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Corporations' Investment, Market Value, and Involuntary Unemployment in a Stock Market Overlapping Generations Model: A Purely Theoretical Exercise

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Abstract

It is the aim of this purely theoretical paper to model corporations' investment decision under market value maximization and managers' "beliefs" about sales of future production output such that involuntary unemployment in a stock market overlapping generations model occurs. In contrast to New-Keynesian macro-models, unemployment is not traced back to inflexible prices and wage rates, but to inflexible aggregate investment based on corporation managers' expectations regarding future sales of the production output enabled by present investment. After setting up the stock market model, sufficient conditions for the existence and dynamic stability of a steady state with involuntary unemployment are presented and the comparative dynamics of this steady state is investigated. Both the rise in investors' optimism and the decline of the savings rate decrease unemployment in the short and in long run.

Keywords

Market Value, Corporation Investment, Involuntary Unemployment, Investors' Beliefs, Stock Market OLG Model

1. Introduction

Corporations' investment decisions are not explicitly modeled in Diamond's (1965) seminal overlapping generations (OLG) model of production and capital accumulation. As Magnani (2015) aptly observes: investment is "macro-founded" such that aggregate savings of households determine aggregate investment which is equal to the sum of perfectly flexible corporations' investments. This perfect flexibility of individual and aggregate investment precludes, however, that aggregate demand falls short of aggregate supply. The absence of "aggregate demand failures" or in Keynes' (1936) words the validity of Say's Law prevents that involuntary unemployment occurs due to lacking aggregate demand.

In contrast to mainstream New-Keynesian stochastic dynamic general equilibrium (DSGE) models in which involuntary unemployment is referred to sticky prices and wage rates, Morishima (1977) and more recently Magnani (2015) trace involuntary unemployment back to inflexible aggregate investment formed independently of aggregate savings. However, Magnani (2015) in Solow's (1956) neo-classical growth model and Farmer and Kuplen (2018) in Diamond's (1965) OLG model simply assume that aggregate investment is somewhat determined by "animal spirits" of investors without any connections to corporations' production technologies.

1.1. Separation of Savers and Investors and Alignment of Managers and Shareholders' Interests

To provide production-theoretic foundations for an independent aggregate investment function it is necessary that savings and investment decisions are undertaken by different, self-interested agents. This is the case when households do not hold physical capital by themselves but only shares in corporations whose managers decide on production of goods and on investment in productive capital. On the other hand, the personal separation of wealth allocation and of production and investment questions the alignment of the interests of owners and managers of firms. While this problem was high on the agenda of microeconomic general equilibrium modelers in the 1970s and 1980s, the new-classical revolution in macroeconomics with its focus on the infinitely-lived, representative agent who directly holds physical capital and decides on its use as production factor and on how much to accumulate of it over time, contributed to the fact that the alignment problem got somewhat out of sight since then in macroeconomic modeling.

1.2. Market Value Maximization and Shareholder Unanimity

However, no rule without exceptions. In line with the microeconomic general equilibrium models of the 1970s and 1980s firstly Devereux and Lockwood (1991) and more recently Cunha (2012) addressed the alignment problem in macroeconomic stock-market OLG models. Core to the alignment problem is the question of whether market-value maximization of corporations is in the unanimous interest of their shareholders (DeAngelo, 1981). While the mainstream of microeconomic, general equilibrium research of 1970s and 1980s holds that this is in general not the case (e.g. Dreze, 1985 and Forsythe/Suchanek, 1987), Makowski (1980, 1983) proves that market value maximization of corporations is in the unanimous interest of corporations' shareholders in perfectly competitive stock-market economies. Without reference to Makowski (1983), Devereux and Lockwood (1991) and Cunha (2012: p. 6) claim that market value maximization of corporations is in the unanimous interest of their shareholders. While this claim would need a careful discussion (see Farmer, 1989), we abstain from it in this paper, and follow Cunha's (2012: p. 6) statement that market value maximization of perfectly competitive corporations is in the unanimous interest of their shareholders when agency problems are absent.

1.3. Involuntary Unemployment and Degenerate Belief Function

Besides the objectives of corporations' managers in line with the interests of their owners there is the problem how to model involuntary unemployment in perfectly competitive stock-market economies. Cunha (2012) like Miyashita (2000) and Magill and Qunizii (2003) assume full employment in their stock market economies with market-value maximizing corporations. Farmer (2023a, 2023b) endogenizes the unemployment rate[1] in line with Magnani (2015) in Magill and Qunizii's (2003) stock-market OLG model and closes the system of the intertemporal equilibrium equations by a degenerate belief function à la R. Farmer (2020). Hereby, each investor forms a quantity belief about his/her investment demand. As R. Farmer (2020) forcefully argues, this belief function is to be seen as a primitive in addition to households' preferences, firms' production technologies and economy's resources. However, the degeneracy of the belief function remains problematic, and arouses our first research question of whether a non-degenerate belief function is feasible in a stock market OLG model with involuntary unemployment. Provided the answer to this question turns out to be positive, our second research question concerns the implications of a non-degenerate belief function for the intertemporal equilibrium dynamics and their steady states in a stock-market OLG model with involuntary unemployment.

1.4. Endogenous Unemployment Rate and Non-Degenerate Belief Function

Against this research background, the main objective of the present paper is to modify Cunha's (2012) perfectly competitive, stock-market OLG model such that the unemployment rate becomes endogenous, and a non-degenerate belief function of corporation managers is compatible with market-value maximization of investing corporations. Instead of assuming that corporations' investment quantity is determined by investors' degenerate beliefs about the investment demand here we assume that corporations' managers form beliefs about future sales of production output from which the required investment quantity is derived using corporations' production functions. Moreover, investing managers adjust their beliefs about the future sales of production output over time towards their expected level in the long run (like Freitas and Serrano, 2015 for the desired capital-output ratio).

Our first contribution to the literature is thus to show how the structure of the

[1]Farmer's (2023a) OLG model can be considered as complementary to Tanaka's (2020) three-period OLG model of involuntary unemployment without real capital and investment.

intertemporal equilibrium dynamics derived from households' and firms' optimization conditions, from government's budget constraint and the intertemporal market-clearing conditions changes when corporations' investment quantities are both optimally indeterminate and determined by managers' beliefs about expected sales of future production output.

Our second contribution to the literature consists in proving the existence of a steady state and in investigating the dynamic stability of the steady state of the intertemporal equilibrium dynamics in our novel stock-market OLG model with involuntary unemployment.

Our third contribution to the literature is to derive analytically the steady-state effects of main parameter changes on endogenous variables as the capital-output ratio, the equity price, and the unemployment rate. This is completed by a numerical calculation of the intertemporal equilibrium paths of these endogenous variables and the expected sales of production output from corporations' investments in productive capital in response to small parameter changes.

In contradistinction to Farmer (2023a, 2023b), there are three novelties in the following paper: 1) Physical capital is not durable but depreciates within one period. 2) The production technology exhibits not constant, but decreasing returns which generate profits which are to be distributed to shareholders. 3) There is a non-degenerate belief function of corporation managers which changes substantially the intertemporal equilibrium dynamics and the steady state.

The structure of the paper is as follows. The next section presents the model set-up. This is followed by derivation of the intertemporal-equilibrium dynamics and demonstration of sufficient conditions for the existence and dynamic stability of steady states. We then investigate the comparative dynamics of the steady-state responses of the capital-output ratio, the equity price, and the unemployment rate to main parameter changes. A numerical specification of all model parameters is then used to calculate numerically the intertemporal-equilibrium paths of these dynamic variables in response to small parameter changes. The main conclusions are drawn in the final section of the paper.

2. The Stock Market Olg Model with Involuntary Unemployment

As in Cunha (2012), we consider an economy of infinite horizon which is composed of infinitely lived firms and finitely lived households. In addition to the former author, we also assume an infinitely lived government with a balanced budget from period to period. In each period $t = 0, 1, 2, \cdots$. A new generation, called generation t, enters the economy. A continuum of $L > 0$ units of identical agents comprises the generation entering in period t.

In line with Cunha (2012) we assume no population growth and no growth in labor productivity.

