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On Continuity of Browder Type Fixed Points

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Abstract

To study the stability of Browder type fixed points, two kinds of usual metrics for set-valued mappings are discussed. Noting that the set of fixed points for a Browder type set-valued mapping may be noncompact, a metric is introduced to construct a complete metric space on a kind of these mappings. Some results for continuity of Browder type fixed points are obtained, and we prove that each fixed point for each Browder type set-valued mapping is essentially stable.

Keywords: Fixed points; Stability; Metric continuity; Completeness; Browder

2010 Mathematics Subject Classification: 47H04; 47H10

1 Introduction

The stability of fixed points is an important problem in nonlinear fields. The stability for many kinds of fixed points is studied, which has a wide range of applications such as vector equilibrium problems, Nash equilibrium problems, coincidence points ([1] [2] [3] [4] [5]).

Particularly, by introducing essential stabilities, Fort gave some seminal results for fixed points of continuous functions (Brouwer type fixed points) in [6]. Unfortunately, it is not necessary that each continuous function on a compact convex set has an essential fixed point. In hyperconvex metric spaces, the generic stability of fixed points for upper semi-continuous mappings (Fan-Glicksberg type fixed points) was shown ([7]). The concept of an essential component of fixed points was introduced in [8], and the existence of such essential components for each continuous function was proved on a compact convex set. Essential stabilities were used to analyze many kinds of solutions, such as KKM points, equilibrium points, maximal elements ([9] [10] [11] [12] [13] [14] [15]).

One of significant feature of the set of solutions for most problems above mentioned is compact, which plays a key role in the analysis of the existence of essential sets, minimal essential sets and essential components. A well known fixed point theorem for a set-valued mapping was proved by Browder in [16] (Browder type fixed points), where the set-valued mapping requires some lower semi-continuities. Nowadays, we can find many generalizations of this fixed point theorem (e.g., see [17] [18]). However, for the set of Browder type fixed points of a set-valued mapping on a compact convex

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set may not be compact, and the set-valued image of a point can also be noncompact. To some extent, this leads to difficulties in the research of stability of this type of fixed points.

In this paper, two kinds of usual topologies for the space of set-valued mappings are discussed, and some examples are given to show that the metric space, consisting of Browder type set-valued mappings, can not be complete. We consider a kind of space of Browder type set-valued mappings on a compact convex set, and prove it is complete, and the solution mapping for fixed points is metric continuous on a dense subset of the space. In addition, we show that each Browder type fixed point is essential for each Browder type set-valued mapping.

2 Preliminaries and Motivations

Let X be a compact and convex subset of a metric linear space (E, d). Let $S : X \to 2^X$ be a set-valued mapping, where 2^X denotes the collection of all subsets of X. Suppose that $S(x)$ is nonempty and convex for each $x \in X$, and $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X for each $y \in X$. Denote by M all such set-valued mappings S (Browder type set-valued mappings) on X. Then for each $S \in M$, by the well known fixed point theorem of Browder ([16]), there exists a fixed point x^* of S; that is, $x^* \in S(x^*)$. In the whole paper, we call these fixed points as **Browder type fixed points**. For each $S \in M$, let $F(S) = \{x \in X : x \in S(x)\}$. Then F defines a set-valued mapping from M to X. Clearly, for each $S \in M$, we have $F(S) \neq \emptyset$. The following example shows that the fixed point set $F(S)$ for each $S \in M$ is not necessary closed, and not compact. In fact, the lack of compactness of $F(S)$ may results in some difficulties in study of stability of these fixed points.

Example 2.1. *Let $X = [0, 1] \subset \mathbb{R}$ and $S \in M$ such that*

$$S(x) = \begin{cases} 1, & x = 0, \\ [0, 1], & x \in (0, 1), \\ 0, & x = 1. \end{cases}$$

Obviously, for each $x \in X$, $S(x)$ is convex, closed and nonempty. For the inverse image of S, we have

$$S^{-1}(y) = \begin{cases} (0, 1], & y = 0, \\ (0, 1), & y \in (0, 1), \\ [0, 1), & y = 1. \end{cases}$$

Then we know that $S^{-1}(y)$ is open in X for each $y \in X$. Hence, $S \in M$. Definitly, we can check that $F(S) = (0, 1)$, which is not compact.

