British Journal of Mathematics & Computer Science
4(16): 2352-2360, 2014

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Positive Solutions of Singular Periodic Boundary Value Problems for Second Order Impulsive Differential Equations

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Original Research Article

Received: 26 May 2014
Accepted: 17 June 2014
Published: 24 June 2014

Abstract

This paper is devoted to study the existence of positive solutions for singular second-order periodic boundary value problem with impulse effects. Existence is established via the theory of fixed point theorem in cones.

Keywords: Positive solution, periodic boundary value problem, second-order impulsive differential equations, fixed point theorem.

MR (2000) Subject Classifications: $34B15$.

1 Introduction

This paper is devoted to study of the existence of positive solutions for the following singular periodic boundary value problems with impulsive effects:

$$\begin{cases} -x'' + Mx = f(t,x), & t \neq t_k, \quad t \in J \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \cdots, l, \\ \Delta x|_{t=t_k} = \overline{I}_k(x(t_k)), & k = 1, 2, \cdots, l, \\ x(0) = x(2\pi), \ \ x'(0) = x'(2\pi) \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = 2\pi, M > 0, J = [0, 2\pi], f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+), \overline{I_k} \in C(R^+, R), R^+ = [0, \infty)$, with $-\frac{1}{m} I_k(x) < \overline{I}_k(x) < \frac{1}{m} I_k(x), x \in R^+$,

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$m=\sqrt{M}$, $\Delta x'|_{t=t_k}=x'(t_k^+)-x'(t_k^-), \Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-)$, $x'(t_k^+), x(t_k^+)$ ($x'(t_k^-), x(t_k^-)$) denote the right limit (left limit) of $x'(t)$ *and* $x(t)$ at $t=t_k$, $f(t,x)$ may be singular at $x=0$.

It is well known that there are abundant results about the existence of positive solutions of boundary value problems for second order ordinary differential equations. Some works can be found in $[1-3]$ and references therein. They mainly investigated the case without impulse actions. Recently, singular Dirichlet boundary problems of second order impulsive differential equations have been studied in $[4-6]$. Motivated by the work above, this paper attempts to study the existence of positive solutions for periodic boundary value problems. The techniques we employ here involve an application of the fixed point theorem in cones to second order boundary value problem with impulse action.

Moreover, for the simplicity in the following discussion, we introduce the following hypotheses.

(H_1): There exists an $\varepsilon_0>0$ such that $f(t,x)$ and $I_k(x)$ are non increasing in $x\le\varepsilon_0$, for each fixed $t\in[0,2\pi]$

(H_2): For each fixed $0<\theta\le\varepsilon_0$, $0<\int_0^{2\pi} f(s,\theta\sigma))ds<\infty.$

(H_3):

$$f^{\infty}+\frac{\sum_{k=1}^{l} I^{\infty}(k)}{\sigma 2\pi}<M,$$

where $f^{\infty}=\limsup_{x\to+\infty}\max_{t\in[0,2\pi]}\frac{f(t,x)}{x}$, $I^{\infty}(k)=\limsup_{x\to+\infty}\frac{I_k(x)}{x}, \sigma=min\{\frac{G(0)}{G(\pi)},\frac{1}{e^{2m\pi}}\}$, $G(0)=\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}, G(\pi)=\frac{2e^{m\pi}}{2m(e^{2m\pi}-1)}$, $m=\sqrt{M}$.

Theorem 1. Assume that $(H_1)-(H_3)$ are satisfied. Then problem (1.1) has at least one positive solution x .Moreover, there exists a $\theta^*>0$ such that

$$x(t)\ge\theta^*\sigma, \qquad t\in[0,2\pi].$$

2 Preliminary

In order to define the solution of (1.1) we shall consider the following space.