Each household consists of one agent and the agent is intergenerationally egoistic: the old agent has no concern for the young agent and the young agent has no concern for the old agent. They live two periods long, namely youth (adult) and old age. In contradistinction to the original Diamond (1965) OLG model and to Cunha's (2012) full-employment, stock-market model, in our model economy there are employed and (involuntarily) unemployed households. All households are endowed with one unit of labor but only the employed households can sell it inelastically to firms. In exchange for the labor supply each employed household of generation t obtains the real wage rate w_t, which denotes the units of the produced good per unit of labor. Thus, the labor supply in period t is not equal to L_t, but only to $(1-u_t)L_t$, where $0 \le u_t < 1$ denotes the unemployment rate. The number of unemployed households (= people) is thus $u_t L_t$. Since the unemployed are unable to obtain any labor income from the market, they are supported by the government through the unemployment benefit ς_t (per household) in each period.

To finance the unemployment benefit, the government collects taxes on wages, quoted as a fixed proportion of wage income, $\tau_t w_t h_t$, $0 < \tau_t < 1$. The unemployed do not pay any taxes. Young, employed agents, denoted by superscript E, split the net wage income $(1-\tau_t)w_t$ each period between current consumption $c_t^{1,E}$ and savings s_t^E. Savings of the employed are invested in the shares of firms, where a share $\theta_t^{j,E}$ of firm $j = 1,\cdots,J$ in period t is bought in the stock market at price Q_t^j by the younger households from the older households. Moreover, the younger households also invest their savings in bonds emitted by firms $j(=1,\cdots,J)$, denoted by $b_{t+1}^{j,E}$, with a rate of return i_{t+1}.

In old age, the employed household sells the shares at the price $Q_{t+1}^{j,E}$ to the then younger household in period $t+1$. The revenues from asset sales and the returns from holding assets one period long,

$(1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} \theta_t^{j,E}\left(D_{t+1}^j / \Psi^j + Q_{t+1}^j\right)$, are used to finance retirement

consumption $c_{t+1}^{2,E}$, where D_t^j denotes the dividend paid by firm j in period t and Ψ^j denotes the outstanding shares of corporation j. In line with Cunha (2012) we assume that the number of outstanding shares with corporation j is time-stationary, or in other words: there is no equity financing of corporation j's investment. In old age, the previously young employed households consume

their gross return on assets: $c_{t+1}^{2,E} = (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} \theta_t^{j,E}\left(D_{t+1}^j / \Psi^j + Q_{t+1}^j\right)$.

This is also true for the unemployed households who finance their retirement consumption through the returns on equity purchases and firm bonds in youth financed by unemployment benefits:

$c_{t+1}^{2,U} = (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J} \theta_t^{j,U}\left(D_{t+1}^j / \Psi^j + Q_{t+1}^j\right)$, where $c_{t+1}^{2,U}$, represents

consumption of the unemployed in old age. To keep it all as simple as possible, we assume that the revenues from equity sales and dividends are not taxed.

The typical younger, employed household maximizes the following intertemporal utility function subject to the budget constraints of the active period (1) and of the retirement period (2):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,E} + \delta \ln c_{t+1}^{2,E}$$

subject to:

(1) $c_t^{1,E} + \sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} Q_t^J \theta_t^{j,E} = w_t (1-\tau_t)$,

(2) $c_{t+1}^{2,E} = (1+i_{t+1}) \sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J} \theta_t^{j,U} \left(D_{t+1}^j / \Psi^j + Q_{t+1}^j \right)$.

Here, $0 < \varepsilon \le 1$ depicts the utility elasticity of employed household's consumption in youth and $0 < \delta < 1$ denotes the subjective future utility discount factor. The intertemporally additive utility function involves the natural logarithm of employed household's consumption in youth weighted by ε, and the natural logarithm of employed household's consumption in old age weighted by $0 < \delta < 1$.

To obtain the first-order conditions for a maximum of the intertemporal utility function subject to the constraints (1) and (2), we form the following Lagrangian:

$$L_t^E \equiv \varepsilon \ln c_t^{1,E} + \delta \ln c_{t+1}^{2,E} - \lambda_t^E \left(c_t^{1,E} + \sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} Q_t^j \theta_t^{j,E} - w_t (1-\tau_t) \right)$$
$$- \lambda_{t+1}^E \left(c_{t+1}^{2,E} - (1+i_{t+1}) \sum_{j=1}^{J} b_{t+1}^{j,E} - \sum_{j=1}^{J} \theta_t^{j,E} \left(D_{t+1}^j / \Psi^j + Q_{t+1}^j \right) \right).$$

Differentiating the Lagrangian with respect to $c_t^{1,E}, c_{t+1}^{2,E}, b_{t+1}^{j,E}, \theta_t^{j,E}, j = 1, \cdots, J$ yields the following first-order conditions for an intertemporal utility maximum:

$$c_t^{1,E} = \frac{\varepsilon}{\varepsilon + \delta} (1-\tau_t) w_t, \tag{1}$$

$$\frac{D_{t+1}^j / \Psi^j + Q_{t+1}^j}{Q_t^j} = 1 + i_{t+1}, \; j = 1, \cdots, J, \tag{2}$$

$$c_{t+1}^{2,E} = \frac{\delta}{\varepsilon + \delta} (1+i_{t+1})(1-\tau_t) w_t, \tag{3}$$

$$s_t^E = \frac{\delta}{\varepsilon + \delta} w_t (1-\tau_t), \; s_t^E \equiv \sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} \theta_t^{j,E} Q_t^j. \tag{4}$$

The typical younger, unemployed household maximizes the following intertemporal utility function subject to the budget constraints of the active period (i) and the retirement period (2):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,U} + \delta \ln_t c_{t+1}^{2,U}$$

subject to:

(1) $c_t^{1,U} + \sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J} Q_t^J \theta_t^{j,U} = \varsigma_t$,

(2) $c_{t+1}^{2,U} = (1+i_{t+1}) \sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J} \theta_t^{j,U} \left(D_{t+1}^j / \Psi^j + Q_{t+1}^j \right)$.

Again, $0 < \varepsilon \le 1$ denotes the utility elasticity of consumption in unemployed youth, while $0 < \delta < 1$ depicts the subjective future utility discount factor and ς_t denotes the unemployment benefit percapita unemployed.

Performing similar intermediate steps as above with respect to the younger, employed household yields the following first-order conditions for a constrained intertemporal utility maximum:

$$c_t^{1,U} = \frac{\varepsilon}{\varepsilon + \delta} \varsigma_t, \tag{5}$$

$$\frac{D_{t+1}^j / \Psi^j + Q_{t+1}^j}{Q_t^j} = 1 + i_{t+1},\ j = 1, \cdots, J, \tag{6}$$

$$c_{t+1}^{2,U} = \frac{\delta}{\varepsilon + \delta} \left(1 + i_{t+1}\right) \varsigma_t, \tag{7}$$

$$s_t^U = \frac{\delta}{\varepsilon + \delta} \varsigma_t,\ s_t^U \equiv \sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J} \theta_t^{j,U} Q_t^j. \tag{8}$$

All corporations are endowed with an identical Cobb-Douglas production function which reads as follows:

$$Y_t^j = M \left(N_t^j\right)^{\beta} \left(K_t^j\right)^{\alpha},\ j = 1, \cdots, J,\ 0 < \alpha < 1,\ 0 < \beta < 1,\ \alpha + \beta < 1, M > 0. \tag{9}$$

Here, Y_t^j denotes production output of firm $j = 1, \cdots, J$, $M > 0$ stands for total factor productivity (equal for all firms), N_t^j represents the number of employed laborers with firm *j*, while K_t^j denotes the input of capital services of firm *j*, all in period *t*, and $\beta(\alpha)$ depicts the production elasticity (= production share) of labor (capital) services, also equal for all firms. In line with Cunha (2012), we assume that $\alpha + \beta < 1$, i.e. that corporation *j*'s technology exhibits decreasing returns to scale enabling pure profits. Also, in line with Cunha (2012), we assume that (physical) capital depreciates completely within one period and needs to be installed one period before it is used. Thus, capital K_{t+1}^j used by corporation *j* equals gross investment of period *t*: $K_{t+1}^j = I_t^j$.