We recall some notions for set-valued mappings ([19] [20]). Let H be a metric space, $F : E \to 2^H$ be a set-valued mapping. Then:

(i) F is said to be upper semi-continuous at $h \in E$, iff for each open set U with $U \supset F(h)$, there exists an open neighborhood $O(h)$ of h such that $U \supset F(h')$ for any $h' \in O(h)$.

(ii) F is lower semi-continuous at $h \in E$, iff for each open set U with $U \cap F(h) \neq \emptyset$, there exists an open neighborhood $O(h)$ of h such that $U \cap F(h') \neq \emptyset$ for any $h' \in O(h)$.

(iii) F is continuous at $h \in E$, iff it is both upper semi-continuous and lower semi-continuous at h.

(iv) F is said to be metric upper semi-continuous at $h \in E$, if for each $\varepsilon > 0$ there exists an open set U with $U \supset h$ such that $F(h') \subset B_\varepsilon(F(h))$ for all $h' \in U$, where $B_\varepsilon(F(h))$ denotes the ε-neighborhood of $F(h)$.

(v) F is said to be metric lower semi-continuous at h, if for each $\varepsilon > 0$ there exists an open set U with $U \supset h$ such that $F(h) \subset B_\varepsilon(F(h))$ for all $h \in U$.

Remark 2.1. In general, if F is upper semi-continuous at h, then F is metric upper semi-continuous at h. If F is metric lower semi-continuous at h, then F is lower semi-continuous at h, while the converse is not true in general. If $F(h)$ is totally bounded, then F is lower semi-continuous at h iff F is metric lower semi-continuous at h.

If we adopt uniform metric induced by d on M to measure the metric between two elements in M; then, for each $S \in M$, naturally, we should require that $S(x)$ is closed for each $x \in X$, and this constrains our discussion on M_c, where M_c is written as

$$M_c = \{S \in M : S(x) \text{ is closed for each } x \in X\}.$$

That is, define the metric between any S_1 and S_2 in M_c as

$$\rho_1(S_1, S_2) = sup_{x\in X} h(S_1(x), S_2(x)),$$

where $h(A, B)$ is the Hausdorff metric induced by d. Then (M_c, ρ_1) is a metric space. However, this subspace of M using the uniform metric is not complete. See the following example.

Example 2.2. *Let $X = [0, 1] \subset \mathbb{R}$. Let $S_n \in M_c, n = 1, 2, \cdots$, such that*

$$S_1(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ [0, \frac{1}{2}], & x \in (\frac{1}{2}, 1], \end{cases}$$

and for $n = 2, 3, \cdots$

$$S_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ S_{n-1}(x), & x \in (\frac{1}{2}, 1 - \frac{1}{2^n}], \\ [0, 1 - \frac{1}{2^n}], & x \in (1 - \frac{1}{2^n}, 1]. \end{cases}$$

Clearly, $S_n(x)$ is nonempty, closed and convex for each $x \in X$, $n = 1, 2, \cdots$. For the inverse image of S_n, we have

$$S_1^{-1}(y) = \begin{cases} X, & y = 0, \\ (1 - \frac{1}{2}, 1], & y \in (0, \frac{1}{2}], \\ \emptyset, & y \in (\frac{1}{2}, 1]. \end{cases}$$

and for $n = 2, 3, \cdots$

$$S_n^{-1}(y) = \begin{cases} S_1^{-1}(y), & y = [0, \frac{1}{2}], \\ (1 - \frac{1}{2^i}, 1], & y \in (1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}], i = 2, 3, \cdots, n \\ \emptyset, & y \in (\frac{1}{2^n}, 1]. \end{cases}$$

Then $S^{-1}(y)$ is open in X for each $y \in X$. Therefore, $S_n \in M_c$, $n = 1, 2, \cdots$.

Define $S(x) = \cup_{n=1}^{\infty} S_n(x)$ for each $x \in X$. Let $N(x) = [log_2^{\frac{1}{1-x}}], x > \frac{1}{2}$. It can be checked that for each $x \in (\frac{1}{2}, 1)$ if $n \le N(x)$, we have $S_n(x) = S_{N(x)}(x)$, while $S_n(x) \subset S_{N(x)}(x)$ if $n > N(x)$, and $S(1) = [0, 1]$. Then we obtain that

$$S(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ S_{N(x)}(x), & x \in (\frac{1}{2}, 1), \\ [0, 1], & x = 1. \end{cases}$$

Then $S : X \to 2^X$ is convex and closed with nonempty values. Furthermore, we have $sup_{x\in X} h(S_n(x), S(x)) \to 0$ as $n \to \infty$. However, for the special point $y = 1$, the set $S^{-1}(y) = 0$ is not open in X. Then $S \notin M$, certainly, $S \notin M_c$. Hence (M_c, ρ_1) is not a complete metric space.