Let $J'=J\setminus\{t_1,t_2,\cdots,t_l\}$,

$$PC(I,R)=\{x\in C(I,R); x|_{(t_k,t_{k+1})}\in C(t_k,t_{k+1}),\ x(t_k^-)=x(t_k),\ \exists\, x(t_k^+),\ k=1,2,\cdots,l\}$$

$$PC'(I,R)=\{x\in C(I,R); x'|_{(t_k,t_{k+1})}\in C(t_k,t_{k+1}),\ x'(t_k^-)=x'(t_k),\ \exists\, x'(t_k^+),\ k=1,2,\cdots,l\}$$

With the norm $\|x\|_{PC}=\sup_{t\in[0,2\pi]}|x(t)|, \|x\|_{PC'}=\max\{\|x\|_{PC},\|x'\|_{PC}\}$. Then $PC(I,R)$, $PC'(I,R)$ are Banach spaces.

Definition 2.1: A function $x\in PC'(J,R)\cap C^2(J',R)$ is a solution of (1.1) if it satisfies the differential equation

$$-x''+Mx=f(t,x),\quad t\in J'$$

and the function x satisfies the conditions $\Delta x'|_{t=t_k}=x'(t_k^+)-x'(t_k^-)=-I_k(x(t_k))$, $\Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-)=\overline{I}_k(x(t_k))$ and the periodic boundary conditions $x(0)=x(2\pi)$, $x'(0)=x'(2\pi)$.

Lemma 2.1: If x is a solution of the equation

$$x(t)=\int_0^{2\pi}G(t,s)f(s,x(s))ds+\sum_{k=1}^{l}G(t,t_k)I_k(x(t_k))+\sum_{k=1}^{l}\frac{\partial G(t,s)}{\partial s}|_{s=t_k}\ \overline{I}_k(x(t_k)),\qquad t\in J \tag{2.1}$$

then x is a solution of (1.1), where G(t, s) is the Green's function to the periodic boundary value problem $-x''+Mx=0, x(0)=x(2\pi), x'(0)=x'(2\pi)$ and

$$G(t,s):=\frac{1}{\Gamma}\begin{cases}e^{m(t-s)}+e^{m(2\pi-t+s)}, & 0\le s\le t\le 2\pi,\\ e^{m(s-t)}+e^{m(2\pi-s+t)}, & 0\le t\le s\le 2\pi.\end{cases}$$

here $\Gamma=2m(e^{2m\pi}-1)$.

One can find that

$$\frac{2e^{m\pi}}{2m(e^{2m\pi}-1)}=G(\pi)\le G(t,s)\le G(0)=\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}. \tag{2.2}$$

For every positive solution of problem (1.1), one has

$$\|x\|_{PC}=\sup_{t\in[0,2\pi]}|x(t)|$$

Without loss of generality, we assume $\lim_{t\to\xi}|x(t)|=\|x\|_{PC},\xi\in[t_k,t_{k+1}],k\in\{0,1,\cdots l\},$ then by (2.2)

$$\begin{aligned}\|x\|_{PC} &\le G(0)\int_0^{2\pi} f(s,x(s))ds+\lim_{t\to\xi}\left\{\sum_{i=1}^{l}G(t,t_i)I_i(x(t_i))+\sum_{i=1}^{l}\frac{\partial G(t,s)}{\partial s}|_{s=t_i}\ \overline{I}_i(x(t_i))\right\}\\
&=G(0)\int_0^{2\pi} f(s,x(s))ds\\
&+\frac{1}{\Gamma}\left\{\sum_{i=1}^{k}[e^{m(\xi-t_i)}+e^{m(2\pi-\xi+t_i)}]I_i(x(t_i))+\sum_{i=k+1}^{l}[e^{m(t_i-\xi)}+e^{m(2\pi-t_i+\xi)}]I_i(x(t_i))\right\}\\
&+\frac{1}{\Gamma}\left\{\sum_{i=1}^{k}[-me^{m(\xi-t_i)}+me^{m(2\pi-\xi+t_i)}]\overline{I}_i(x(t_i))+\sum_{i=k+1}^{l}[me^{m(t_i-\xi)}-me^{m(2\pi-t_i+\xi)}]\overline{I}_i(x(t_i))\right\}\\
&=G(0)\int_0^{2\pi} f(s,x(s))ds\\
&+\frac{1}{\Gamma}\left\{\sum_{i=1}^{k}[e^{m(\xi-t_i)}(I_i(x(t_i))-m\overline{I}_i(x(t_i)))+e^{m(2\pi-\xi+t_i)}(I_i(x(t_i))+m\overline{I}_i(x(t_i)))]\right\}\\
&+\frac{1}{\Gamma}\left\{\sum_{i=k+1}^{l}[e^{m(t_i-\xi)}(I_i(x(t_i))+m\overline{I}_i(x(t_i)))+e^{m(2\pi-t_i+\xi)}(I_i(x(t_i))-m\overline{I}_i(x(t_i)))]\right\}\end{aligned}$$

