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# On Generalized Bigollo Numbers

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***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

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## Abstract

It is aimed to describe generalized Bigollo sequences and to examine Bigollo and Bigollo-Lucas sequences in this article. For this purpose, we give Binet's formulas, Simson formulas, generating functions and we introduce a lot of main features of these sequences. Also, we get some formulas and give special matrices for these generalized sequences. Finally, we have determined some close relationships between Bigollo and Bigollo-Lucas numbers and Mersenne, Mersenne-Lucas numbers.

*Keywords: Mersenne numbers; bigollo numbers; bigollo-lucas numbers; mersenne-lucas numbers; tribonacci numbers.*

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# 1 Introduction

The Mersenne number $M_n$ is given by $M_n = 2^n - 1$ and the Mersenne sequence $\{M_n\}_{n\geq 0}$ is defined recurrence relation:

$$M_n = 3M_{n-1} - 2M_{n-2} \tag{1.1}$$

for $M_0 = 0, M_1 = 1$. In addition a Mersenne-Lucas number $H_n$ , is defined by $H_n = 2^n + 1$, extensively also the Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ is given recursively by,

$$H_n = 3H_{n-1} - 2H_{n-2} \tag{1.2}$$

for $H_0 = 2, H_1 = 3$. $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ are the sequences with numbers $A000225$, $A000051$ in the OEIS respectively [1].

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ are given to negative subscripts with defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ....$ So, (1.1) and (1.2) are true for every integer $n$.

It should be noted that Mersenne-Lucas numbers are known as Fermat numbers. Actually, there are two definitions of the Fermat numbers. The number in the form $2^n + 1$ whose first few terms are $2, 3, 5, 9, 17, 33, ...$ is less widely known (OEIS $A000051$ ). However, the more common Fermat numbers are a special case, defined by $F_n = 2^{2^n} + 1$ . Some of them are as follows: $3, 5, 17, 257, 65537, ...$ (OEIS $A000215$ ).

Many authors have worked on the Mersenne sequence and for more detail on this sequence the following resources can be preferred: [2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22].

Now, we define two sequences associated with Mersenne, Mersenne-Lucas numbers. Bigollo and Bigollo-Lucas numbers are defined by

$$B_n = 3B_{n-1} - 2B_{n-2} + 1, \quad \text{for } B_0 = 0, B_1 = 1, \quad n \geq 2,$$

and

$$C_n = 3C_{n-1} - 2C_{n-2}, \quad \text{for } C_0 = 3, C_1 = 4, \quad n \geq 2,$$

respectively. Also, the first few Bigollo and Bigollo-Lucas numbers are

$$0, 1, 4, 11, 26, 57, 120, 247, ...$$

and

$$3, 4, 6, 10, 18, 34, 66, 130, \ldots$$

respectively. A third-order linear recurrence relation for $\{B_n\}$ and $\{C_n\}$ sequences can be given as:

$$\begin{aligned} B_n &= 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \quad B_0 = 0, B_1 = 1, B_2 = 4, \\ C_n &= 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \quad C_0 = 3, C_1 = 4, C_2 = 6. \end{aligned}$$

The identities giving the important interrelationships between Bigollo, Bigollo-Lucas and Mersenne, Mersenne-Lucas numbers can be written as follows:

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1, \end{aligned}$$

and

$$\begin{aligned} B_n &= 4H_{n+1} - 6H_n - n, \\ 2C_n &= 2M_{n+1} - 2M_n + 4. \end{aligned}$$

This article intends to describe the generalization of these sequence of numbers (i.e., Bigollo, Bigollo-Lucas numbers). Now, let's remember the basic structure of generalized Tribonacci numbers.

The generalized Tribonacci sequence

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined by

$$W_0 = a, W_1 = b, W_2 = c \text{ and for } n \geq 3,\ W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} \tag{1.3}$$

where $r, s, t \in \mathbb{R}$ and $a, b, c$ are real or complex numbers (arbitrary).