Corporations are owned by shareholders and are managed to maximize the payoff for their current owners. These are the older households who own the shares of corporation *j* endowed with a capital of K_t^j. The owners are entitled to obtain $D_t^j / \Psi^j + Q_t^j$ from which should be derived an objective for corporation *j*'s managers which is aligned with the interests of corporation *j*'s owners. However, as it stands, $D_t^j / \Psi^j + Q_t^j$ cannot be used as corporation *j*'s objective. In line with Makowski (1983) and Farmer (1988) we assume that corporation *j* has a (fixed) conjectural function $V^j\left(D_{t+1}^j / \Psi^j\right)$ "about how the value of its shares will vary as it varies its market plan" (Makowski, 1983: p. 309).[2] Moreover, in any t-period stock-market equilibrium corporations' conjectures are verified which means that $Q_t^j = V^j\left(D_{t+1}^j / \Psi^j\right)$. Farmer (1988: p. 493) shows that if $V^j(.)$ is linear and the demand for *j*'s shares is perfectly elastic, maximizing $D_t^j / \Psi^j + v^j\left(D_{t+1}^j / \Psi^j\right)$, $v^j = \text{constant} > 0$, $j = 1, \cdots, J$ is in the unanimous interest of corporation *j*'s owners.

To proceed, dividends D_t^j need to be defined. We assume that all corpora-

[2]In contrast to Makowski (1983), Farmer (1988: p. 6) assumes that the conjectured market value of corporation *j* depends on planned next-period dividends per share, and not on current net production.

tion earnings are distributed to its shareholders and investment expenditures are financed by issuance of one-period corporations' bonds, $I_t^j = B_t^j$, $j = 1, \cdots, J$. Thus, $D_t^j = Y_t^j - w_t N_t^j - (1+i_t) I_{t-1}^j$.

In contrast to Devereux and Lockwood (1991) and Cunha (2012), not households but the mangers of corporation *j* decide on its investment signifying the presumption that investment is decided independently of savings. Inserting the definition of dividends into the objective of corporation *j*, the decision calculus of its managers reads as follows:

$$\underbrace{\max}_{\{N_t^j, I_t^j, N_{t+1}^j\}} \left\{ \frac{M\left(K_t^j\right)^\alpha \left(N_t^j\right)^\beta - w_t N_t^j - \left(1+i_t\right) I_{t-1}^j}{\Psi^j} + \nu^j \frac{\left[\left(K_{t+1}^j\right)^\alpha \left(N_{t+1}^j\right)^\beta - w_{t+1} N_{t+1}^j - \left(1+i_{t+1}\right) I_t^j\right]}{\Psi^j} \right\} \quad (10)$$

s.t.: $K_{t+1}^j = I_t^j$, $j = 1, \cdots, J$.

Maximization of corporation *j*'s objective function subject to the accumulation equation in (10) implies the following first-order conditions:

$$\beta M \left(K_t^j\right)^\alpha \left(N_t^j\right)^{\beta-1} = w_t, \quad (11)$$

$$\alpha M \left[K_{t+1}^j\right]^{\alpha-1} \left(N_{t+1}^j\right)^\beta = 1 + i_{t+1}, \quad (12)$$

$$\beta M \left[K_{t+1}^j\right]^\alpha \left(a_{t+1} N_{t+1}^j\right)^{\beta-1} = w_{t+1}. \quad (13)$$

As in Diamond (1965), the government does not optimize, but is subject to the following budget constraint period by period:

$$L_t u_t \varsigma_t = \tau_t \left(1-u_t\right) w_t L_t, \quad (14)$$

where, for the sake of simplicity, it is assumed that the government does not have any other expenditures than the unemployment benefits and that there is no government debt.

As Magnani (2015: pp. 13-14) rightly states, aggregate investment in Solow's (1956) neoclassical growth model is not micro-, but macro-founded since it is determined by aggregate savings. The same holds true in Diamond's (1965) OLG model of neoclassical growth where perfectly flexible aggregate investment is also determined by aggregate savings of households. As already mentioned in the Introduction above, and as the first-order conditions for optimal investment of corporations (15) show, optimal investment is indeterminate and thus also perfectly flexible in our stock market model thus far.

Morishima (1977), and more recently Magnani (2015), both deviate from neoclassical growth models in maintaining that an independent investment function is needed to determine the level of investment in intertemporal-equilibrium models of involuntary unemployment. The big question, however, is where does this function come from in a general equilibrium model with an active stock market and an explicit corporation calculus to find optimal investment quantities?

One promising avenue to answer this seminal question is provided by R. Farmer's (2013, 2020) "belief" function which he sees as synonymous with the neo-classical fundamentals like consumer preferences, corporation technologies and the resource endowment of an economy. R. Farmer and collaborators suggest different expected price or income variables about which investors form beliefs (see for an overview Farmer, 2023a).

Here, we assume that corporation managers form beliefs about future expected sales of production output $Y_{t+1}^{ex,j}$ resulting from current period investment I_t^j and future input of labor services N_{t+1}^j. Hereby, the managers consider corporation's production technology $M\left(I_t^j\right)^{\alpha}\left(N_{t+1}^j\right)^{\beta}$ to find out that corporation's investment is determined as follows:

$$I_t^j = M^{-1/\alpha}\left(Y_{t+1}^{j,ex}\right)^{-1/\alpha}\left(N_{t+1}^j\right)^{-\beta/\alpha},\ j = 1,\cdots,J,\ \forall t. \tag{15}$$

Like Freitas and Serrano (2015) for the desired capital-output ratio, we assume a partial adjustment of the corporation *j*'s belief about the expected outputsales in period $t+1$ towards its long-run expectedlevel $\hat{Y}^{j,ex}, j = 1,\cdots,J$:

$$Y_{t+1}^{j,ex} = (1-\varphi)Y_t^{j,ex} + \varphi\hat{Y}^{j,ex},\ 0 < \varphi \le 1,\ j = 1,\cdots,J,\ \forall t. \tag{16}$$

Inserting the expectation adjustment Equation (16) into Equation (15), we finally obtain the following beliefs-founded equation determining corporation *j*'s investment:

$$I_t^j = M^{-1/\alpha}\left((1-\varphi)Y_t^{j,ex} + \varphi\hat{Y}^{,ex}\right)^{-1/\alpha}\left(N_{t+1}^j\right)^{-\beta/\alpha},\ j = 1,\cdots,J,\ \forall t. \tag{17}$$

Equation (17) does not appear in Cunha's (2012) stock market model, since he assumes full employment of the labor force, which is equivalent to $u_t = 0, \forall t$ in our model. For $u_t > 0$ and u_t being endogenous, Equation (17) features as the intertemporal equilibrium condition which makes the whole set of intertemporal equilibrium equations determinate. In contrast to Morishima (1977: pp. 117-119) and Magnani (2015: p. 14), inflexible aggregate investment is not simply assumed to be macro-founded but turns out to be consistent with an indeterminate, market-value maximizing investment quantity of firm *j*. In this restricted sense, we are entitled to claim that inflexible investment is production-founded in our modified stock-market OLG model of involuntary unemployment.

In addition to the restrictions imposed by household and corporations' optimizations as well as corporations' beliefs and by the government budget constraint, markets for labor, corporation bonds, and equity, ought to clear in all periods (the market for the output of production is cleared by means of Walras' law[3]).

$$L_t(1-u_t) = \sum_{j=1}^{J} N_t^j = N_t, \forall t. \tag{18}$$

$$L(1-u_t)b_{t+1}^E + Lu_t b_{t+1}^U = \sum_{j=1}^{J} b_{t+1}^j,\ \forall t. \tag{19}$$

The demand of the younger employed and the unemployed households for corporation bonds (left-hand side of Equation (19)) balances with their supply

[3]The proof of Walras' law can be obtained upon request from the author.