It follows from $-\frac{1}{m}I_i(x)<\overline{I}_i(x)<\frac{1}{m}I_i(x),$ that $I_i(x)-m\overline{I}_i(x)>0,I_i(x)+m\overline{I}_i(x)>0.$ So

$$\|x\|_{PC}\le G(0)\int_0^{2\pi} f(s,x(s))ds+\frac{2e^{2m\pi}}{\Gamma}\sum_{i=1}^{l}I_i(x(t_i)). \tag{2.3}$$

For any $t\in[0,2\pi],$ without loss of generality, we assume that $t\in[t_k,t_{k+1}),$ then

$$\begin{aligned}x(t)&\ge G(\pi)\int_0^{2\pi} f(s,x(s))ds+\sum_{i=1}^{l}G(t,t_i)I_i(x(t_i))+\sum_{i=1}^{l}\frac{\partial G(t,s)}{\partial s}|_{s=t_i}\ \overline{I}_i(x(t_i))\\
&=G(\pi)\int_0^{2\pi} f(s,x(s))ds\\
&+\frac{1}{\Gamma}\sum_{i=1}^{k}[e^{m(t-t_i)}(I_i(x(t_i))-m\overline{I}_i(x(t_i)))+e^{m(2\pi-t+t_i)}(I_i(x(t_i))+m\overline{I}_i(x(t_i)))]\\
&+\frac{1}{\Gamma}\sum_{i=k+1}^{l}[e^{m(t_i-t)}(I_i(x(t_i))+m\overline{I}_i(x(t_i)))+e^{m(2\pi-t_i+t)}(I_i(x(t_i))-m\overline{I}_i(x(t_i)))]\end{aligned}$$

It follows from $-\frac{1}{m}I_i(x)<\overline{I}_i(x)<\frac{1}{m}I_i(x),$ that $I_i(x)-m\overline{I}_i(x)>0,I_i(x)+m\overline{I}_i(x)>0.$ So

$$x(t) \geq G(\pi)\int_0^{2\pi} f(s,x(s))ds + \frac{2}{\Gamma}\sum_{i=1}^{l} I_i(x(t_i))$$

$$\geq \frac{G(\pi)}{G(0)} \bullet G(0)\int_0^{2\pi} f(s,x(s))ds + \frac{1}{e^{2m\pi}}\frac{2e^{2m\pi}}{\Gamma}\sum_{i=1}^{l} I_i(x(t_i))$$

$$\geq min\{\frac{G(\pi)}{G(0)}, \frac{1}{e^{2m\pi}}\}\|x\|_{PC} := \sigma\|x\|_{PC}. \quad (2.4)$$

3 Main Results

Lemma 3.1:Let $E=(E,\|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E ,and $\|\cdot\|$ be increasing with respect to K .Also, r , R are constants with $0<r<R$. Suppose that $A:(\overline{\Omega}_R \setminus \Omega_r)\bigcap K \to K$ ($\Omega_R = \{x \in E, \|x\| < R\}$) is a continuous, compact map and assume that the conditions are satisfied:

(i) $\| Ax \| > x$, for $x \in \partial\Omega_r \bigcap K$

(ii) $x \neq \mu A(x)$, for $\mu \in [0,1)$ and $x \in \partial\Omega_R \bigcap K$

Then A has a fixed point in $K \bigcap \{x \in E : r \leq \|x\| \leq R\}$.

Proof. In applications below, we take $E = C(I,R)$ and define

$$K = \{x \in C(I,R) : x(t) \geq \sigma\|x\|, t \in [0,2\pi]\}.$$

One may readily verify that K is a cone in E . Now, let $r>0$ such that

$$r < min\{\varepsilon_0, G(\pi)\int_0^{2\pi} f(s,\varepsilon_0)ds + \frac{2}{\Gamma}\sum_{k=1}^{l} I_k(\varepsilon_0)\} \quad (3.1)$$

and let $R>r$ be chosen large enough later.