Many authors have examined different features of this sequence, see for example [23]. $\{W_n\}_{n\geq 0}$ is adapted to negative subscripts by defining them as follows:

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $t \neq 0$ and $n \in \{1, 2, 3, ...\}$. So, (1.3) gets for each integer $n$. The characteristic equation of $\{W_n\}$, which has a third-order recurrence sequence, is as follows:

$$x^3 - rx^2 - sx - t = 0 \tag{1.4}$$

where the roots $\alpha, \beta$ and $\gamma$;

$$\begin{aligned} \alpha &= \frac{r}{3} + A + B, \\ \beta &= \frac{r}{3} + \omega A + \omega^2 B, \\ \gamma &= \frac{r}{3} + \omega^2 A + \omega B, \end{aligned}$$

and

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3},\ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}, \\ \Delta &= \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4},\ \ \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

Binet's formula is given using the recurrence relation and roots in the next theorem for generalized Tribonacci numbers:

**Theorem 1.** *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$ )*

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n \tag{1.5}$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2}, \\ A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)}, \\ A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}. \end{aligned}$$

# 2 Generalized Bigollo Sequence

Throughout this article, we deal with the case of $r = 4, s = -5, t = 2$. A generalized Bigollo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is given by the third-order recurrence relations

$$W_n = 4W_{n-1} - 5W_{n-2} + 2W_{n-3} \tag{2.1}$$

for $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. Using the negative subscripts we write next formula:

$$W_{-n} = \frac{5}{2}W_{-(n-1)} - 2W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n \in \{1, 2, 3, ...\}$. Hence, (2.1) is valid for each integer $n$.

It is obtained using the Binet formula (1.5) for generalized Bigollo numbers (two distinct roots case: $\alpha \neq \beta = \gamma$) by

$$W_n = (A_1 + A_2 n) \times \beta^n + A_3 \times \alpha^n = (A_1 + A_2 n) + A_3 \times 2^n$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\beta W_1 - \alpha(2\beta - \alpha)W_0}{(\beta - \alpha)^2} = -W_2 + 2W_1, \\ A_2 &= \frac{W_2 - (\beta + \alpha)W_1 + \beta\alpha W_0}{\beta(\beta - \alpha)} = -W_2 + 3W_1 - 2W_0, \\ A_3 &= \frac{W_2 - 2\beta W_1 + \beta^2 W_0}{(\beta - \alpha)^2} = W_2 - 2W_1 + W_0, \end{aligned}$$

i.e.

$$W_n = ((-W_2 + 2W_1) + (-W_2 + 3W_1 - 2W_0)n) + (W_2 - 2W_1 + W_0) \times 2^n.$$

Here, $\alpha, \beta$ and $\gamma$ are the roots of the characteristic equation

$$x^3 - 4x^2 + 5x - 2 = (x - 2)(x - 1)(x - 1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= 2, \\ \beta &= 1, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 4, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 5, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

Using the positive and negative subscript, we present the first few generalized Bigollo numbers in next table.

**Table 1. A few generalized Bigollo numbers**

| $n$ | $W_n$ | $W_{-n}$ |
|---|---|---|
| 0 | $W_0$ | $W_0$ |
| 1 | $W_1$ | $\frac{1}{2}(5W_0 - 4W_1 + W_2)$ |
| 2 | $W_2$ | $\frac{1}{4}(17W_0 - 18W_1 + 5W_2)$ |
| 3 | $2W_0 - 5W_1 + 4W_2$ | $\frac{1}{8}(49W_0 - 58W_1 + 17W_2)$ |
| 4 | $8W_0 - 18W_1 + 11W_2$ | $\frac{1}{16}(129W_0 - 162W_1 + 49W_2)$ |
| 5 | $22W_0 - 47W_1 + 26W_2$ | $\frac{1}{32}(321W_0 - 418W_1 + 129W_2)$ |
| 6 | $52W_0 - 108W_1 + 57W_2$ | $\frac{1}{64}(769W_0 - 1026W_1 + 321W_2)$ |
| 7 | $114W_0 - 233W_1 + 120W_2$ | $\frac{1}{128}(1793W_0 - 2434W_1 + 769W_2)$ |
| 8 | $240W_0 - 486W_1 + 247W_2$ | $\frac{1}{256}(4097W_0 - 5634W_1 + 1793W_2)$ |
| 9 | $494W_0 - 995W_1 + 502W_2$ | $\frac{1}{512}(9217W_0 - 12802W_1 + 4097W_2)$ |
| 10 | $1004W_0 - 2016W_1 + 1013W_2$ | $\frac{1}{1024}(20481W_0 - 28674W_1 + 9217W_2)$ |
| 11 | $2026W_0 - 4061W_1 + 2036W_2$ | $\frac{1}{2048}(45057W_0 - 63490W_1 + 20481W_2)$ |
| 12 | $4072W_0 - 8154W_1 + 4083W_2$ | $\frac{1}{4096}(98305W_0 - 139266W_1 + 45057W_2)$ |
| 13 | $8166W_0 - 16343W_1 + 8178W_2$ | $\frac{1}{8192}(212993W_0 - 303106W_1 + 98305W_2)$ |

