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On Natural Characterizations of Tensor Integrability

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Abstract

We revisit the definition of the tensor integrability introduced in [1], and prove some useful characterizations that provide a foundation for studying relevant properties to the integration theory. As applications, we obtain some representation theorems for the projective and injective tensor products of the space of scalar integrable functions with Banach spaces.

Keywords: Vector integral; tensor integral; tensor product.

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1 Introduction

Several authors have attempted to establish a theory of integration of vector-valued functions with respect to vector-valued measures since the early days of Banach spaces (see [2]). The most known method is perhaps that of [3], which was subsequently generalized by Dobrakov (see [4]). Contributions on this subject can be seen in [5],[6],[7],[8],[9], and references therein. Recently, a more comprehensive approach was introduced in [1] using tensor product in tandem with the Moore-Smith limit. Such an approach strengthens the various existing classical concepts of integral and provides a continuous thread tying the subject matter together. Such a technique offers two major advantages. Firstly, the approach does not require any measurability condition. Secondly, this

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approach is completely independent of the topological or the algebraic structure of the domain space.

The aim of this note is to try to make the reader familiar with the basic ideas of the above mentioned approach by revisiting the definition and deriving some natural, systematic and useful characterizations of the notion of tensor integrability. Our results provide a foundation for studying relevant properties to the integration theory. By way of example, we shall prove that the completed projective tensor product $I^1(\Omega)\hat{\otimes}_\pi X$ is naturally represented by the space $I^1(\Omega, X, \mu)$ of (classes of) X-valued μ-norm-integrable functions, where X is a Banach space and $\mu : \Sigma \to [0, \infty]$ is a subadditive set function. Likewise, we shall also show that the completed injective tensor product of the form $I^1[\Omega]\hat{\otimes}_\epsilon X$ can be naturally identified to the space (all classes of) X-valued μ-integrable functions when normed with an appropriate norm.

2 Extended Definition of the Integral

Throughout this paper U,V and W will denote normed spaces over the same scalar field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). By a tensor, we mean a continuous bilinear mapping $\tau : U \times V \to W$ that satisfies

$$\|u\|_U = \sup\left\{\|\tau(u, v)\|_W : \|v\|_v \le 1\right\}$$

for every $u \in U$.

Let Ω be a nonempty set, andΣ a ring of subsets of Ω. A set function $\mu : \Sigma \to V$ is said to be **subadditive** if μ satisfies

1. $\mu(\emptyset) = 0$;
2. $\left\|\mu(\bigcup_{i=1}^n A_i)\right\|_V \le \sum_{i=1}^n \|\mu(A_i)\|_V$ for every finite family $\{A_i : i = 1, ..., n\}$ in Σ.

Clearly, any finitely additive set function is subadditive.

By a Σ**-subpartition** of a set $A \in 2^\Omega$, we mean any finite pairwise disjoint collection $P = \{I_i : I_i \subset A, i = 1, 2 \ldots, n\} \subset \Sigma$ satisfying $\|\mu(I_i)\|_V < \infty$. We denote by $\bigsqcup P$ the subset of A obtained by taking the union of all elements of P. A Σ-subpartition $P = \{I_i : i = 1, \ldots, n\}$ is said to be *tagged* if a point $t_i \in I_i$ is chosen for each $i \in \{1, \ldots, n\}$. We denote by $\Pi(A, \Sigma)$ the collection of all tagged Σ-subpartitions of the set A. The *mesh* or the *norm* of $P \in \Pi(A, \Sigma)$ is defined to be $\|P\| = \max\{\|\mu(I_i)\|_V : I_i \in P\}$.

If $P, Q \in \Pi(A, \Sigma)$, we say that Q is a **refinement** of P and we write $Q \succ P$ if $\|Q\| \le \|P\|$ and $\bigsqcup P \subset \bigsqcup Q$. It is readily seen that such a relation does not depend on the tagging points. It is also easy to see that the relation $\succ$ is transitive on $\Pi(A, \Sigma)$. If $P, Q \in \Pi(A, \Sigma)$, we denote

$$P \vee Q := \{I \setminus J, I \cap J, J \setminus I : \ I \in P, \ J \in Q\}.$$

Clearly, $P \vee Q \in \Pi(A, \Sigma)$, $P \vee Q \succ P$ and $P \vee Q \succ Q$. Thus the relation $\succ$ has the upper bound property on $\Pi(A, \Sigma)$. We then infer that the set $\Pi(A, \Sigma)$ is directed (in the sense of Moore-Smith as described in [10], by the binary relation $\succ$.

Given a function $f : A \subset \Omega \to U$, and a tagged Σ-subpartition $P = \{(I_i, t_i) : \ i \in \{1, \ldots, n\}\} \in \Pi(A, \Sigma)$, we define the **Riemann-tensor-sum** of f at P with respect to an additive measure $\mu : \Sigma \to V$ to be the element of W given by

$$f_{\mu,\tau}(P) = \sum_{i=1}^n \tau(f(t_i), \mu(I_i)).$$

Thus the function $P \mapsto f_{\mu,\tau}(P)$ is a W-valued net defined on the directed set $(\Pi(A, \Sigma), \succ)$. For convenience, we are going to denote the net-limit by

$$\int_A \tau(f, d\mu) := \lim_{(\Pi(A,\Sigma),\succ)} f_{\mu,\tau}(\cdot)$$

whether or not such a limit exists.