(right-hand side of Equation (19)). Corporations finance their investments by the sales of bonds:

$$\sum_{j=1}^{J} I_t^j = \sum_{j=1}^{J} b_{t+1}^j,\ \forall t.. \tag{20}$$

The shares of employed and unemployed younger households add up to the number of outstanding shares of corporation *j*:

$$L(1-u_t)\theta_t^{j,E} + Lu_t\theta_t^{j,U} = \Psi^j,\ j=1,\cdots,J,\ \forall t. \tag{21}$$

The sales of equity shares by employed and unemployed older households are equal to the share purchases of employed and unemployed younger households:

$$L_{t-1}(1-u_{t-1})\theta_{t-1}^{j,E} = L_t(1-u_t)\theta_t^{j,E},\ j=1,\cdots,J,\ \forall t, \tag{22}$$

$$L_{t-1}(1-u_{t-1})\theta_{t-1}^{j,U} = L_t(1-u_t)\theta_t^{j,U},\ j=1,\cdots,J,\ \forall t. \tag{23}$$

Using the definition of savings for younger employed households in (4) and younger unemployed households in (8), together with the bond market clearing condition (19), the investment financing constraint (20) and condition (21), leads us to the following aggregate savings/investment equality:

$$L_t(1-u_t)s_t^E + L_t u_t s_t^U = \sum_{j=1}^{J} I_t^j + \sum_{j=1}^{J} Q_t^j \Psi^j. \tag{24}$$

3. Intertemporal Equilibrium

To start with, assume in line with Magill and Quinzii (2003: p. 249) a balanced-growth intertemporal equilibrium in which firms always exhibit the same relative sizes and stock-market values. Then, consider initial conditions $\left(K_0^j, Q_0^j\right) = \phi^j\left(K_0, Q_0\right)$ with $\phi^j > 0$ and $\sum_{j=1}^{J}\phi^j = 1$. If, for the sequence of (real) wage and interest rates $\left(w_t, i_{t+1}\right)_{t\geq 0}$, equity prices $\left(Q_t\right) \geq 0$ and employment-investment decisions $\left(N_t, I_t\right)_{t\geq 0}$ satisfy the Equations (11), (12), (13), (15), (16), (17), then $\left(Q_t^j, N_t^j, I_t^j\right) = \phi^j\left(Q_t, N_t, I_t\right)$ also satisfy Equations (11), (12), (13), (15), (16), and (17), such that for each firm $\left(N_t^j, I_t^j\right)$ is market-value maximizing, its market value is larger than zero, and the return on bonds equals i_{t+1}. Hence, the optimal choices of individual firms can be depicted by the market-value maximizing choice of aggregate employment and capital.

Sinceall firms have the same production function (see Equation (9)), the optimal capital-labor ratio will be the same for all firms: $\frac{K_t^j}{N_t^j} = \frac{K_t^{j'}}{N_t^{j'}} = \frac{K_t}{N_t}$, $j \neq j' = 1,\cdots,J$. Moreover, since the number of employed workers is $N_t \equiv \sum_{j=1}^{J} N_t^j = L(1-u_t)$, we can rewrite the first-order conditions (11)-(13) by setting $\alpha + \beta = \gamma$ as follows:

$$\beta M\left(K_t\right)^{\alpha}\left(N_t\right)^{\beta-1} = \beta M\left(K_t / L(1-u_t)\right)^{\alpha}\left(L(1-u_t)\right)^{\gamma-1} = w_t, \tag{25}$$

$$\alpha M\left(K_{t+1}\right)^{\alpha-1}\left(N_{t+1}\right)^{\beta} = \alpha M\left[K_{t+1} / \left(L(1-u_{t+1})\right)\right]^{\alpha-1}\left(L(1-u_{t+1})\right)^{\gamma-1} = 1 + i_{t+1}, \tag{26}$$

$$\beta M\left(K_{t+1}\right)^{\alpha}\left(N_{t+1}\right)^{\beta-1}=\beta M\left(K_{t+1}/\left[L\left(1-u_{t+1}\right)\right]\right)^{\alpha}\left(L\left(1-u_{t+1}\right)\right)^{\gamma-1}=w_{t+1}. \quad (27)$$

Finally, the GDP function can be rewritten as follows:

$$\begin{aligned}Y_t&=M\left(K_t\right)^{\alpha}\left(L\left(1-u_t\right)\right)^{\beta}=M\left(K_t\right)^{\alpha}\left(L\left(1-u_t\right)\right)^{\gamma-\alpha}\\&=M\left(K_t/\left(L\left(1-u_t\right)\right)\right)^{\alpha}\left(L\left(1-u_t\right)\right)^{\gamma}.\end{aligned} \quad (28)$$

Introducing the aggregate capital-per-labor quantities $k_t \equiv K_t/L$, we rewrite the first-order conditions (25)-(27) as follows:

$$\beta M\left(k_t\right)^{\alpha}L^{\gamma-1}\left(1-u_t\right)^{\beta-1}=w_t, \quad (29)$$

$$\alpha M\left(k_{t+1}\right)^{\alpha-1}L^{\gamma-1}\left(1-u_{t+1}\right)^{\beta}=1+i_{t+1}, \quad (30)$$

$$\beta M\left(k_{t+1}\right)^{\alpha}L^{\gamma-1}\left(1-u_{t+1}\right)^{\beta-1}=w_{t+1}. \quad (31)$$

Using the aggregate version of Equation (24), the savings/investment equality can be rewritten as follows:

$$L\left(1-u_t\right)s_t^{E}+Lu_ts_t^{U}=I_t+Q_t\Psi,\ \Psi\equiv\sum_{1}^{J}\Psi^{j}. \quad (32)$$

Next, insert into Equation (32) the optimal savings functions (4) and (8), and the government balanced budget condition (14):

$$\begin{aligned}&L\left(1-u_t\right)\sigma w_t\left(1-\tau_t\right)+Lu_t\sigma\varsigma_t=L\left(1-u_t\right)\sigma w_t\left(1-\tau_t\right)+L\left(1-u_t\right)\sigma w_t\tau_t\\&=L\left(1-u_t\right)\sigma w_t=I_t+Q_t\Psi,\ \sigma\equiv\beta/\left(\varepsilon+\beta\right).\end{aligned} \quad (33)$$

Inserting into Equation (33) the first-order condition (29), and dividing the resulting equation on both sides by L, we obtain:

$$\begin{aligned}&\left(L\left(1-u_t\right)\sigma\beta M\left(k_t\right)^{\alpha}L^{\gamma-1}\left(1-u_t\right)^{\beta-1}=I_t+Q_t\Psi\right)\frac{1}{L}\\&\Leftrightarrow\sigma\beta M\left(k_t\right)^{\alpha}L^{\gamma-1}\left(1-u_t\right)^{\beta}=\frac{I_t}{L}+Q_t\frac{\Psi}{L}=\frac{I_t}{L}+Q_t\psi,\ \psi\equiv\frac{\Psi}{L}.\end{aligned} \quad (34)$$

By using the capital-output ratio

$\kappa_t\equiv K_t/Y_t=K_t/\left[ML^{\gamma}\left(1-u_t\right)^{\beta}\left(k_t\right)^{\alpha}\right]=\left(k_t\right)^{1-\alpha}/\left[ML^{\gamma-1}\left(1-u_t\right)^{\beta}\right]$ or

$k_t=L^{(\gamma-1)/(1-\alpha)}M^{1/(1-\alpha)}\left(\kappa_t\right)^{1/(1-\alpha)}\left(1-u_t\right)^{\beta/(1-\alpha)}$ and Equation (17), Equation (34) can be transformed into Equation (35):

$$\begin{aligned}&\sigma\beta L^{\alpha(\gamma-1)/(1-\alpha)}M^{1/(1-\alpha)}\left(\kappa_t\right)^{\alpha/(1-\alpha)}\left(\omega_t\right)^{\alpha\beta/(1-\alpha)}\\&=L^{(1-\gamma)/\alpha}M^{-1/\alpha}\left(y_{t+1}^{ex}\right)^{1/\alpha}\left(\omega_{t+1}\right)^{-\beta/\alpha}+Q_t\psi,\ y_{t+1}^{ex}\equiv Y_{t+1}^{ex}/L,\ \omega_t\equiv1-u_t,\ \forall t.\end{aligned} \quad (35)$$

Equation (35) represents the first difference equation of the intertemporal equilibrium in our stock-market OLG model of involuntary unemployment.