Let us define an operator $A:(\overline{\Omega}_R \setminus \Omega_r)\bigcap K \to K$ by

$$(Ax)(t) = \int_0^{2\pi} G(t,s)f(s,x(s))ds + \sum_{k=1}^{l} G(t,t_k)I_k(x(t_k)) + \sum_{k=1}^{l} \frac{\partial G(t,s)}{\partial s}|_{s=t_k} \bar{I}_k(x(t_k)), t \in J.$$

First we show that A is well defined. To see this, notice that if $x \in (\overline{\Omega}_R \setminus \Omega_r) \bigcap K$ then $r \le \|x\| \le R$ and $x(t) \ge \sigma \|x\| \ge \sigma r, 0 \le t \le 2\pi.$ Also notice by (H_1) that

$$f(t, x(t)) \le f(t, r\sigma), \qquad when\ 0 \le x(t) \le r,$$

and

$$f(t, x(t)) \le \max_{r \le x \le R} \max_{0 \le t \le 2\pi} f(t, x) \qquad when\ r \le x(t) \le R.$$

These inequalities with (H_2) guarantee that $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ is well defined.

Next we show that $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ .If $x \in (\overline{\Omega}_R \setminus \Omega_r) \bigcap K$, then we have

$$\|Ax\|_{PC} \le G(0) \int_0^{2\pi} f(s, x(s)) ds + \frac{2e^{2m\pi}}{\Gamma} \sum_{k=1}^{l} I_k(x(t_k)),$$

$$(Ax)(t) \ge min\{\frac{G(\pi)}{G(0)}, \frac{1}{e^{2m\pi}}\} \|Ax\|_{PC} := \sigma \|Ax\|_{PC}. \quad t \in [0, 2\pi]$$

i.e. $Ax \in K$ so $A : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K.$

It is clear that A is continuous and completely continuous.

We now show that

$$\| Ax \| > \|x\|, \qquad for\ x \in \partial\Omega_r \bigcap K \tag{3.2}$$

To see that, let $x \in \partial\Omega_r \bigcap K$, then $\|x\| = r$ and $x(t) \ge \sigma r$ for $t \in [0, 2\pi]$. So by (H_1) and (3.1) we have

$$\begin{aligned}(Ax)(t) &= \int_0^{2\pi} G(t,s) f(s, x(s)) ds + \sum_{k=1}^{l} G(t, t_k) I_k(x(t_k)) + \sum_{k=1}^{l} \frac{\partial G(t,s)}{\partial s}\Big|_{s=t_k} \bar{I}_k(x(t_k)) \\ &\ge G(\pi) \int_0^{2\pi} f(s, x(s)) ds + \frac{2}{\Gamma} \sum_{k=1}^{l} I_k(x(t_k)) \\ &\ge G(\pi) \int_0^{2\pi} f(s, r) ds + \frac{2}{\Gamma} \sum_{k=1}^{l} I_k(r) \\ &\ge G(\pi) \int_0^{2\pi} f(s, \varepsilon_0) ds + \frac{2}{\Gamma} \sum_{k=1}^{l} I_k(\varepsilon_0) \\ &> r = \|x\|.\end{aligned}$$

so (3.2) is satisfied.

On the other hand, from (H_3), there exist $0<\varepsilon<M-f^{\infty}$ and $H>r$ such that

$$(M-\varepsilon-f^{\infty})2\pi\sigma>\sum_{k=1}^{l}(I^{\infty}(k)+\varepsilon). \tag{3.3}$$

$$f(t,x)\le(f^{\infty}+\varepsilon)x, I_k(x)\le(I^{\infty}(k)+\varepsilon)x \quad \forall\ t\in[0,2\pi], x\ge H.$$