The following definition slightly extends the definition of the notion of tensor integrability introduced in [1], in the sense that here the additivity of the vector valued set function $\mu : \Sigma \to V$ is replaced by subadditivity.

Definition 2.1. Let U,V and W be normed spaces, and let $\tau : U \times V \to W$ be a tensor. Let Ω be a nonempty set and $\Sigma \subset 2^{\Omega}$ and let $\mu : \Sigma \to V$ be a subadditive set function. We say that a function $f : \Omega \to U$ is **Σ, μ, τ-integrable over a set A with respect to μ** if the limit $\int_A \tau(f, d\mu)$ represents a vector in W. The vector $\int_A \tau(f, d\mu)$ is then called the **Σ, μ, τ-integral** of f relative to μ over the set A.

In other words, $f : \Omega \to U$ is Σ, μ, τ-integrable over the set A with Σ, μ, τ-integral $\int_A \tau(f, d\mu)$ if for every $\epsilon > 0$, there exists a Σ-subpartition P_0 of the set A such that for every $P \succ P_0$ in $\Pi(A, \Sigma)$ we have

$$\left\| \int_A \tau(f, d\mu) - f_{\mu,\tau}(P) \right\|_W < \epsilon. \tag{2.1}$$

We shall denote by $\mathcal{I}^{\tau}(A, \Sigma, \mu, U, V, W)$ the (patently) linear space of all functions $f : \Omega \to U$ that are Σ, μ, τ-integrable over a given subset A of Ω. It is easily checked that the mapping

$$f \mapsto \sup \{\|f_{\mu}(P)\| : P \in \Pi(A, \Sigma)\}$$

defines a seminorm on $\mathcal{I}^{\tau}(A, \Sigma, \mu, U, V, W)$.

It should be noticed that if the set A is such that $\mu(A) = 0$, then for all subpartitions $P \in \Pi(A)$, $f_{\mu,\tau}(P) = 0$, and thus $\int_A \tau(f, d\mu) = 0$. It follows that

$$\int_A \tau(f, d\mu) = \int_A \tau(g, d\mu) \text{ whenever } \mu\{x \in A : f(x) \neq g(x)\} = 0.$$

We write $f \overset{\mu}{\sim} g$, if $\mu\{x \in A : f(x) \neq g(x)\} = 0$. It is readily seen that the relation $f \overset{\mu}{\sim} g$ is an equivalence relation on $\mathcal{I}^{\tau}(A, \Sigma, \mu, U, V, W)$. We shall then denote by $I^{\tau}(A, \Sigma, \mu, U, V, W)$ the quotient space $\mathcal{I}^{\tau}(A, \Sigma, \mu, U, V, W)/ \overset{\mu}{\sim}$. For $1 \le p < \infty$, we shall denote by $I^{\tau,p}(A, \Sigma, \mu, U, V, W)$ the subspace of $I^{\tau}(A, \Sigma, \mu, U, V, W)$ consisting of functions f such that the function $s \mapsto \|f(s)\|_V^p$ is μ-integrable. The space $I^{\tau,p}(A, \Sigma, \mu, U, V, W)$ shall be normed by $f \mapsto \|f\|_p = (\int_{\Omega} \|g\|_V^p \, d\mu)^{\frac{1}{p}}$. The space $I^{\tau,\infty}(A, \Sigma, \mu, U, V, W)$ is defined to be

$$I^{\tau,\infty}(A, \Sigma, \mu, U, V, W) = \left\{f \in \mathfrak{F}(\Omega, V) : \mu\text{-esssup}\|f\|_V < \infty\right\}$$

where $\mathfrak{F}(\Omega, V)$ denotes the set of all V-valued function defined on Ω.

In all of the above notations, to save notations, once understood, we may drop Σ if there is no risk of confusion about the ring of subspace.

Example 2.1. *Recall that $\mu : \Sigma \to \mathbb{R}$ is a size function if $\mu(A) \ge 0$, $\mu(\emptyset) = 0$, $\mu(A) \le \mu(B)$ whenever $A \subset B$, and $\mu(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu(A_i)$ for every countable family $\{A_i : i \in \mathbb{N}\}$ in Σ. Thus if $V = \mathbb{K}$ is the scalar field, consider $\tau : U \times \mathbb{K} \to U$ is the* 'reverse scaling tensor' *defined by $\tau(u, \alpha) = \alpha u$. Then the space $\mathcal{I}^{\tau}(A, \mu, U, \mathbb{K}, U)$ corresponds exactly to the space of vector valued integrable functions in the sense of the definition introduced in [11] (see also [12]). For $p \in [1, \infty]$, we shall denote*

$$I^{\tau,p}(A, \mu, U, \mathbb{K}, U) = I^p(A, \mu, U)$$

and if U is the scalar field, we further simply write: $I^p(A, \mu, V) = I^p(A, \mu)$

Particularly, if μ is the Lebesgue measure defined on the Borel σ-algebra $\mathcal{B}$ of a nonempty set Ω, then $I^p(A, \mu, U)$ contains the space of all Lebesgue-Bochner U-valued p-integrable functions over the set A.