Now we define two special cases of $\{W_n\}$ .Using the third-order recurrence relations, Bigollo sequence $\{B_n\}_{n\geq 0}$ and Bigollo-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined,

$$B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \quad B_0 = 0, B_1 = 1, B_2 = 4, \tag{2.2}$$

$$C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \quad C_0 = 3, C_1 = 4, C_2 = 6. \tag{2.3}$$

The sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ is extended to negative subscripts by defining

$$B_{-n} = \frac{5}{2}B_{-(n-1)} - 2B_{-(n-2)} + \frac{1}{2}B_{-(n-3)},$$

$$C_{-n} = \frac{5}{2}C_{-(n-1)} - 2C_{-(n-2)} + \frac{1}{2}C_{-(n-3)},$$

for $n \in \{1, 2, 3, ...\}$ . That is, (2.2)-(2.3) are valid for each integer $n$.

$B_n$ and $C_n$ are the sequences A000295 (Eulerian numbers), A052548 in [1].

Now, for positive and negative subscripts we give some values of the Bigollo and Bigollo-Lucas numbers.

**Table 2. The first few values of the special third-order numbers**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_n$ | 0 | 1 | 4 | 11 | 26 | 57 | 120 | 247 | 502 | 1013 | 2036 | 4083 | 8178 | 16369 |
| $B_{-n}$ | | 0 | $\frac{1}{2}$ | $\frac{5}{4}$ | $\frac{17}{8}$ | $\frac{49}{16}$ | $\frac{129}{32}$ | $\frac{321}{64}$ | $\frac{769}{128}$ | $\frac{1793}{256}$ | $\frac{4097}{512}$ | $\frac{9217}{1024}$ | $\frac{20481}{2048}$ | $\frac{45057}{4096}$ |
| $C_n$ | 3 | 4 | 6 | 10 | 18 | 34 | 66 | 130 | 258 | 514 | 1026 | 2050 | 4098 | 8194 |
| $C_{-n}$ | | $\frac{5}{2}$ | $\frac{9}{4}$ | $\frac{17}{8}$ | $\frac{33}{16}$ | $\frac{65}{32}$ | $\frac{129}{64}$ | $\frac{257}{128}$ | $\frac{513}{256}$ | $\frac{1025}{512}$ | $\frac{2049}{1024}$ | $\frac{4097}{2048}$ | $\frac{8193}{4096}$ | $\frac{16\,385}{8192}$ |

For every $n \in \mathbb{Z}$, using Binet's formulas Bigollo and Bigollo-Lucas numbers can be written as

$$B_n = 2^{n+1} - n - 2,$$

$$C_n = 2^n + 2.$$

Also, Binet's formulas of Mersenne and Mersenne-Lucas numbers, are

$$M_n = 2^n - 1,$$

$$H_n = 2^n + 1,$$

and so

$$
\begin{aligned}
B_n &= 2M_n - n, && (2.4)\\
C_n &= H_n + 1. && (2.5)
\end{aligned}
$$

Now, we present the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of $W_n$.