The second dynamic equation equals the aggregate per-capita version of Equation (16):

$$y_{t+1}^{ex}=\left(1-\varphi\right)y_t^{ex}+\varphi\hat{y}^{ex},\ y_0^{ex}=\underline{y}^{ex}>0,\forall t. \quad (36)$$

The third dynamic equation results from the capital accumulation equation

$$I_t = K_{t+1} \quad \text{or} \quad \frac{I_t}{L} = \frac{K_{t+1}}{L} = k_{t+1}:$$

$$L^{(\gamma-1)/(1-\alpha)} M^{1/(1-\alpha)} \left(\kappa_{t+1}\right)^{1/(1-\alpha)} \left(\omega_{t+1}\right)^{\beta/(1-\alpha)} = \left(y_{t+1}^{ex}\right)^{1/\alpha} M^{-1/\alpha} L^{(1-\gamma)/\alpha} \left(\omega_{t+1}\right)^{-\beta/\alpha}, \forall t. \quad (37)$$

The fourth dynamic equation pops up when the definition of dividends per share and the first-order conditions (29)-(31) are inserted into the no-arbitrage condition (2) respective (6):

$$Q_{t+1} = \alpha\left(\kappa_{t+1}\right)^{-1} Q_t - \psi^{-1} L^{(\gamma-1)/(1-\alpha)} M^{1/(1-\alpha)} (1-\gamma)\left(\kappa_{t+1}\right)^{1/(1-\alpha)} \left(\omega_{t+1}\right)^{\beta/(1-\alpha)}, \forall t. \quad (38)$$

4. Existence of Steady States

The steady states of the equilibrium dynamics depicted by the difference Equations (34)-(38) are defined as $\lim_{t\to\infty} \kappa_t = \kappa$, $\lim_{t\to\infty} q_t = q$, $\lim_{t\to\infty} y_t^{ex} = \hat{y}^{ex}$, and $\lim_{t\to\infty} \omega_t = \omega$. Acknowledging these definitions within the dynamic system (35) - (38), leads us to the following proposition 1:

Proposition 1. Suppose that $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma = \alpha + \beta < 1$ and $\left(\hat{y}^{ex}\right)^{1-\alpha} L^{1-\gamma} \le M\alpha^\alpha$. Then, the following steady-statesolution for $(\kappa, q) > 0$ and $0 < u < 1$ exists:

$$y^{ex} = \hat{y}^{ex}, \quad (39)$$

$$\sigma\beta M(\kappa)^{\alpha-1}\left(\hat{y}^{ex}\right)^{\alpha-1} = 1 + \frac{1-\gamma}{\alpha-\kappa}, \quad (40)$$

$$\omega = 1 - u = \left(\hat{y}^{ex}\right)^{(1-\alpha)/\beta} L^{(1-\gamma)/\beta} M^{-1/\beta} \kappa^{-\alpha/\beta}, \quad (41)$$

$$q = \frac{\sigma\beta M\left(\hat{y}^{ex}\right)^\alpha \kappa^\alpha - \hat{y}^{ex}\kappa}{\psi}. \quad (42)$$

Proof. While y^{ex}, ω and q can be explicitly solved as Equations (39), (41) and (42) show, to ensure a positive and unique solution of Equation (40) the intermediate value theorem needs to be applied. For this sake, denote the left-hand side of Equation (40) by $LHS(\kappa)$ and the right-hand side of this equation by $RHS(\kappa)$. For $\kappa \to 0$, $LHS(\kappa) \to \infty$ while $RHS(0) = 1 + (1-\gamma)/\alpha < \infty$. On the other hand, if $\kappa \to \alpha$, $RHS(\kappa) \to \infty$, while $LHS(\kappa) = \beta\sigma M\alpha^{\alpha-1}\left(y^{ex}\right)^{\alpha-1} < \infty$. Since both $LHS(\kappa)$ and $RHS(\kappa)$ are continuous functions on $0 < \kappa < \alpha$, there must exist a strictly positive solution of Equation (40) for $0 < \kappa < \alpha$. Consequently, $\omega > 0 \Leftrightarrow u < 1$ and due to the assumption $\left(\hat{y}^{ex}\right)^{1-\alpha} L^{1-\gamma} \le M\alpha^\alpha$ is $\omega < 1 \Leftrightarrow u > 0$. Moreover, $q > 0$ since $\sigma\beta M\left(\hat{y}^{ex}\right)^\alpha \kappa^\alpha - \hat{y}^{ex}\kappa = \hat{y}^{ex}\kappa(1-\gamma)/(\alpha-\kappa) > 0$ for $\kappa < \alpha$.

5. Dynamic Stability of the Steady State

The next step is to investigate the local dynamic stability of the unique steady-state solution (39) - (42). To make the algebraic analysis a little clearer, we assume $\varphi = 1$ an assumption through which the equilibrium dynamics becomes three-dimensional instead of four-dimensional. The intertemporal equilibrium Equations (35), (37) (38) are then totally differentiated with respect to

$\kappa_{t+1},\omega_{t+1},q_{t+1},\kappa_t,\omega_t,q_t$. Then, the Jacobian matrix $J(\kappa,\omega,q)$ of all partial differentials with respect to κ_t, ω_t and q_t is formed as follows:

$$J(\kappa,\omega,q)\equiv\begin{bmatrix}\frac{\partial\kappa_{t+1}}{\partial\kappa_t}(\kappa,\omega,q) & \frac{\partial\kappa_{t+1}}{\partial\omega_t}(\kappa,\omega,q) & \frac{\partial\kappa_{t+1}}{\partial q_t}(\kappa,\omega,q)\\ \frac{\partial\omega_{t+1}}{\partial\kappa_t}(\kappa,\omega,q) & \frac{\partial\omega_{t+1}}{\partial\omega_t}(\kappa,\omega,q) & \frac{\partial\omega_{t+1}}{\partial q_t}(\kappa,\omega,q)\\ \frac{\partial q_{t+1}}{\partial\kappa_t}(\kappa,\omega,q) & \frac{\partial q_{t+1}}{\partial\omega_t}(\kappa,\omega,q) & \frac{\partial q_{t+1}}{\partial q_t}(\kappa,\omega,q)\end{bmatrix}, \tag{43}$$

with

$$\frac{\partial\kappa_{t+1}}{\partial\kappa_t}\equiv j_{11}=\frac{\alpha\beta\sigma\kappa^{-1+\frac{\alpha}{1-\alpha}}L^{\frac{-\alpha(1-\gamma)}{1-\alpha}}M^{\frac{1}{1-\alpha}}\omega^{\frac{\alpha\beta}{1-\alpha}}}{(1-\alpha)\hat{y}^{ex}},$$

$$\frac{\partial\kappa_{t+1}}{\partial\omega_t}\equiv j_{12}=\frac{\alpha\beta^2\sigma\kappa^{\frac{\alpha}{1-\alpha}}L^{\frac{-\alpha(1-\gamma)}{1-\alpha}}M^{\frac{1}{1-\alpha}}\omega^{-1+\frac{\alpha\beta}{1-\alpha}}}{(1-\alpha)\hat{y}^{ex}},$$

$$\frac{\partial\kappa_{t+1}}{\partial q_t}\equiv j_{13}=-\frac{\psi}{\hat{y}^{ex}},$$

$$\frac{\partial\omega_{t+1}}{\partial\kappa_t}\equiv j_{21}=-\frac{\alpha^2\sigma\kappa^{-1+\frac{\alpha}{1-aa}}L^{\frac{-(1-(1-\alpha)\alpha)(1-\gamma)}{(1-\alpha)\alpha}}M^{\frac{1}{\alpha(1-\alpha)}}\omega^{\frac{\gamma}{\alpha}+\frac{\alpha\beta}{1-\alpha}}\left(\hat{y}^{ex}\right)^{-\frac{1}{\alpha}}}{1-\alpha},$$

$$\frac{\partial\omega_{t+1}}{\partial\omega_t}\equiv j_{22}=-\frac{\alpha^2\beta\sigma\kappa^{\frac{\alpha}{1-aa}}L^{\frac{-(1-(1-\alpha)\alpha)(1-\gamma)}{(1-\alpha)\alpha}}M^{\frac{1}{\alpha(1-\alpha)}}\omega^{\frac{\beta}{\alpha}+\frac{\alpha\beta}{1-\alpha}}\left(\hat{y}^{ex}\right)^{-\frac{1}{\alpha}}}{1-\alpha},$$

$$\frac{\partial\omega_{t+1}}{\partial q_t}\equiv j_{23}=\frac{\alpha\psi L^{-\frac{1-\gamma}{\alpha}}M^{\frac{1}{\alpha}}\omega^{\frac{\gamma}{\alpha}}\left(\hat{y}^{ex}\right)^{-\frac{1}{\alpha}}}{\beta},$$

$$\frac{\partial q_{t+1}}{\partial\kappa_t}\equiv j_{31}=-\frac{\alpha^2\beta\sigma\kappa^{-1+\frac{\alpha}{1-\alpha}}L^{-\frac{(2-(2-\alpha)\alpha)(1-\gamma)}{(1-\alpha)\alpha}}M^{\frac{2-\alpha}{(1-\alpha)\alpha}}\omega^{\frac{2\beta}{\alpha}+\frac{\alpha\beta}{1-\beta}}q\left(\hat{y}^{ex}\right)^{-\frac{2-\alpha}{\alpha}}}{1-\alpha},$$