Example 2.2. *If $U = \mathbb{K}$ is the scalar field, and $\tau : \mathbb{K} \times V \to V$ is the scaling tensor defined by $\tau(\alpha, v) = \alpha v$, then the space $\mathcal{I}^\tau(A, \mu, \mathbb{K}, V, V)$ is the space of all scalar valued functions, integrable with respect to a vector valued subadditive set function $\mu : \Sigma \to V$ and shall be simply denoted by $\mathcal{I}(A, V)$.*

We now prove some results that follow immediately from the definition. First we say that two subpartitions P and Q are **disjoint from each other** if no element in P interesect an element in Q.

Proposition 2.1. *If $f \in \mathcal{I}^\tau(A, \mu, U, V, W)$ then for every $\varepsilon > 0$, there exists $P_0 \in \Pi(A)$ such that $\|f_{\mu,\tau}(Q)\|_W < \varepsilon$ for every $Q \in \Pi(A)$ that is disjoint from P_0 and such that $\|Q\| \leq \|P_0\|$.*

Proof. Fix $\varepsilon > 0$. Let $P_1 \in \Pi(A)$ be such that for every $P \succ P_1$ in $\Pi(A)$ we have

$$\left\| \int_A \tau(f, d\mu) - f_{\mu,\tau}(P) \right\|_W < \varepsilon/2.$$

Fix $P_0 \succ P_1$. Then for every $Q \in \Pi(A)$ that is disjoint from intersect P_0, and such that $\|Q\| \leq \|P_0\|$, we have $P_0 \vee Q \succ P_1$, and therefore

$$\left\| \int_A \tau(f, d\mu) - f_{\mu,\tau}(P_0 \vee Q) \right\|_W < \varepsilon/2.$$

It follows that

$$\begin{aligned} \|f_{\mu,\tau}(Q)\|_W &= \|f_{\mu,\tau}(P_0 \vee Q) - f_{\mu,\tau}(P)\|_W \\ &\leq \left\| f_{\mu,\tau}(P_0 \vee Q) - \int_A \tau(f, d\mu) \right\|_W \\ &\quad + \left\| f_{\mu,\tau}(P_0) - \int_A \tau(f, d\mu) \right\|_W < \varepsilon. \end{aligned}$$

The proof is complete. □

The above proposition suggests the following definition.

Definition 2.2. Let U be a normed vector space. A function $f : \Omega \to U$ is said to satisfy the **Cauchy criterion for integrability** on A if for every $\varepsilon > 0$, there exists $P_0 \in \Pi(A)$ such that $\|f_{\mu,\tau}(P)\|_W < \varepsilon$ for every $P \in \Pi(A)$ disjoint from P_0.

Thus Proposition 2.1 states that every tensor integrable function satisfies the Cauchy criterion for integrability. Converely, we notice that if $P, Q \in \Pi(A)$ is such that no set in P intersects a set in Q, then

$$\|f_{\mu,\tau}(P \vee Q) - f_{\mu,\tau}(P)\|_W = \|f_{\mu,\tau}(Q)\|_W$$

It is then quickly seen that the Cauchy criterion for integrability of a function f is equivalent to the Cauchy condition for the net $P \mapsto f_{\mu,\tau}(P)$. It is a well known fact that Cauchy nets taking values in a Banach space is convergent (see [10]). Hence we have the following characterization theorem.

Theorem 2.3. *Let U, V, W be Banach spaces and let $\mu : \Sigma \to V$ be subadditive. Then $f \in \mathcal{I}^\tau(A, \mu, U, V, W)$ if and only if it satisfies the Cauchy criterion for integrability on A.*

We end this section with the following lemma that will also be used for later result.

Lemma 2.4. *Let $\mu : \Sigma \subset 2^{\Omega} \to [0,\infty]$ a subadditive set function be such that $\mu(\Omega) < \infty$. Let $f \in I(\Omega,\Sigma,\mu,X)$ and let E be a closed subset of X such that for every $A \in \Sigma$ with $\mu(A) > 0$,*

$$\frac{1}{\mu(A)}\int_A f d\mu \in E.$$

Then $f(\omega) \in E$ for μ-a.e. $\omega \in \Omega$.

Proof. We first notice that since μ can be extended to a subadditive set function $\tilde{\mu}$ defined on the whole power set 2^{Ω} by setting

$$\tilde{\mu}(A) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(I_n) : \ A \subset \bigcup_{n\in\mathbb{N}} I_n, \ I_n \in \Sigma\right\}$$

for all $A \in 2^{\Omega}$, no generality is lost in assuming that μ is defined on 2^{Ω}.