**Lemma 2.** *Let $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Bigollo sequence $\{W_n\}_{n\geq 0}$. In this case, $\sum_{n=0}^{\infty} W_n x^n$ is obtained with*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 4W_0)x + (W_2 - 4W_1 + 5W_0)x^2}{1 - 4x + 5x^2 - 2x^3}.$$

Proof. If $r = 4, s = -5, t = 2$ are chosen in [23, Lemma 1.1], we get this equality. □

The following results can be obtained from the previous lemma.

**Corollary 3.** *Generated functions of Bigollo and Bigollo-Lucas numbers are*

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_n x^n &= \frac{x}{1 - 4x + 5x^2 - 2x^3},\\
\sum_{n=0}^{\infty} C_n x^n &= \frac{3 - 8x + 5x^2}{1 - 4x + 5x^2 - 2x^3},
\end{aligned}
$$

*respectively.*

# 3 Simson Formulas

The Simson formula of the Fibonacci sequence $\{F_n\}$ is:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

and in 1753 it was established by R. Simson then this formula is called as Cassini formula. It can be given by

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

Now we give such formulas for the generalized Bigollo sequence $\{W_n\}_{n\geq 0}$.

**Theorem 4** (Simson Formula for Generalized Bigollo Numbers)**.** *For every integers $n$,*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -2^{n-2}(W_2 - 2W_1 + W_0)(W_2 - 3W_1 + 2W_0)^2.$$

*is written.*

Proof. If we take $r = 4, s = -5, t = 2$ in [24, Theorem 2.2], we have this equality. □

From the Theorem 4, we write the following special cases.

**Corollary 5.** *For every $n \in \mathbb{N}$, Simson formula for Bigollo and Bigollo-Lucas numbers are obtained following*

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1},$$

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = 0.$$

# 4 Some Identities

Now, we give some identities of Bigollo and Bigollo-Lucas numbers. Firstly, a few important relationships between $\{W_n\}$ and $\{B_n\}$ will be expressed.

**Lemma 6.** *For every integers $n$, the following equalities are valid :*

**(a)** $8W_n = (49W_0 - 58W_1 + 17W_2)B_{n+4} + 2(98W_1 - 81W_0 - 29W_2)B_{n+3} + (129W_0 - 162W_1 + 49W_2)B_{n+2}.$

**(b)** $4W_n = (17W_0 - 18W_1 + 5W_2)B_{n+3} - 2(29W_0 - 32W_1 + 9W_2)B_{n+2} + (49W_0 - 58W_1 + 17W_2)B_{n+1}.$

**(c)** $2W_n = (5W_0 - 4W_1 + W_2)B_{n+2} - 2(9W_0 - 8W_1 + 2W_2)B_{n+1} + (17W_0 - 18W_1 + 5W_2)B_n.$

**(d)** $W_n = W_0B_{n+1} + (-4W_0 + W_1)B_n + (5W_0 - 4W_1 + W_2)B_{n-1}.$

**(e)** $W_n = W_1B_n + (W_2 - 4W_1)B_{n-1} + 2W_0B_{n-2}.$

**(f)** $2(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = -(-5W_1^2 - W_2^2 + 2W_0W_1 + 4W_1W_2)W_{n+4} + 2(-9W_1^2 - 2W_2^2 + 4W_0W_1 - W_0W_2 + 8W_1W_2)W_{n+3} + (4W_0^2 + 25W_1^2 + 5W_2^2 - 20W_0W_1 + 8W_0W_2 - 22W_1W_2)W_{n+2}.$

**(g)** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (W_1^2 - W_0W_2)W_{n+3} + (2W_0^2 - 5W_0W_1 + 4W_0W_2 - W_1W_2)W_{n+2} + (5W_1^2 + W_2^2 - 2W_0W_1 - 4W_1W_2)W_{n+1}.$

**(h)** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{n+2} + (W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{n+1} - 2(-W_1^2 + W_0W_2)W_n.$

**(i)** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (8W_0^2 + 16W_1^2 + W_2^2 - 22W_0W_1 + 5W_0W_2 - 8W_1W_2)W_{n+1} - (10W_0^2 + 18W_1^2 - 25W_0W_1 + 2W_0W_2 - 5W_1W_2)W_n + 2(2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{n-1}.$