$$\frac{\partial q_{t+1}}{\partial\omega_t}\equiv j_{32}=-\frac{\alpha^2\beta^2\sigma\kappa^{\frac{\alpha}{1-\alpha}}L^{-\frac{(2-(2-\alpha)\alpha)(1-\gamma)}{(1-\alpha)\alpha}}M^{\frac{2-\alpha}{(1-\alpha)\alpha}}\omega^{-1+\frac{2\beta}{\alpha}+\frac{\alpha\beta}{1-\beta}}q\left(\hat{y}^{ex}\right)^{-\frac{2-\alpha}{\alpha}}}{1-\alpha},$$

$$\frac{\partial q_{t+1}}{\partial q_t}\equiv j_{33}=\alpha L^{-\frac{2-\gamma}{\alpha}}M^{\frac{1}{\alpha}}\omega^{\frac{\beta}{\alpha}}\left(\hat{y}^{ex}\right)^{-\frac{2-\alpha}{\alpha}}\left(L^{\frac{\gamma}{\alpha}}M^{\frac{1}{\alpha}}\omega^{\frac{\beta}{\alpha}}\psi q+L^{\frac{1}{\alpha}}\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\right).$$

A glance on the entries of Jacobian (43) reveals that the second column is a $\beta\kappa\omega^{-1}$ multiple of the first column. Thus, we know that the determinant of Jacobian (43) is zero. This implies that one of the eigenvalues of the Jacobian, λ_i, $i=1,2,3$,is zero, too: say $\lambda_1=0$.

Due to the rather complex entries of Jacobian (43) no clear expression for the other two eigenvalues can be expected. To shed some light on them, we assume for the sake of simplicity that $M=L=\psi=1$. Then, the remaining eigenvalues read as follows:

$$\lambda_2 = \frac{1}{2}\left(\frac{\alpha\beta\kappa^{-1+\frac{\alpha}{1-\alpha}}\omega^{\frac{\alpha\beta}{1-\alpha}}\sigma}{(1-\alpha)\hat{y}^{ex}} - \frac{\alpha^2\beta\kappa^{\frac{\alpha}{1-\alpha}}\omega^{-1+\frac{\alpha\beta}{1-\beta}+\frac{\gamma}{\alpha}}\sigma\left(\hat{y}^{ex}\right)^{-\frac{1}{\alpha}}}{1-\alpha} + \alpha\omega^{\frac{2\beta}{\alpha}}q\left(\hat{y}^{ex}\right)^{\frac{-2+\alpha}{\alpha}} + \alpha\omega^{\frac{\beta}{\alpha}}\left(\hat{y}^{ex}\right)^{\frac{-1+\alpha}{\alpha}} \right)$$

$$-\frac{1}{2}\left(\sqrt{\frac{1}{\left(\hat{y}^{ex}(1-\alpha)\right)^2}\alpha^2\left(\hat{y}^{ex}\right)^{-\frac{4}{\alpha}}\left(4(-1+\alpha)\beta\kappa^{-1+\frac{\alpha}{1-\alpha}}\omega^{-1+\frac{\beta}{\alpha}+\frac{\alpha\beta}{1-\alpha}}\sigma\left(\hat{y}^{ex}\right)^{2+\frac{2}{\alpha}}\left(-\alpha\kappa\omega^{\frac{\gamma}{\alpha}}\hat{y}^{ex} + \omega\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\right)\right.} \right. \quad (44)$$

$$\left. \left. + \sqrt{\left(-(1-\alpha)\omega^{\frac{\beta}{\alpha}}\hat{y}^{ex2}\left(\omega^{\frac{\beta}{\alpha}}q + (\hat{y}^{ex})^{\frac{1}{\alpha}}\right) + \beta\kappa^{-2+\frac{1}{1-\alpha}}\omega^{-1+\frac{\alpha\beta}{1-\alpha}}\sigma\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\left(\alpha\kappa\omega^{\frac{\gamma}{\alpha}}\hat{y}^{ex} - \omega\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\right)\right)^2} \right) \right)$$

$$\lambda_3 = \frac{1}{2}\left(\frac{\alpha\beta\kappa^{-1+\frac{\alpha}{1-\alpha}}\omega^{\frac{\alpha\beta}{1-\alpha}}\sigma}{(1-\alpha)\hat{y}^{ex}} - \frac{\alpha^2\beta\kappa^{\frac{\alpha}{1-\alpha}}\omega^{-1+\frac{\alpha\beta}{1-\beta}+\frac{\gamma}{\alpha}}\sigma\left(\hat{y}^{ex}\right)^{-\frac{1}{\alpha}}}{1-\alpha} + \alpha\omega^{\frac{2\beta}{\alpha}}q\left(\hat{y}^{ex}\right)^{\frac{-2+\alpha}{\alpha}} + \alpha\omega^{\frac{\beta}{\alpha}}\left(\hat{y}^{ex}\right)^{\frac{-1+\alpha}{\alpha}} \right)$$

$$+\frac{1}{2}\left(\sqrt{\frac{1}{\left(\hat{y}^{ex}(1-\alpha)\right)^2}\alpha^2\left(\hat{y}^{ex}\right)^{-\frac{4}{\alpha}}\left(4(-1+\alpha)\beta\kappa^{-1+\frac{\alpha}{1-\alpha}}\omega^{-1+\frac{\beta}{\alpha}+\frac{\alpha\beta}{1-\alpha}}\sigma\left(\hat{y}^{ex}\right)^{2+\frac{2}{\alpha}}\left(-\alpha\kappa\omega^{\frac{\gamma}{\alpha}}\hat{y}^{ex} + \omega\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\right)\right.} \right. \quad (45)$$

$$\left. \left. + \sqrt{\left(-(1-\alpha)\omega^{\frac{\beta}{\alpha}}\hat{y}^{ex2}\left(\omega^{\frac{\beta}{\alpha}}q + \left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\right) + \beta\kappa^{-2+\frac{1}{1-\alpha}}\omega^{-1+\frac{\alpha\beta}{1-\alpha}}\sigma\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\left(\alpha\kappa\omega^{\frac{\gamma}{\alpha}}\hat{y}^{ex} - \omega\left(\hat{y}^{ex}\right)^{\frac{1}{\alpha}}\right)\right)^2} \right) \right)$$

While in general the sign and the magnitude of the eigenvalues λ_2 and λ_3 are impossible to determine, their structure reveals that 2 times the term in front of the square root equals the trace of the Jacobian (43) since $\lambda_2 + \lambda_3 = TrJ$.

Proposition 2. Suppose the assumptions of Proposition 1 and $TrJ > 0$ hold. Then for a broad set of numerical parameter combinations in line with the parameter restrictions in Proposition 1 the eigenvalues of the Jacobian (43) $\lambda_i, i = 1,2,3$ are real, $\lambda_1 = 0$, $\lambda_2 > 1$ while $\lambda_3 < 1$.

In other words: the equilibrium dynamics with initial values $\kappa_0 = \underline{\kappa} > 0$ and $\omega_0 = \underline{\omega} > 0$ in the neighborhood of the steady-state solution in our stock-market OLG model with involuntary unemployment is non-oscillating and converges along a saddle-path towards the steady state (40) - (42) as time approaches infinity.

6. Comparative Dynamics of the Steady-State Solution and the Intertemporal Equilibrium Dynamics

As a next step it is apt to investigate firstly the comparative dynamics of the steady-state solution (39) - (42). The effects of infinitesimal, isolated parameter changes on the steady-state solution (39) - (42) are summarized in the following Proposition 3.