We want to show that $\mu\left(\{\omega \in \Omega : f(\omega) \notin E\}\right) = 0$. To see this, we write the open set E^c as countable union of open balls $\bigcup_{n=1}^{\infty} B(x_n, r_n)$. Thus

$$\{\omega \in \Omega : f(\omega) \notin E\} = \left\{\omega \in \Omega : f(\omega) \in \bigcup_{n=1}^{\infty} B(x_n, r_n)\right\} = \bigcup_{n=1}^{\infty} f^{-1}(B(x_n, r_n)).$$

Let $A_n = f^{-1}(B(x_n, r_n)$.Assume that for every n, $\mu(A_n) > 0$. Then

$$\begin{aligned} \left\| \frac{1}{\mu(A_n)} \int_{A_n} f d\mu - x_n \right\| &= \left\| \frac{1}{\mu(A_n)} \int_{A_n} (f - x_n) d\mu \right\| \\ &\leq \frac{1}{\mu(A_n)} \int_{A_n} \|f - x_n\| \, d\mu \leq r_n. \end{aligned}$$

Thus $\frac{1}{\mu(A_n)}\int_{A_n} f d\mu \in B(x_n, r_n) \subset E^c$. This contradiction proves the lemma. □

3 Characterizations of Tensor Integrability

Building on the result obtained in Theorem 2.3, we prove in this section further natural characterizations of integrability of Banach space valued functions.

In what follows, U, V, W are Banach spaces, $\tau : U \times V \to W$ is a tensor, and $\mu : \Sigma \to V$ is a subadditive set function where Σ is a ring of subsets of a nonempty set Ω.

Theorem 3.1. *Let U,V and W be Banach spaces, let $\tau : U \times V \to W$ be a tensor, and let $f : \Omega \to U$. The following conditions are equivalent:*

1. *$f \in \mathcal{I}^{\tau}(A, \mu, U, V, W)$.*
2. *For any injection $\varphi : \Gamma \to \Omega$, the function $\gamma \mapsto f(\varphi(\gamma))$ is in $\mathcal{I}^{\tau}(\varphi^{-1}(A), \varphi^{-1}(\Sigma), \mu, U, V, W)$.*
3. *For every $\epsilon : \Omega \to \{-1, 1\}$, the function $\omega \mapsto \epsilon(\omega) f(\omega)$ is in $\mathcal{I}^{\tau}(A, \mu, U, V, W)$.*
4. *For every μ-essentially bounded function $\phi : \Omega \to \mathbb{K}$, the function $\omega \mapsto \phi(\omega) f(\omega)$ is in $\mathcal{I}^{\tau}(A, \mu, U, V, W)$.*

Proof. It is clear that $4. \Rightarrow 3. \Rightarrow 1.$ and also $2. \Rightarrow 1.$. To see that $1. \Rightarrow 2.$, suppose that f is in $\mathcal{I}^{\tau}(A, \mu, U, V, W)$ and let $\varepsilon > 0$. By Theorem 2.3, there exists $P_0 \in \Pi(A)$ such that $\|f_{\mu,\tau}(P)\| < \varepsilon$ whenever $P \in \Pi(A, \Sigma)$ is disjoint from P_0. Let $\varphi : \Gamma \to \Omega$ be an injective mapping. Let $Q_0 = \varphi^{-1}(P_0)$. By injectivity of φ, if $Q \cap Q_0 = \emptyset$, then $\varphi(Q) \cap P_0 = \emptyset$. It follows that whenever $Q = \{(t_i, I_i) : i = 1, ..., n\} \in \Pi(\varphi^{-1}(A))$ is disjoint from Q_0, we have

$$\begin{aligned} \|(f \circ \varphi)_{\mu,\tau}(Q))\|_W &= \left\| \sum_{i=1}^{n} \tau(f(\varphi(t_i)), \mu(\varphi(I_i)) \right\|_W \\ &= \|f_{\mu,\tau}(\varphi(Q))\|_W < \varepsilon \end{aligned}$$

Hence, the function $\gamma \mapsto f(\varphi(\gamma))$ is in $\mathcal{I}^{\tau}(\varphi^{-1}(A), \varphi^{-1}(\Sigma), \mu, U, V, W)$. We have established that $1. \Rightarrow 2.$.

