(x, t)]x = 0 && \forall x \in E, \quad t > 0 \\ \Longrightarrow \quad & N(H(x), t) - N(x, t) = 0 && \forall x \in E, \quad t > 0 \end{aligned}$$

Therefore, $N(H(x), t) = N(x, t)$ and (E, N) is a fuzzy Banach $*$-algebra. Finally, we assume that H satisfies (8). Then we have

$$\begin{aligned} & N_k(([N(2^{-n} f(2^n x_1)x_1, 2^{-2n} st) - N(2^{-n} f(2^n x_1), 2^{-n} s)N(x_1, 2^{-n} t)]x_1, ..., \\ & [N(2^{-n} f(2^n x_k)x_k, 2^{-2n} st) - N(2^{-n} f(2^n x_k), 2^{-n} s)N(x_k, 2^{-n} t)]x_k), 2^{-n} t) \\ & \geq \frac{t}{t + \psi(2^n x_1, 2^n x_1, ..., 2^n x_k, ..., 2^n x_k)}. \end{aligned}$$

Thus,

$$\begin{aligned} & N_k(([N(2^{-n} f(2^n x_1)x_1, st) - N(2^{-n} f(2^n x_1), s)N(x_1, t)]x_1, ..., \\ & [N(2^{-n} f(2^n x_k)x_k, st) - N(2^{-n} f(2^n x_k), s)N(x_k, t)]x_k), t) \\ & \geq \frac{2^n t}{2^n t + 2^n L^n \psi(x_1, x_1, ..., x_k, ..., x_k)}. \end{aligned}$$

Again, similarly to the above it can be concluded

$$
\begin{aligned}
& N_k(([N(H(x_1)x_1,st)-N(H(x_1),s)N(x_1,t)]x_1,...,\\
& [N(H(x_k)x_k,st)-N(H(x_k),s)N(x_k,t)]x_k),t)=1 \qquad \forall x_1,...,x_k\in E, \quad s,t>0\\
\Longrightarrow \quad & N([N(H(x)x,st)-N(H(x),s)N(x,t)]x,t)=1 \qquad \forall x\in E, \quad s,t>0\\
\Longrightarrow \quad & [N(H(x)x,st)-N(H(x),s)N(x,t)]x=0 \qquad \forall x\in E, \quad s,t>0\\
\Longrightarrow \quad & N(H(x)x,t)-N(H(x),s)N(x,t)=0. \qquad \forall x\in E, \quad s,t>0.
\end{aligned}
$$

Therefore, $N(H(x)x,st)=N(H(x),s)N(x,t)$, then E is a C^*-algebra with involution $x^*=H(x)$, for all $x\in E$. $\square$

Theorem 3. *Let (E,N) be a fuzzy Banach algebra and $\{(E^K,N_k),k\in\mathbb{N}\}$ be a multi-fuzzy Banach algebra. In addition, suppose that $\psi:E^{2k}\to[0,\infty)$ is a given function and there exists a constant L, $0<L<\frac{1}{2}$, such that*

$$\psi(x_1,y_1,...,x_k,y_k)\le\frac{L}{2}\psi(2x_1,2y_1,...,2x_k,2y_k), \tag{11}$$

$$N_k((D_{\lambda\gamma}f(x_1,y_1),D_{\lambda\gamma}f(x_2,y_2),...,D_{\lambda\gamma}f(x_k,y_k)),t)\ge\frac{t}{t+\psi(x_1,y_1,...,x_k,y_k)}, \tag{12}$$

$$N_k((f(x_1y_1)-f(y_1)f(x_1),...,f(x_ky_k)-f(y_k)f(x_k)),t)\ge\frac{t}{t+\psi(x_1,y_1,...,x_k,y_k)}, \tag{13}$$

$$N-\lim_{n\to\infty}2^nf(2^{-n}(N-\lim_{n\to\infty}2^nf(2^{-n}x)))=x, \tag{14}$$

for all $x_1,...,x_k,y_1,...,y_k\in E$, all $t>0$ and all $\lambda,\gamma\in\mathbb{T}^1_{\frac{1}{m_0}}$, then there exists a unique involution $H:E\to E$ such that

$$H(x):=N-\lim_{n\to\infty}2^nf(2^{-n}x)$$

and

$$N_k((f(x_1)-H(x_1),...,f(x_k)-H(x_k)),t)\ge\frac{(1-L)t}{(1-L)t+\psi(x_1,0,...,x_k,0)}. \tag{15}$$