Proposition 3. Suppose that the assumptions of Propositions 1 and 2 hold. Then, the effects of infinitesimal, isolated changes of main model parameters on the steady-state solution (39) - (42) read as follows:

$$\frac{\partial \kappa}{\partial \hat{y}^{ex}} = -\frac{(1-\alpha)\beta M \sigma \kappa^{\alpha-1}\left(\hat{y}^{ex}\right)^{\alpha-2}}{\frac{1-\gamma}{(\alpha-\kappa)^2} + (1-\alpha)\beta M \sigma \kappa^{\alpha-2}\left(\hat{y}^{ex}\right)^{\alpha-1}} < 0,$$

$$\frac{\partial \kappa}{\partial \sigma} = \frac{\beta M \kappa^{\alpha-1}\left(\hat{y}^{ex}\right)^{\alpha-1}}{\frac{1-\gamma}{(\alpha-\kappa)^2} + (1-\alpha)\beta M \sigma \kappa^{\alpha-2}\left(\hat{y}^{ex}\right)^{\alpha-1}} > 0, \tag{46}$$

$$\frac{\partial \omega}{\partial \hat{y}^{ex}} = \frac{(1-\alpha)\kappa^{-\frac{\alpha}{\beta}} L^{\frac{1-\gamma}{\beta}} M^{-\frac{1}{\beta}} \left(\hat{y}^{ex}\right)^{-\frac{1-\gamma}{\beta}} \left((1-\gamma)\kappa^2 \hat{y}^{ex} + \beta(\alpha-\kappa)^2 \kappa^{\alpha} M \sigma \left(\hat{y}^{ex}\right)^{\alpha}\right)}{\beta\left((1-\gamma)\kappa^2 \hat{y}^{ex} + (1-\alpha)\beta(\alpha-\kappa)^2 \kappa^{\beta} M \sigma \left(\hat{y}^{ex}\right)^{\alpha}\right)} > 0,$$

$$\frac{\partial \omega}{\partial \sigma} = -\frac{\alpha \kappa^{-2+\alpha-\frac{\alpha}{\beta}} L^{\frac{1-\gamma}{\beta}} M^{\frac{-(1-\beta)}{\beta}} \left(\hat{y}^{ex}\right)^{\frac{(1+\alpha)(1+\beta)}{\beta}}}{\frac{1-\gamma}{(\alpha-\kappa)^2} + (1-\gamma)\beta \kappa^{-2+\alpha} M \sigma \left(\hat{y}^{ex}\right)^{\alpha-1}} < 0, \tag{47}$$

$$\frac{\partial q}{\partial \hat{y}^{ex}} = -\frac{(1-\gamma)\kappa^2\left(\kappa \hat{y}^{ex} - \alpha\beta\kappa^{\alpha} M \sigma \left(\hat{y}^{ex}\right)^{\alpha}\right)}{(1-\gamma)\kappa^2 \hat{y}^{ex} + (1-\alpha)\beta(\alpha-\kappa)^2 \kappa^{\alpha} M \sigma \left(\hat{y}^{ex}\right)^{\alpha}},$$

$$\frac{\partial q}{\partial \sigma} = -\frac{\beta \kappa^{\alpha} M \left(\hat{y}^{ex}\right)^{\alpha}\left(\kappa\left(\alpha^2 - 2\alpha\kappa - \kappa(1-\gamma-\kappa)\right)\hat{y}^{ex} - \beta(\alpha-\kappa)^2 \kappa^{\alpha} M \sigma \left(\hat{y}^{ex}\right)^{\alpha}\right)}{(1-\gamma)\kappa^2 \hat{y}^{ex} + (1-\alpha)\beta(\alpha-\kappa)^2 \kappa^{\alpha} M \sigma \left(\hat{y}^{ex}\right)^{\alpha}}. \tag{48}$$

Considering the results of the comparative-dynamics experiment in (46) - (48) with respect to marginal changes of the expected sales of production output and the savings rate we observe that the effects on capital-output ratio and on one minus the unemployment rate are qualitatively determinate while the effects on the equity price are in general indeterminate. It turns out that higher expected output sales ($d\hat{y}^{ex} > 0$) impacts negatively both the capital-output ratio ($d\kappa < 0$) and the unemployment rate ($du < 0$) while a lower savings rate ($d\sigma < 0$) decreases the capital-output ratio ($d\kappa < 0$) and the unemployment rate ($du < 0$). More optimistic manager expectations regarding future output sales decrease the capital-output ratio and the unemployment rate. Moreover, a lower savings rate by younger households decreases the capital-output ratio but also decreases the unemployment rate. Notice these typical "Keynesian" results in our neo-classical stock-market OLG model: more optimistic investors and less thrifty consumers reduce the steady-state unemployment rate. Also notice that in contrast to neo-classical growth theory (Solow, 1956; Diamond, 1965) an altered saving rate does not only impact the short-run unemployment rate but also the unemployment rate in the long run as in post-Keynesian studies like Fazzari et al. (2020).

So far, we assumed for the purpose of algebraic tractability that corporation managers' expectations immediately adjust towards their long-run values. Moreover, the qualitatively indeterminate impacts of altered long-run output-sales expectations and changes in the savings rate on the equity price suggest a numerical specification of our stock-market OLG model with involuntary unemployment. For our purely theoretical exercise, the main model parameters are chosen such that the assumptions of Propositions 1 and 2 hold. Moreover, we calibrate a parameter set which implies a steady-state capital-output ratio and an unemployment rate which accord rather well with their medium-term empirical values for the global economy averaged over the period between 1990 and 2020 (see IMF, 2008, 2014, 2020): $\kappa = 0.14$, $u = 0.06$. In line with the assumption $M = 1$, $L = 1$, $\psi = 1$ which simplify the entries of the Jacobian matrix (43) we then calibrate the numerical values of $\hat{y}^{ex}$ and β under fixing $\alpha = 0.2$ and $\varepsilon = \delta = 0.851$,to obtain $\kappa = 0.14$, $u = 0.06$. The calibration exercise delivers the following result: $\beta = 0.7131$, $\hat{y}^{ex} = 0.5789$, $q = 0.1173$. The coefficient which governs expectation adjustment will be set at $\varphi = 0.5$.

Before we calculate the equilibrium dynamics after marginal parameter shocks, we find the following numerical values for the trace of the Jacobian (43), its eigen values and the effects of more optimistic output-sales expectations and a lower savings rate on the steady-state equity price, respectively: $TrJ = 0.386$, $\lambda_2 = 3.8$, $\lambda_3 = 0.1839$, $\partial q / \partial \hat{y}^{ex} = -0.045$, $\partial q / \partial \sigma = 0.39$.

Starting from the same steady-state solution as before, suppose managing investors become more optimistic about output sales in the long run, i.e.
$\hat{y}^{ex} = 0.5789$ increases towards $\hat{y}^{ex}_{new} = 0.58$ while all other parameters remain on their pre-shock values. The effects of this small, positive investment shock on the capital-output ratio, the equity price, the unemployment rate and on the expected output sales along the intertemporal-equilibrium path towards the new steady state ($t = 40$) are depicted in Table 1.

As Table 1 reveals, the positive shock on investment decreases temporarily and permanently the capital-output ratio and (rather starkly) the unemployment rate, while the equity price slightly declines along the saddle-path towards the slightly lower new steady state. At first sight it seems to be surprising that higher expected output sales negatively impact the equity price. However, the reason is that better output-sales expectations raise dividends per share which compete with the future equity price according to the no-profitable arbitrage condition (2) respective (6). That the unemployment rate declines along the intertemporal equilibrium path towards its lower new steady-state level sounds Keynes-like in our neo-classical stock-market OLG model.

Consider now a small negative and unexpected shock on $\varepsilon = \delta$ from 0.851 towards 0.8505 implying a small decrease of the savings rate. Then, the following Table 2 exhibits the intertemporal equilibrium path of main endogenous variables towards the new steady state($t = 40$): $\kappa = 0.13995$, $q = 0.11725$,
$u = 0.05985$.

A glance on Table 2 reveals that a small reduction of the savings rate induces a reduction of the capital-output ratio, the unemployment rate, and the equity price along the saddle-path towards the new steady state with lower values for all three dynamic variables. That the unemployment rate temporarily (in the short-term) and permanently decreases with a lower saving rate sounds again

Table 1. Intertemporal equilibrium path of $\left(\kappa_t, q_t, u_t, y_t^{ex}\right)_{t>1}$ after a small positive output-sales-expectation shock.

t	0	1	2	3	4	5	…	40
κ_t	0.139973	0.139946	0.139906	0.139889	0.139882	0.1398789	…	0.1398754
q_t	0.117273	0.117260	0.117257	0.1172569	0.1172567	0.1172567	…	0.1172566
u_t	0.05989	0.05920	0.05844	0.05807	0.05788	0.05779	…	0.05770
y_t^{ex}	0.5789	0.57925	0.57963	0.57981	0.5799	0.57993		0.5800

Source: Author's own calculation.