Let $S_1, S_2 \in M$. If we measure the metric between S_1 and S_2 by

$$\rho_2(S_1, S_2) = sup_{y\in X} h(X \backslash S_1^{-1}(y), X \backslash S_2^{-1}(y)).$$

Then, for each $S \in M$, the requirement $X \backslash S^{-1}(y) \neq \emptyset$ for each $y \in X$ is natural. This restrains our discussion on M_e, where (M_e, ρ_2) is written as

$$M_e = \{S \in M : S^{-1}(y) \neq X, \text{ for each } y \in X\}.$$

Certainly, we have $M_e \not\subset M_c$ and $M_c \not\subset M_e$. The subspace (M_e, ρ_2) of M is also not complete, see the following example.

Example 2.3. *Let $X = [0, 1] \subset \mathbb{R}$. Let $S_n \in M, n = 1, 2, \cdots$, such that*

$$S_n(x) = \begin{cases} 1, & x = 0, \\ [0, 1], & x \in (0, \frac{1}{n}), \\ 0, & x \in [\frac{1}{n}, 1]. \end{cases}$$

Easily, we can check that $S_n \in M_e, n = 1, 2, \cdots$. Then

$$X \backslash S_n^{-1}(y) = \begin{cases} 0, & y = 0, \\ 0 \cup [\frac{1}{n}, 1], & y \in X \backslash \{0, 1\}, \\ [\frac{1}{n}, 1], & y = 1. \end{cases}$$

Define a set-valued mapping S on X satisfying

$$S(x) = \begin{cases} \emptyset, & x = 0, \\ 0, & x \in X \backslash 0. \end{cases}$$

Then it holds that

$$X \backslash S^{-1}(y) = \begin{cases} 0, & y = 0, \\ X, & y \in X \backslash 0. \end{cases}$$

Clearly, for each $y \in X$, we have $h(X \backslash S_n^{-1}(y), X \backslash S^{-1}(y)) \to 0$. However, $S(0) = \emptyset$, that is, $S \notin M$ and $S \notin M_e$, hence, (M_e, ρ_2) is also not complete.

Noting the facts in Examples, given a point $y_0 \in X$ and a nonempty closed subset B in X, we consider the space

$$M(y_0, B) = \{S \in M : y_0 \in S(x), \forall x \in X; S(x) = y_0, \forall x \in B\},$$

Obviously, we have that $M(y_0, B)$, M_c and M_e can not include each other. Note that the set of fixed points, $F(S)$, may be noncompact for a set-valued mapping $S \in M(y_0, B)$. For instance, in Example 2.1, if we set $S'(1) = 1$ and $S'(x) = S(x)$ for $x \in X \backslash 1$. Let $y_0 = 1 \in X$ and $B = \{0\}$, then $y_0 \in S'(x), \forall x \in X$ and $S' \in M(y_0, B)$. We can check that $F(S')$ is equal to $(0, 1]$, and not a compact set.

Define the metric between any S_1 and S_2 in $M(y_0, B)$ as

$$\rho(S_1, S_2) = sup_{y \in X \backslash y_0} h(X \backslash S_1^{-1}(y), X \backslash S_2^{-1}(y)) + h(S_1^{-1}(y_0), S_2^{-1}(y_0)).$$

Let $S \in M(y_0, B)$, $y \in X \backslash y_0$. If $S^{-1}(y) = X$, then $x \in S^{-1}(y), \forall x \in X$, that is, $y \in S(x), \forall x \in X$. However, we know that $S(x) = y_0 \neq y, x \in B$, a contradiction. Consequently, it holds that $X \backslash S^{-1}(y) \neq \emptyset$, hence, the metric ρ on $M(y_0, B)$ is well defined. That is, $(M(y_0, B), \rho)$ is definitely a metric space.

Another space that we will consider is $M(y_0)$ for a given point $y_0 \in X$, where

$$M(y_0) = \{S \in M : y_0 \in S(x), \forall x \in X; S^{-1}(y) \neq X, \forall y \in X \backslash y_0\},$$

Clearly, $(M(y_0), \rho)$ is also a metric space with $M(y_0, B) \subset M(y_0)$ for any point $y_0 \in X$ and nonempty closed set $B \subset X$. In the next section, we will prove that they are all complete.