Let $C=\max_{r\le x\le H}\max_{0\le t\le 2\pi} f(t,x)+\sum_{k=1}^{l}\max_{r\le x\le H} I_k(x),$ it is clear that

$$f(t,x)\le f(t,r\sigma)+C+(f^{\infty}+\varepsilon)x, I_k(x)\le I_k(r\sigma)+C+(I^{\infty}(k)+\varepsilon)x, \forall\ t\in[0,2\pi], x\ge 0.$$

Next we show that if R is large enough ,then $\mu Ax\ne x$ for any $x\in K\bigcap\partial\Omega_R$ and $0\le\mu<1$. If this is not true ,then there exist $x_0\in K\bigcap\partial\Omega_R$ and $0\le\mu_0<1$ such that $\mu_0 Ax_0=x_0$. Thus $\|x_0\|=R>r$ and $x_0(t)\ge\sigma R.$ Note that $x_0(t)$ satisfies

$$\begin{cases} -x_0''(t)+Mx_0(t)=\mu_0 f(t,x_0(t)), & t\in J', \\ -\Delta x_0'|_{t=t_k}=\mu_0 I_k(x_0(t_k)), & k=1,2,\cdots,l, \\ \Delta x_0|_{t=t_k}=\mu_0\bar{I}_k(x_0(t_k)), & k=1,2,\cdots,l, \\ x_0(0)=x_0(2\pi), & \\ x'_0(0)=x'_0(2\pi). & \end{cases} \tag{3.4}$$

Integrate from 0 to 2π , using integration by parts in the left side, notice that

$$\begin{aligned}\int_0^{2\pi}[-x_0''(t)+Mx_0(t)]dt&=\sum_{k=1}^{l}\Delta x_0'(t_k)+M\int_0^{2\pi}x_0(t)dt\\ &=-\mu_0\sum_{k=1}^{l}I_k(x_0(t_k))+M\int_0^{2\pi}x_0(t)dt\end{aligned}$$

So we obtain

$$
\begin{aligned}
M\int_0^{2\pi} x_0(t)dt &= \mu_0 \sum_{k=1}^{l} I_k(x_0(t_k)) + \mu_0 \int_0^{2\pi} f(t, x_0(t))dt \\
&\le \sum_{k=1}^{2\pi} (I^\infty(k)+\varepsilon)x_0(t_k) + Cl + \sum_{k=1}^{l} I_k(r\sigma) \\
&+(f^\infty+\varepsilon)\int_0^{2\pi} x_0(t)dt + C2\pi + \int_0^{2\pi} f(t, r\sigma)dt
\end{aligned}
$$

Consequently, we obtain that

$$
\begin{aligned}
(M - f^\infty - \varepsilon)\int_0^{2\pi} x_0(t)dt &\le \sum_{k=1}^{l} (I^\infty(k)+\varepsilon)x_0(t_k) + \int_0^{2\pi} f(t, r\sigma)dt \\
&+C(l+2\pi) + \sum_{k=1}^{l} I_k(r\sigma) \\
&\le \|x_0\| \sum_{k=1}^{l} (I^\infty(k)+\varepsilon) + \int_0^{2\pi} f(t, r\sigma)dt \\
&+C(l+2\pi) + \sum_{k=1}^{m} I_k(r\sigma)
\end{aligned}
$$

We also have

$$\int_0^{2\pi} x_0(t)dt \ge \sigma \| x_0 \| 2\pi$$

Thus

$$\| x_0 \| \le \frac{\int_0^{2\pi} f(t, r\sigma)dt + C(l+2\pi) + \sum_{k=1}^{l} I_k(r\sigma)}{\sigma 2\pi(M - f^\infty - \varepsilon) - \sum_{k=1}^{l} (I^\infty(k)+\varepsilon)} =: \overline{R}$$

Let $R > max\{\overline{R}, H\}$, then for any $x \in K \bigcap \partial\Omega_R$ and $0 \le \mu < 1$, we have $\mu Ax \ne x$. Hence all the assumptions of Lemma 3.1 are satisfied, A has a fixed point x in $K \bigcap \{x \in E : r \le \| x \| \le R\}$, $x(t) \ge \sigma r \quad \forall t \in [0, 2\pi]$. Let $\theta^* := r$, this complete the proof of Theorem 1.

Competing Interests

Author has declared that no competing interests exist.

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