Further, if for all $x_1,...,x_k\in E$ and for all $t>0$

$$N_k(([N(f(x_1),t)-N(x_1,t)]x_1,...,[N(f(x_k),t)-N(x_k,t)]x_k),t)\ge\frac{t}{t+\psi(x_1,x_1,...,x_k,x_k)}, \tag{16}$$

then (E,N) is a fuzzy Banach $*$-algebra.
Moreover, if for all $x_1,...,x_k\in E$ and for all $s,t>0$

$$
\begin{aligned}
& N_k(([N(f(x_1)x_1,st)-N(f(x_1),s)N(x_1,t)]x_1,...,\\
& [N(f(x_k)x_k,st)-N(f(x_k),s)N(x_k,t)]x_k),t)\ge\frac{t}{t+\psi(x_1,x_1,...,x_k,x_k)},
\end{aligned} \tag{17}
$$

then (E,N) is a fuzzy C^*-algebra with involution $x^*=H(x)$ for all $x\in E$.

Proof. Let (S,d) be the complete generalized metric space defined in the proof of Theorem 2. Consider the linear mapping $J:S\to S$ by

$$Jg(x):=2g(\frac{x}{2}) \qquad \forall x\in E.$$

Putting $\lambda=1$ and $y_1=...=y_k=0$ in (12), we have

$$N_k((2f(\frac{x_1}{2})-f(x_1),...,2f(\frac{x}{k})-f(x_k)),t)\ge\frac{t}{t+\psi(x_1,0,...,x_k,0)},$$

for all $x_1, ..x_k \in E$ and all $t > 0$. Therefore $d(Jf, f) \leq 1$ and thus

$$d(f, H) \leq \frac{1}{1-L},$$

which implies that the inequality (15) holds. The rest of the proof is similar to the proof of Theorem 2. $\square$

Corollary 1. *Let (E, N) be a fuzzy Banach algebra and $\{(E^K, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach algebra. In addition, let $p \in (0,1)$ and $\theta \in [0, \infty)$ be real numbers. Suppose that $f : E \to E$ with $f(1) = 1$, satisfies satisfying*

$$N_k((D_{\lambda\gamma}f(x_1, y_1), D_{\lambda\gamma}f(x_2, y_2), ..., D_{\lambda\gamma}f(x_k, y_k)), t) \geq \frac{t}{t + \theta \sum_{i=1}^{k}(||x_i||^p + ||y_i||^p)},$$

$$N_k((f(x_1y_1) - f(y_1)f(x_1), ..., f(x_ky_k) - f(y_k)f(x_k)), t) \geq \frac{t}{t + \theta \sum_{i=1}^{k}(||x_i||^p + ||y_i||^p)},$$

$$N - \lim_{k\to\infty} 2^{-k}f(2^k(N - \lim_{k\to\infty} 2^{-k}f(2^k x))) = x,$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$, all $t > 0$ and all $\lambda, \gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$. Then there exists a unique involution $H : E \to E$ such that

$$H(x) := N - \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$$

and

$$N_k((f(x_1) - H(x_1), ..., f(x_k) - H(x_k)), t) \geq \frac{(1 - 2^{p-1})t}{(1 - 2^{p-1})t + 2^{p-1}\theta \sum_{i=1}^{k} ||x_i||^p}.$$