Definition 2.1. For each $S \in M(y_0)$, a set $e(S)$ is called an essential fixed point set of S with respect to $M(y_0)$ iff it satisfies the following conditions:

(1) $e(S)$ is nonempty subset of $F(S)$;

(2) for any open set U with $U \supset e(S)$, there exists an open neighborhood $O(S)$ in $M(y_0)$ such that $U \cap F(S') \neq \emptyset$, for any $S' \in O(S)$.

If $e(S)$ is a singleton set $\{x^*\}$, x^* is called an essential fixed point of S with respect to $M(y_0)$.

Remark 2.2. An essential fixed point x^* of S means that for each Browder type set-valued mapping near S there is a Browder type fixed point near the point x^*.

For the proof of main results of the paper, we need the following well known result and a generic continuity result in [21].

Lemma 2.4. *Let E, H be two metric spaces. If the set-valued mapping $F : E \to 2^H$ satisfies that for each $y \in H$, the set $S^{-1}(y) = \{x \in E : y \in S(x)\}$ is open in E, then F is lower semi-continuous on E.*

Lemma 2.5. *([21]) Let P be a complete metric space, Y be a metric space and $F : P \to 2^Y$ be metric lower semi-continuous. Then there exists a dense residual set $Q \subset P$ such that F is metric upper semi-continuous at each $x \in Q$.*

3 The Continuity of Browder Type Fixed Points

Theorem 3.1. *For each given point $y_0 \in X$ and a nonempty closed subset $B \subset X$, the metric space $(M(y_0, B), \rho)$ is complete.*

Proof. Let $\{S_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $M(y_0, B)$. Then for any $\varepsilon > 0$, there exists a number N such that $\rho(S_n, S_m) < \varepsilon$ for any $n, m > N$. That is,

$$h(X \backslash S_n^{-1}(y), X \backslash S_m^{-1}(y)) < \varepsilon, \forall y \in X \backslash y_0,$$

and $h(S_n^{-1}(y_0), S_m^{-1}(y_0)) < \varepsilon$. Therefore, $\{X \backslash S_n^{-1}(y)\}_{n=1}^{\infty}$ and $\{S_n^{-1}(y_0)\}_{n=1}^{\infty}$ are two Cauchy sequences in $K(X)$, for each $y \in X$, where $K(X)$ denotes all nonempty and compact subsets of X. Since X is compact, $K(X)$ is complete. Then for each $y \in X \backslash y_0$ there is a set $A(y) \in K(X)$ such that $h(X \backslash S_n^{-1}(y), A(y)) \to 0$ $(n \to \infty)$. For $y_0 \in X$, there is a set $A(y_0) \in K(X)$ such that $h(S_n^{-1}(y_0), A(y_0)) \to 0$. Since $S_n^{-1}(y_0) = X$, $n = 1, 2 \cdots$, we have $A(y_0) = X$.

Define a set-valued mapping S as

$$S(x) = \{y \in X \backslash y_0 : x \notin A(y)\} \cup \{y_0\}, \forall x \in X,$$

next, we need to show that $S \in M(y_0, B)$.

(a) For each $x \in X$, since $y_0 \in S(x)$, we have $S(x) \neq \emptyset$.

(b) Suppose that there exists a point $x \in B$ and a point $y \in X$ with $y \neq y_0$ such that $y \in S(x)$. Then $x \notin A(y)$. Since $h(X \backslash S_n^{-1}(y), A(y)) \to 0$, we have $x \notin X \backslash S_n^{-1}(y)$ as n is large enough. That is, there is a number N' such that $x \in S_n^{-1}(y)$ for each $n > N'$, hence, $y \in S_n(x)$. However, we know that $S_n \in M$, it holds that $S_n(x) = y_0$, a contradiction with $y \in S_n(x)$. Therefore, $S(x) = y_0, \forall x \in B$.

(c) For each $y \in X \backslash y_0$, we have

$$S^{-1}(y) = \{x : y \in S(x)\} = \{x \in X : x \notin A(y)\} = X \backslash A(y).$$

Noting that $A(y) \in K(X)$, $S^{-1}(y)$ is open for each $y \in X \backslash y_0$. For $y = y_0$, clearly, we have $S^{-1}(y_0) = X$, then $S^{-1}(y_0)$ is also open.