To show that $2. \Rightarrow 3.$, let $\Gamma_1 = \epsilon^{-1}(1)$ and $\Gamma_{-1} = \epsilon^{-1}(-1)$. Then $\Omega = \Gamma_1 \cup \Gamma_{-1}$ and $\Gamma_1 \cap \Gamma_{-1} = \emptyset$. Let $\varphi_1 : \Gamma_1 \to \Omega$ and $\varphi_{-1} : \Gamma_{-1} \to \Omega$ be respectively, the canonical injection respectively of Γ_1 and Γ_{-1} into Ω. Then by $2.$

$$\gamma \in \Gamma_1 \mapsto f(\varphi_1(\gamma)) = f(\gamma)$$
$$\gamma \in \Gamma_{-1} \mapsto f(\varphi_{-1}(\gamma)) = f(\gamma)$$

are respectively in $\mathcal{I}^{\tau}(\Gamma_1 \cap A, \Sigma_{\Gamma_1}, \mu, U, V, W)$ and $\mathcal{I}^{\tau}(\Gamma_2 \cap A, \Sigma_{\Gamma_2}, \mu, U, V, W)$, where

$$\Sigma_{\Gamma_i} = \{E \cap \Gamma_i : E \in \Sigma\}.$$

for $i = 1, 2$. It follows that the function

$$\gamma \mapsto \epsilon(\gamma) f(\gamma) = 1_{\Gamma_1}(\gamma) f(\gamma) + 1_{\Gamma_{-1}}(\gamma) f(\gamma)$$

is in $\mathcal{I}^{\tau}(A, \mu, U, V, W)$.

$3. \Rightarrow 4.$ We give the proof for real case. The changes for complex spaces are straightforward. Assume that W is a real vector space. Fix $P = \{(t_i, I_i) : i = 1, ..., n\} \in \Pi(A)$. Pick an $x^* \in W^*$ so that

$$\sum_{i=1}^{n} \phi(t_i) x^* \tau(f(t_i), \mu(I_i)) = \left\| \sum_{i=1}^{n} \phi(t_i) \tau(f(t_i), \mu(I_i)) \right\|.$$

Let $\epsilon : \Omega \to \{-1, 1\}$ be defined by

$$\epsilon(\omega) = \begin{cases} 1 & \text{if } x^* \tau(f(t_i), \mu(I_i)) \geq 0 \\ -1 & \text{if } x^* \tau(f(t_i), \mu(I_i)) < 0. \end{cases}$$

Then

$$\begin{aligned} \left\| \sum_{i=1}^{n} \phi(t_i) \tau(f(t_i), \mu(I_i)) \right\| &\leq \sum_{i=1}^{n} |\phi(t_i)| \, |x^* \tau(f(t_i), \mu(I_i))| \\ &\leq \|\phi\|_{\infty} \sum_{i=1}^{n} \epsilon(t_i) x^* \tau(f(t_i), \mu(I_i)) \\ &\leq \|\phi\|_{\infty} x^* \left(\sum_{i=1}^{n} \epsilon(t_i) \tau(f(t_i), \mu(I_i)) \right) \\ &\leq \|\phi\|_{\infty} \left\| \sum_{\omega \in P} \epsilon(\omega) \tau(f(t_i), \mu(I_i)) \right\|. \end{aligned}$$

The desired result follows. The proof is complete. $\square$

Remark 3.1. Property 4. of the characterization theorem 3.1 suggests an alternative way to norming the space of tensor integrable functions.

Indeed, the mapping

$$f \mapsto |||f||| := \sup \left\{ \left\| \int_A \tau(\phi f, d\mu) \right\|_W \right\}$$

where the supremum is taken over the set of all μ-essentially bounded functions $\phi : \Omega \to \mathbb{K}$ such that $\|\phi\|_\infty \leq 1$, is quickly seen to define a seminorm on the space $\mathcal{I}^\tau(A, \mu, U, V, W)$ of τ-integrable function.

We shall use $I[A, \mu, U, V, W]$ to denote the completion of the space $I^\tau(A, \mu, U, V, W)$ when it is normed with $f \mapsto |||f|||$.

4 The Projective Tensor Product $I^1(\Omega, \mu)\hat{\otimes}_\pi X$

In this section, we shall restrict ourselves to the tensor integral generated by the reverse scaling tensor as in Example 2.2. We shall focus mainly on the space of μ-norm integrable X-valued functions $I^1(\Omega, \mu, X)$, where X is a Banach space.

Recall that the projective norm for an element u of a tensor product $X \otimes Y$ is defined by

$$\pi(u) = \inf \left\{ \sum_{i=1}^n \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

The tensor product of $I^1(\Omega, \mu)$ with X gives a representation of the space of all (classes of)μ-norm integrable functions.

Theorem 4.1. *Let X be a Banach space and $\mu : 2^\Omega \to \mathbb{R}$ a nonnegative subsadditive set function. Then the completed projective tensor product $I^1(\Omega, \mu)\hat{\otimes}_\pi X$ is isometrically isomorphic to the space $I^1(\Omega, \mu, X)$.*

Proof. We note that every element in $I^1(\Omega, \mu) \otimes X$ can be viewed as X-valued function: for each $\varphi \in I^1(\Omega, \mu)$ and $x \in X$, the elementary tensor $\varphi \otimes x$ corresponds to the function $\omega \mapsto \varphi(\omega)x$. Since $\varphi \in I^1(\Omega, \mu)$, for every $\epsilon > 0$, there exists $P_0 \in \Pi(\Omega)$ such that if $P = \{(t_i, I_i) : i = 1, ..., n\} \in \Pi(\Omega)$ does not intersect P_0, then $\sum_{i=1}^n |\varphi(t_i)| \mu(I_i) < \epsilon / \|x\|$. It follows that for such P

$$\sum_{i=1}^n \|\varphi(t_i)\mu(I_i)x\| \leq \|x\| \sum_{i=1}^n |\varphi(t_i)| \mu(I_i) < \epsilon.$$