Further, if

$$N_k(([N(f(x_1), t) - N(x_1, t)]x_1, ..., [N(f(x_k), t) - N(x_k, t)]x_k), t) \geq \frac{t}{t + 2\theta \sum_{i=1}^{k} ||x_i||^p},$$

for all $x_1, ..., x_k \in E$ and for all $t > 0$, then (E, N) is a fuzzy Banach $*$-algebra.
Moreover, if

$$\begin{aligned} &N_k(([N(f(x_1)x_1, st) - N(f(x_1), s)N(x_1, t)]x_1, ..., \\ &[N(f(x_k)x_k, st) - N(f(x_k), s)N(x_k, t)]x_k), t) \geq \frac{t}{t + 2\theta \sum_{i=1}^{k} ||x_i||^p}, \end{aligned}$$

for all $x_1, ..., x_k \in E$ and for all $s, t > 0$, then (E, N) is a fuzzy C^*-algebra with involution $x^* = H(x)$ for all $x \in E$.

Proof. It follows from Theorem 2 by putting

$$\psi(x_1, y_1, ..., x_k, y_k) = \theta \sum_{i=1}^{k} ||x_i||^p + ||y_i||^p),$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$ and $L = 2^{p-1}$. $\square$

Corollary 2. *Let (E, N) be a fuzzy Banach algebra and $\{(E^K, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach algebra. In addition, let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose that $f : E \to E$ with $f(1) = 1$, satisfies*

$$N_k((D_{\lambda\gamma}f(x_1,y_1), D_{\lambda\gamma}f(x_2,y_2), ..., D_{\lambda\gamma}f(x_k,y_k)), t) \geq \frac{t}{t + \theta\sum_{i=1}^{k}\sum_{j=1}^{k}||x_i y_j||^p},$$

$$N_k((f(x_1y_1) - f(y_1)f(x_1), ..., f(x_ky_k) - f(y_k)f(x_k)), t) \geq \frac{t}{t + \theta\sum_{i=1}^{k}\sum_{j=1}^{k}||x_i y_j||^p},$$

$$N - \lim_{k\to\infty} 2^{-k} f(2^k(N - \lim_{k\to\infty} 2^{-k} f(2^k x))) = x,$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$, all $t > 0$ and all $\lambda, \gamma \in \mathbb{T}^1_{\frac{1}{m_0}}$. Then f is an involution on E. Moreover, if

$$N_k(([N(f(x_1),t) - N(x_1,t)]x_1, ..., [N(f(x_k),t) - N(x_k,t)]x_k), t) \geq \frac{t}{t + \theta(\sum_{i=1}^{k}||x_i||^p)^2},$$

for all $x_1, ..., x_k \in E$ and for all $t > 0$, then (E, N) is a fuzzy Banach $*$-algebra. Furthermore, if

$$\begin{aligned} &N_k(([N(f(x_1)x_1, st) - N(f(x_1),s)N(x_1,t)]x_1, ..., \\ &[N(f(x_k)x_k, st) - N(f(x_k),s)N(x_k,t)]x_k), t) \geq \frac{t}{t + \theta(\sum_{i=1}^{k}||x_i||^p)^2}, \end{aligned}$$

for all $x_1, ..., x_k \in E$ and for all $s, t > 0$, then (E, N) is a fuzzy C^*-algebra with involution $x^* = H(x)$ for all $x \in E$.

Proof. We put

$$\psi(x_1, y_1, ..., x_k, y_k) := \theta \sum_{i=1}^{k}\sum_{j=1}^{k} ||x_i y_j||^p$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$ and $L = 2^{2p-1}$ in Theorem 2, and then, as a result, the sentence is obtained. □

3. Conclusions

We define multi-fuzzy Banach algebra and then prove the Hyers–Ulam–Rassias stability of involution on multi-fuzzy Banach algebra by fixed point method and find some conditions for which a multi-Banach algebra with approximate involution is a C^*-algebra.

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