(d) For each $x \in X$, let $y_1, y_2 \in S(x)$ with $y_1 \neq y_2$. For each $\lambda \in [0, 1]$, let $y_\lambda = \lambda y_1 + (1 - \lambda) y_2$. We show that $y_\lambda \in S(x)$, that is, $S(x)$ is convex by the following steps.

(i) If $y_\lambda = y_0$, naturally, $y_\lambda \in S(x)$ by the fact that $S^{-1}(y_0) = X$.

(ii) If $y_1 \neq y_0$ and $y_2 \neq y_0$, then $x \notin A(y_1)$ and $x \notin A(y_2)$. For the case that $y_\lambda \neq y_0$, by way of contradiction, suppose that $x \in X \backslash S^{-1}(y_\lambda) = A(y_\lambda)$. Since $h(X \backslash S_n^{-1}(y_\lambda), A(y_\lambda)) \to 0$, there exists a point $x_n \in X \backslash S_n^{-1}(y_\lambda)$ such that $x_n \to x$. If $y_1 \in S_n(x_n)$ and $y_2 \in S_n(x_n)$, then $y_\lambda \in S_n(x_n)$ by the convexity of $S_n(x_n)$, hence, $x_n \in S_n^{-1}(y_\lambda)$, a contradiction with $x_n \in X \backslash S_n^{-1}(y_\lambda)$. Therefore, for each $n = 1, 2, \cdots$, we argue that at least one of the followings holds:

$$y_1 \notin S_n(x_n), \text{ or } y_2 \notin S_n(x_n).$$

Then, $x_n \in X\backslash S_n^{-1}(y_1)$ or $x_n \in X\backslash S_n^{-1}(y_2)$, hence, $x \in A(y_1)$ or $x \in A(y_2)$ as n gets close to infinity, a contradiction. Therefore, we have $x \notin A(y_\lambda)$, that is, $y_\lambda \in S(x)$.

(iii) If $y_1 \neq y_0$ and $y_2 = y_0$, then $x \notin A(y_1)$. For the case that $y_\lambda \neq y_0$, suppose that $x \in X\backslash S^{-1}(y_\lambda) = A(y_\lambda)$. Similar with (ii), there is a point $x_n \in X\backslash S_n^{-1}(y_\lambda)$ such that $x_n \to x$. If $y_1 \in S_n(x_n)$, then $y_\lambda \in S_n(x_n)$ because $y_2 = y_0 \in S_n(x_n)$ and $S_n(x_n)$ is convex. Then $x_n \in S_n^{-1}(y_\lambda)$. This contradicts to $x_n \in X\backslash S_n^{-1}(y_\lambda)$. Therefore, we have $y_1 \notin S_n(x_n)$, hence, $x_n \in X\backslash S_n^{-1}(y_1)$. This results in the fact that $x \in X\backslash S^{-1}(y_1) = A(y_1)$ as n tends to infinity, a contradiction. Therefore, we have $y_\lambda \in S(x)$. If $y_1 = y_0$ and $y_2 \neq y_0$, the proof is similar.

Then, $S \in M(y_0, B)$, and the proof is completed. $\square$

Theorem 3.2. *For each $y \in X$, $F^{-1}(y) = \{S \in M(y_0, B) : y \in F(S)\}$ is an open set in $(M(y_0, B), \rho)$ for each $y_0 \in X$ and a nonempty closed set B with $B \subset X$.*

Proof. For each $y \in X$, we only need to prove that $M(y_0, B)\backslash F^{-1}(y)$ is closed in $M(y_0, B)$.

(a) For the case satisfying that $y = y_0$, if there is a $S \in M(y_0, B)$ such that $S \in M(y_0, B)\backslash F^{-1}(y_0)$, then $S \notin F^{-1}(y_0)$, that is, $y_0 \notin F(S)$, hence, $y_0 \notin S(y_0)$, a contradiction with $S \in M(y_0, B)$. Therefore, $M(y_0, B)\backslash F^{-1}(y_0)$ is empty, and closed also.