This shows that the function $\omega \mapsto \|\varphi(\omega)x\|$ satisfies the Cauchy criterion for integrability and therefore the function $\omega \mapsto \varphi(\omega)x$ belongs to the space $I^1(\Omega, \mu, X)$. It follows that there exists a linear mapping $J : I^1(\Omega, \mu) \otimes_\pi X \to I^1(\Omega, \mu, X)$ satisfying $J(\varphi \otimes x) = \varphi(\cdot)x$.

Now if $\sum_{j=1}^{m} \varphi_j \otimes x_j$ is a representation of $u \in I^1(\Omega,\mu) \otimes X$, then

$$\begin{aligned}\left\|\sum_{j=1}^{m} \varphi_j(\cdot)x_j\right\|_1 &= \int_\Omega \left\|\sum_{j=1}^{m} \varphi_j(\cdot)x_j\right\|_X d\mu \\ &\le \int_\Omega \sum_{j=1}^{m} \|\varphi_j(\cdot)x_j\|_X \, d\mu \\ &= \sum_{j=1}^{m} \int_\Omega |\varphi_j| \, d\mu \, \|x_j\|_X \\ &= \sum_{j=1}^{m} \|\varphi_j\|_1 \|x_j\|_X \, .\end{aligned}$$

It follows that $\|Ju\|_1 \le \pi(u)$.

To establish the reverse inequality, let $\sum_{j=1}^{m} \varphi_j \otimes x_j$ be a representation of $u \in I^1(\Omega,\mu) \otimes X$. Then $J(u) = \phi$ where $\phi(\omega) = \sum_{j=1}^{m} \varphi_j(\omega)x_j$. Let Π denote the collection of subsets A of Ω such that $0 < \mu(A) < \infty$. For every $A \in \Pi$, consider the function $e_A := \frac{1}{\mu(A)} 1_A$, where 1_A is the characteristic function of the set A. Then clearly, $e_A \in I^1(\Omega,\mu)$. We claim that for every $\Gamma \in 2^\Pi$, the net $\Gamma \mapsto \sum_{A\in\Gamma} e_A \otimes \int_A \phi d\mu$ directed by refinement converges to u in $I^1(\Omega,\mu) \otimes_\pi X$. We have the following series of equalities

$$\begin{aligned}\pi\left(u - \sum_{A\in\Gamma} e_A \otimes \int_A \phi d\mu\right) &= \pi\left(\sum_{j=1}^{m} \varphi_j \otimes x_j - \sum_{A\in\Gamma} e_A \otimes \int_A \sum_{j=1}^{m} \varphi_j(\cdot)x_j d\mu\right) \\ &= \pi\left(\sum_{j=1}^{m}\left(\varphi_j \otimes x_j - \sum_{A\in\Gamma} e_A \otimes \int_A \varphi_j(\cdot)x_j d\mu\right)\right) \\ &= \pi\left(\sum_{j=1}^{m}\left(\varphi_j \otimes x_j - \sum_{A\in\Gamma} e_A \int_A \varphi_j(\cdot)d\mu \otimes x_j\right)\right) \\ &= \pi\left(\sum_{j=1}^{m}\left(\varphi_j - \sum_{A\in\Gamma} e_A \int_A \varphi_j(\cdot)d\mu\right) \otimes x_j\right) \\ &\le \sum_{j=1}^{m}\left\|\varphi_j - \sum_{A\in\Gamma} e_A \int_A \varphi_j(\cdot)d\mu\right\|_1 \|x_j\| \, .\end{aligned}$$

Our claims follows.

We then have

$$\begin{aligned}\pi(u) &= \pi(\lim_{\Gamma\in 2^{|\Pi|}} \sum_{j=1}^{m} \phi_\Gamma(\varphi_j) \otimes x_j) \le \lim_{\Gamma\in 2^{|\Pi|}} \left\|\sum_{j=1}^{m} \phi_\Gamma(\varphi_j)x_j\right\| \\ &= \left\|\sum_{j=1}^{m} \varphi_j(\cdot)x_j\right\|_1 = \|Ju\|_1 \, .\end{aligned}$$

This shows that the linear mapping $J : I^1(\Omega,\mu) \otimes_\pi X \to I^1(\Omega,\mu,X)$ is an isometry. Since X is a Banach space, $I^1(\Omega,\mu,X)$ is complete and thus J extends to an isometry from the completed projective tensor product $I^1(\Omega,\mu)\hat{\otimes}_\pi X$ into $I^1(\Omega,\mu,X)$.