Table 2. Intertemporal equilibrium path of $\left(\kappa_t, q_t, u_t, y_t^{ex}\right)_{t>1}$ after a small negative savings-rate shock.

t	0	1	2	3	4	…	40
κ_t	0.1399700	0.139955	0.139953	0.1399527	0.13995267	…	0.13995265
q_t	0.1172523	0.1172499	0.1172495	0.11724942	0.11724940	…	0.11724940
u_t	0.059890	0.059856	0.0598519	0.0598510	0.0598509	…	0.0598508
y_t^{ex}	0.5789	0.5789	0.5789	0.5789	0.5789		0.5789

Source: Author's own calculation.

"Keynesian" respective post-Keynesian (Fazzari et al. 2020) in our neo-classical stock-market OLG model with involuntary unemployment.

It remains to be seen whether the intertemporal and steady-state response of the equity price with respect to more optimistic investors and less thrifty consumers depends on the chosen parameter values or not. To this end, we calibrated the labor production share β and the long-run output-sales expectation $\hat{y}^{ex}$ for a higher capital production share $\alpha = 0.3$ with the result: $\beta = 0.557,\ \hat{y}^{ex} = 0.409893$. The pre-shock steady state solution reads as follows: $\kappa = 0.13997,\ \omega = 0.9401\ (u = 0.0599),\ q = 0.051267$.

The results in Table 3 show that the long-run effects of a positive shock of the expected output sales on the equity price are qualitatively like those of a smaller capital-production share and a larger labor-production share. Along the intertemporal equilibrium there seems to occur a positive overshooting with a larger capital-production share which does not occur with smaller capital-production shares. The effects on the capital-output ratio and the unemployment rate are qualitatively like those in the case with a lower capital-production share.

Disregarding the different starting values for the equity price and sales-expectations, Table 4 does not exhibit qualitatively different intertemporal equilibrium paths after a small negative savings-rate shock for the case of a larger capital-production share than for the case in Table 2 with a lower capital-production share.

Thus, we are entitled to conclude on account of the present restricted numerical parameter variations that the intertemporal-equilibrium effects of more optimistic sales expectations and lower savings rates are qualitatively similar for a broad set of capital-production shares and long-run output-sales expectations.

Table 3. Intertemporal equilibrium path of $\left(\kappa_t, q_t, u_t, y_t^{ex}\right)_{t>1}$ after a small positive output-sales-expectation shock under a larger capital-production share.

t	0	1	2	3	4	5	…	40
κ_t	0.139970	0.1393632	0.1391049	0.138988	0.138932	0.138906	…	0.138880
q_t	0.051160[4]	0.0511499	0.0511498	0.05115	0.051152	0.051152	…	0.051153
u_t	0.0599	0.05031	0.045661	0.04337	0.04224	0.04168	…	0.041125
y_t^{ex}	0.409893	0.412446	0.41372	0.41436	0.41436	0.41484		0.4150

Source: Author's own calculation.

Table 4. Intertemporal equilibrium path of $\left(\kappa_t, q_t, u_t, y_t^{ex}\right)_{t>1}$ after a small negative savings-rate shock under larger capital-production share.

t	0	1	2	3	4	…	40
κ_t	0.1399700	0.139941	0.139933	0.139931	0.13993	…	0.13993
q_t	0.0512445	0.0512412	0.051240	0.051240	0.0512399	…	0.0512399
u_t	0.059890	0.059856	0.059763	0.059755	0.059753	…	0.059752
y_t^{ex}	0.409893	0.409893	0.409893	0.409893	0.409893		0.409893

Source: Author's own calculation.

7. Conclusion

This paper introduces an endogenous unemployment rate and investors' beliefs about the expected sales of future production output into Cunha's (2012) stock market OLG model with market-value maximizing corporations. The current investment quantity is derived from the inverse of corporations' intertemporal production functions with expected sales of production output as argument. Moreover, this beliefs-determined investment quantity is consistent with optimally indeterminate firm-level investment. In that sense, inflexible aggregate investment is production-founded in our stock-market OLG model with involuntary unemployment.

In contradistinction to Cunha's (2012) full employment model, in our model the unemployment rate appears as additional dynamic variable with the consequence that the intertemporal-equilibrium dynamics is three- instead of two-dimensional as in Cunha (2012). The step-by-step derivation of the intertemporal-equilibrium equations from the first-order conditions for intertemporal-utility and market-value maxima, the government budget constraint, the non-degenerate belief function of corporations' managers and the market-clearing conditions brings forth a four-dimensional difference-equation system with the capital-output ratio, the unemployment rate, the equity price and the expected sales of production output as dynamic variables.

[4]The attentive reader acknowledges a difference between the pre-shock value for $q = 0.051267$ and $q_0 = 0.0512445$ in **Table 3**. The difference occurs because q_t is a jump variable which immediately responds to the expected-sales shock.

The investigation of the existence of steady-state solutions whereby the capital-output ratio, the unemployment rate, and the equity price do not change over time any longer reveals that there is only one feasible steady state solution with an unemployment rate between zero and one. Sufficient for the existence of the steady state is essential that the aggregate expected output-sales level per capita in the long run is not too large in comparison with the parameters of the aggregate production function, made precise in Proposition 1.

Due to the algebraic complexity of the entries of the three-dimensional Jacobian matrix of the equilibrium dynamics around the steady-state solution we are unable to prove in general local stability of the equilibrium dynamics. However, the numerical specification of a broad set of model parameters in line with the assumptions in Proposition 1 shows that one eigen value of the equilibrium dynamics is larger than one and another is smaller than one. Because for all admissible parameters the determinant of the Jacobian matrix is zero, we know for sure that the third eigenvalue is zero, too. Consequently, the equilibrium dynamics near the steady state is saddle-point stable with the capital-output ratio and the unemployment rate as slowly moving variables and the equity price as jump variable.

Having proven in Propositions 1 existence and shown dynamic stability of the steady state for numerically specified parameters in Proposition 2, we are entitled to perform local, comparative-dynamic experiments whereby we investigate the impacts of infinitesimal changes of expected future sales of production output and the savings rate on the steady-state capital-output ratio, the equity price and on the unemployment rate. We find that better long-run output-sales expectations of investing managers decrease the capital-output ratio and the unemployment rate, while the impact on the equity price is in general ambiguous. This ambiguity suggests again a numerical parameter specification to see how more manager optimism impacts the equity price. Starting from the pre-shock steady state, the equilibrium dynamics along the saddle-path towards the new steady state reveals that the positive shock on managers' optimism impacts slightly negative the equity price. This turns out to be true both for a relatively low and a relatively high capital-production share. In the former case the equity decreases monotonically towards the new, lower steady-state level, in the latter case the equity price initially increases and then decreases towards the new, lower steady-state level. The negative impact of better sales expectations on the equity price occurs because higher expected output sales raise the dividends per share which competes with the correctly expected, future equity price. Most interesting is the response of the steady-state unemployment rate with more manager's optimism: it decreases. And this is true also for intertemporal-equilibrium path towards the new steady state. This last result accords well with short-term Keynesian insights, and it turns out to be valid even in the long run as in post-Keynesian analyses.

In contrast to the insights from neo-classical growth theory (Solow, 1956; Diamond, 1965), variations of the savings rate do impact the steady-state capital-output ratio, the steady-state equity price, and the steady-state unemployment rate. Again, that the steady-state capital-output ratio and the steady-state unemployment rate decrease with a lower savings rate is generally valid, while the steady-state response of the equity price is in general again ambiguous. The numerical specification both for the case of low and high capital-production share reveals that the equity price declines with less consumer thriftiness. Now, the equity price declines monotonically towards the new steady for both cases of capital-production-share combinations. That a lower savings rate lowers the unemployment rate both in the short- and long-term sounds again not only Keynesian, but also post-Keynesian.

Obviously, there is ample space for future research. Highest on the agenda in this respect is the integration of our production-based, non-degenerate belief function into Magill and Quinzii's (2003) stock-market OLG model with non-shiftable, firm-specific physical capital. Moreover, it would be interesting to investigate the intertemporal equilibrium dynamics if investment in firm-specific capital is solely equity financed. Finally, other non-degenerate belief functions which are consistent with intertemporal equilibrium in stock-market OLG models with involuntary unemployment could be searched for.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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