(b) For each $y \in X\backslash y_0$. Let $S_n \in M(y_0, B)\backslash F^{-1}(y), n = 1, 2, \cdots$ and $S_n \to S_0$ as $n \to \infty$. By Theorem 3.1, it holds that $S_0 \in M(y_0, B)$. For each $n = 1, 2, \cdots$, since $S_n \in M(y_0, B)\backslash F^{-1}(y)$, we have $y \notin F(S_n)$, then $y \notin S_n^{-1}(y)$, that is, $y \in X\backslash S_n^{-1}(y)$. Since $h(X\backslash S_n^{-1}(y), X\backslash S_0^{-1}(y)) \to 0$ for each $y \in X\backslash y_0$ by the fact that $S_n \to S_0$, we have $y \in X\backslash S_0^{-1}(y)$. Hence $y \notin S_0^{-1}(y)$, consequently, $y \notin F(S_0)$, that is, $S_0 \in M(y_0, B)\backslash F^{-1}(y)$. Therefore, $M(y_0, B)\backslash F^{-1}(y)$ is closed in $M(y_0, B)$. The proof is completed. $\square$

Theorem 3.3. (1) *For each given point $y_0 \in X$, the metric space $(M(y_0), \rho)$ is complete.* (2) *For each $y \in X$, $F^{-1}(y) = \{S \in M(y_0) : y \in F(S)\}$ is an open set in $(M(y_0), \rho)$ for each $y_0 \in X$.*

Proof. For the part (1), it just follows from the proof of Theorem 3.1 by ignoring the part (b). Noting that $(M(y_0), \rho)$ is complete, similar with Theorem 3.2, the part (2) follows. $\square$

Remark 3.1. A problem is whether there is a space (M', ρ) which can include $(M(y_0), \rho)$ by using the metric ρ to construct a metric space on M. For a point $S \in M$, we know that $S^{-1}(y)$ is open in X. Noting that if $S^{-1}(y_0)$ is nonempty and closed in X (for a given $y_0 \in X$, the metric ρ uses the Hausdorff metric which generally is defined on nonempty, closed and bounded sets), then $S^{-1}(y_0) = X$; that is, $y_0 \in S(x), \forall x \in X$. Thus, to construct a metric space using the metric ρ, $M(y_0)$ is maximal space; that is, there is no (M', ρ) such that $M(y_0) \subset M' \subset M$.

Corollary 3.4. *$F : M(y_0) \to 2^X$ is lower semi-continuous on $(M(y_0), \rho)$ for each $y_0 \in X$.*

Proof. It follows from Lemma 2.4 and Theorem 3.3. $\square$

Theorem 3.5. *Let $y_0 \in X$, B be a nonempty closed subset in X, and $S \in M(y_0, B)$. For each $x^* \in F(S)$, x^* is an essential fixed point of S with respect to $M(y_0, B)$.*

Proof. For each open set U in X with $x^* \in U$, it follows that $F(S) \cap U \neq \emptyset$. By Corollary 3.4, F is lower semi-continuous at each S in $M(y_0)$. Then there exists an open set of $O(S) \subset M(y_0)$ such that $F(S') \cap U \neq \emptyset$ for each $S' \in O(S)$. That is, x^* is an essential fixed point of S with respect to $M(y_0)$. $\square$

Remark 3.2. By Theorem 3.5, for any $S \in M(y_0)$, $F(S)$ itself is an essential fixed point set of S with respect to $M(y_0)$; for each $e(S) \subset F(S)$, $e(S)$ is also an essential fixed point set of S with respect to $M(y_0)$; and each Browder type fixed point has the ability to resist the perturbation of S in $M(y_0)$.

Theorem 3.6. *Let $y_0 \in X$. The soluton mapping $F : M(y_0) \to 2^X$ for Browder type fixed points is metric continuous at almost all $S \in M(y_0)$. That is, there exists a dense residual set $Q \subset M(y_0)$ such that F is metric continuous at each $S \in Q$.*

Proof. Since X is compact, for each $S \in M$, we have that $F(S)$ with $F(S) \subset X$ is relatively compact, hence, $F(S)$ is totally bounded. Then F is metric lower semi-continuous at each S with $S \in M(y_0)$ by Corollary 3.4 and Remark 2.1. From Theorem 3.3, $M(y_0)$ is complete. Then, by Lemma 2.5, we obtain that there exists a dense residual set $Q \subset M(y_0)$ such that F is metric upper semi-continuous at each $S \in Q$. Thus, F is metric continuous at each $S \in Q$. □

4 Conclusions

To measure the metric between two Browder type set-valued mappings, in consideration of completeness, we show that two kinds of usual metrics have some faults. By using a new metric, we obtain generic stability results in relation to continuity. These results hold, though the set of Browder type fixed points may be noncompact.

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Competing Interests

The authors declare that no competing interests exist.

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