Finally, we claim that such an isometry is surjective. To see this, let $f \in I^1(\Omega, \mu, X)$. We notice that

$$\pi\left(\sum_{A\in\Gamma} e_A \otimes \int_A f d\mu\right) \leq \sum_{A\in\Gamma} \int_A \|f\| \, d\mu$$

It follows that the net $\Gamma \mapsto \sum_{A\in\Gamma} e_A \otimes \int_A f d\mu$ is Cauchy, and its limit which obviously belongs to $I^1(\Omega, \mu)\hat{\otimes}_\pi X$, is mapped by J to the function f in $I^1(\Omega, \mu, X)$. The proof is complete. □

Note that for the particular case where μ is the Lebesgue measure on the Borel σ-algebra of subsets of the set Ω, the result of the above theorem coincides with the known fact that the space of Bochner integrable functions $L^1(\Omega, \Sigma, X)$ can be identified as the projective tensor product of $L^1(\Omega)$ with the Banach space X. Also the particular case where $\Omega = \mathbb{N}$, and where μ is the counting measure on $\mathbb{N}$, the result of the above theorem reduces to the special and well-known representation $\ell^1\hat{\otimes}_\pi X = \ell^1(X)$ (see [13] for details).

It is a well known fact that projective tensor products do not, in general, respect subspaces. However, we notice that if Y is a closed subspace of a Banach space X, then it is clear that $I^1(\Omega, \mu, Y)$ is a subspace of $I^1(\Omega, \mu, X)$. Consequently, we have:

Proposition 4.1. *Let X be a Banach space and Y a closed subspace of X. Then $I^1(\Omega, \mu)\hat{\otimes}_\pi Y$ is a subspace of $I^1(\Omega, \mu)\hat{\otimes}_\pi X$.*

5 The Injective Tensor Product $I[\Omega, \mu]\hat{\otimes}_\epsilon X$

Again in this section, the integral is generated by the reverse scaling tensor (Example 2.2). In view of Remark 3.1, we shall consider the norm

$$f \mapsto |||f||| := \sup\left\{\left\|\int_A \phi f d\mu\right\|_X : \|\phi\|_\infty \leq 1\right\}$$

and the space $I[\Omega, \mu, X]$ which is the $|||\cdot|||$-completion of the space $I(\Omega, \mu, X)$.

Recall that the injective norm for an element u of a tensor product $X \otimes Y$ of two Banach spaces X and Y can be given by

$$\epsilon(u) = \inf\left\{\left\|\sum_{i=1}^n x^*(x_i)y_i\right\|_Y : \; x^* \in X^*, \|x^*\| \leq 1\right\}$$

where $\sum_{i=1}^n x_i \otimes y_i$ is a representation of u in $X \otimes Y$ and where X^* is the Banach dual of the normed vector space X.

We begin with the following simple but invaluable observation that integrals in the setting of $I[\Omega, \mu, X]$, interact well with bounded linear operators.

Proposition 5.1. *Let $f \in I[\Omega, \mu, X]$, and let $T : X \to Y$ be a bounded linear operator. Then $Tf \in I[\Omega, \mu, Y]$ and*

$$\int_\Omega T f d\mu = T \int_\Omega f d\mu.$$

Furthermore, $|||Tf||| \leq \|T\| \, |||f|||$.

Proof. Let $y^* \in Y^*$. Then

$$\begin{aligned} y^* \int_\Omega T f d\mu &= y^* \lim_{(\Pi(\Omega,\Sigma),\succ)} \sum \mu(I_i) T f(t_i) \\ &= \lim_{(\Pi(\Omega,\Sigma),\succ)} \sum \mu(I_i) \langle T f(t_i), y^* \rangle \\ &= \lim_{(\Pi(\Omega,\Sigma),\succ)} \sum \mu(I_i) \langle f(t_i), T^* y^* \rangle \\ &= \lim_{(\Pi(\Omega,\Sigma),\succ)} \left\langle \sum \mu(I_i) f(t_i), T^* y^* \right\rangle \\ &= \lim_{(\Pi(\Omega,\Sigma),\succ)} \left\langle T(\sum \mu(I_i) f(t_i)), y^* \right\rangle = y^* T \int_\Omega f d\mu. \end{aligned}$$

Hence we have $\int_\Omega T f d\mu = T \int_\Omega f d\mu$, and it follows that $|||Tf||| \leq \|T\| \, |||f|||$. ☐

Our next result is a representation theorem that plays a role in the proof of our last result, although it can also be seen as of independent interest.

Proposition 5.2. *The dual of $I[\Omega, \mu]$ is naturally identified to $I^\infty(\Omega, \mu)$.*

Proof. It quickly follows from Property 4. of the characterization Theorem 3.1 that for every $\phi \in I^\infty(\Omega, \mu)$, the mapping $\Lambda_\phi : f \mapsto \int_A \phi f d\mu$ defines a bounded linear functional on $I[\Omega, \mu]$ with $\|\Lambda_\phi\| \leq \|\phi\|_\infty$.

Conversely, if $T : I[\Omega, \mu] \to \mathbb{K}$ is a bounded linear functional, then the mapping $A \mapsto T(1_A)$ is quickly seen to be an additive set function that is absolutely continuous with respect to μ. It then follows form the extended Lebesgue-Nikodym theorem (see [12, Theorem 10]), that there exists $\phi \in I(\Omega, \mu)$ such that

$$T(1_A) = \int_\Omega \phi 1_A d\mu.$$

Thus for every $A \in \Sigma$, we have

$$\left| \int_\Omega \phi 1_A d\mu \right| = |T(1_A)| \leq \|T\| \, \mu(A).$$

It follows from Lemma 2.4 that $|\phi(x)| \leq \|T\|$ for μ-almost every x, that is, $\phi \in I^\infty(\Omega, \mu)$ and $\|\phi\|_\infty \leq \|T\|$.

Now if $f \in I[\Omega, \mu]$, then for every $P = \{(t_i, I_i) : i = 1, ..., n\} \in \Pi(\Omega, \Sigma)$, the function

$$f_P = \sum_{i=1}^n 1_{I_i} f(t_i) \in I[\Omega, \mu]$$

and $f_P \to f$ in $I[\Omega, \mu]$. It follows that

$$\begin{aligned} Tf &= \lim_{(\Pi(A,\Sigma),\succ)} T \sum f(t_i) 1_{I_i} = \lim_{(\Pi(A,\Sigma),\succ)} \sum \int_\Omega \phi f(t_i) 1_{I_i} d\mu \\ &= \lim_{(\Pi(A,\Sigma),\succ)} \sum \phi(t_i) f(t_i) \mu(I_i) = \int_\Omega \phi f d\mu. \end{aligned}$$

Thus $T = \Lambda_\phi$ and $\|T\| = \|\Lambda_\phi\|$. ☐

We now state and prove the representation of the injective tensor product $I[\Omega, \mu] \hat{\otimes}_\epsilon X$ as the space $I[\Omega, \mu, X]$.

Theorem 5.1. *Let X be a Banach space. Then the completed injective tensor product $I[\Omega,\mu]\hat{\otimes}_\epsilon X$ is isometrically isomorphic to the space $I[\Omega,\mu,X]$.*

Proof. Let $J : I[\Omega,\mu]\otimes X \to I[\Omega,\mu,X]$ be the canonical mapping that maps the tensor $u = \sum_{j=1}^m \varphi_j \otimes x_j$ to $Ju(\cdot) = \sum_{j=1}^m \varphi_j(\cdot)x_j$. Bearing in mind the result in the Proposition 5.2, we have

$$\begin{aligned} |||Ju||| &= \sup\left\{\left\|\int_\Omega \phi Ju d\mu\right\| : \|\phi\|_\infty \leq 1\right\} \\ &= \sup\left\{\left\|\int_\Omega \sum_{j=1}^m \phi\varphi_j d\mu x_j\right\| : \|\phi\|_\infty \leq 1\right\} \\ &= \sup\left\{\left\|\sum_{j=1}^m \int_\Omega \phi\varphi_j d\mu x_j\right\| : \|\phi\|_\infty \leq 1\right\} = \epsilon(u). \end{aligned}$$

Thus J is an isometry from $I[\Omega,\mu]\otimes_\epsilon X$ into $I[\Omega,\mu,X]$. Now if $f \in I[\Omega,\mu,X]$, then for every $P = \{(t_i,I_i) : i = 1,...,n\} \in \Pi(\Omega,\Sigma)$, the function $f_P = \sum_{i=1}^n 1_{I_i} f(t_i)$ belongs to $I[\Omega,\mu,X]$ and $f_P = J\left(\sum_{i=1}^n 1_{I_i}\otimes f(t_i)\right)$.

On the other hand, since $f \in I[\Omega,\mu,X]$, for every $\epsilon > 0$, there exists $P_0 \in \Pi(\Omega,\Sigma)$ such that if $P \in \Pi(\Omega,\Sigma)$ is disjoint from P_0, then $\|f_\mu(P)\| = \left\|\int_\Omega f_P\right\| < \epsilon$. It follows that $|||f_P||| < \epsilon$. Thus the net $P \mapsto f_P$ is Cauchy in $I[\Omega,\mu,X]$ which is then clearly seen to converge to f. Thus J extends to an isometry from $I(\Omega,\mu)\hat{\otimes}_\epsilon X$ onto $I[\Omega,\mu,X]$. The proof is complete. ☐

We conclude by noticing that the above theorem generalizes the fact that the injective tensor product $L^1(\Omega)$ with a Banach space X can be identified to the completion with respect to the Pettis norm of the space of all classes of X-valued Pettis-integrable strongly measurable functions (see for example [13]).

6 Conclusion

This paper can be considered as a logical continuation of the work of the author published in [1]. It essentially gives useful characterizations of tensor integrability of vector valued functions with repect to subadditive vector valued set-functions. The author believes that the interest of this paper lies not only in the obtained characterization theorems, but also in the light it shed on the very foundation of the study of integration theory.

Competing Interests

The author declares that no competing interests exist.

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