**(j)** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (22W_0^2 + 46W_1^2 + 4W_2^2 - 63W_0W_1 + 18W_0W_2 - 27W_1W_2)W_n - (36W_0^2 + 72W_1^2 + 5W_2^2 - 100W_0W_1 + 25W_0W_2 - 38W_1W_2)W_{n-1} + 2(8W_0^2 + 16W_1^2 + W_2^2 - 22W_0W_1 + 5W_0W_2 - 8W_1W_2)W_{n-2}.$

Proof. We only prove (a), since other equalities is shown similarly. First using

$$W_n = a \times B_{n+4} + b \times B_{n+3} + c \times B_{n+2}$$

we solve the system of equations

$$\begin{aligned} W_0 &= a \times B_4 + b \times B_3 + c \times B_2 \\ W_1 &= a \times B_5 + b \times B_4 + c \times B_3 \\ W_2 &= a \times B_6 + b \times B_5 + c \times B_4 \end{aligned}$$

then we have $a = \frac{1}{8}(49W_0 - 58W_1 + 17W_2), b = \frac{1}{4}(98W_1 - 81W_0 - 29W_2), c = \frac{1}{8}(129W_0 - 162W_1 + 49W_2).$ □

Obviously, all identities in Lemma 6 are also shown using induction.

We present some relationships between $\{W_n\}$ and $\{C_n\}$.

**Lemma 7.** *The following identities are correct.*

**(a)** $4(W_0-2W_1+W_2)(2W_0-3W_1+W_2)C_n = (10W_0-19W_1+9W_2)W_{n+4} - 2(14W_0-27W_1+13W_2)W_{n+3} + (18W_0-35W_1+17W_2)W_{n+2}$.

**(b)** $2(W_0-2W_1+W_2)(2W_0-3W_1+W_2)C_n = (6W_0-11W_1+5W_2)W_{n+3} - 2(8W_0-15W_1+7W_2)W_{n+2} + (10W_0-19W_1+9W_2)W_{n+1}$.

**(c)** $(W_0-2W_1+W_2)(2W_0-3W_1+W_2)C_n = (4W_0-7W_1+3W_2)W_{n+2} - 2(5W_0-9W_1+4W_2)W_{n+1} + (6W_0-11W_1+5W_2)W_n$.

**(d)** $(W_0-2W_1+W_2)(2W_0-3W_1+W_2)C_n = 2(3W_0-5W_1+2W_2)W_{n+1} - 2(7W_0-12W_1+5W_2)W_n + 2(4W_0-7W_1+3W_2)W_{n-1}$.

**(e)** $(W_0-2W_1+W_2)(2W_0-3W_1+W_2)C_n = 2(5W_0-8W_1+3W_2)W_n - 2(11W_0-18W_1+7W_2)W_{n-1} + 4(3W_0-5W_1+2W_2)W_{n-2}$.

Then, we also give some basic relationships between $\{B_n\}$ and $\{C_n\}$.

**Lemma 8.** *The following identities are true*

$$\begin{aligned} 8C_n &= 17B_{n+4} - 50B_{n+3} + 33B_{n+2}, \\ 4C_n &= 9B_{n+3} - 26B_{n+2} + 17B_{n+1}, \\ 2C_n &= 5B_{n+2} - 14B_{n+1} + 9B_n, \\ C_n &= 3B_{n+1} - 8B_n + 5B_{n-1}, \\ C_n &= 4B_n - 10B_{n-1} + 6B_{n-2}. \end{aligned}$$

# 5 Identities Between Special Numbers

Now, we give some identities on Bigollo and Bigollo-Lucas numbers and Mersenne and Mersenne-Lucas numbers. It is known that

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1. \end{aligned}$$

We also note that from Lemma 8, we have

$$2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n$$

and from Soykan [19, Lemma 11], we get

$$M_n = 2H_{n+1} - 3H_n.$$

If we use the above identities, we have

$$\begin{aligned} B_n &= 4H_{n+1} - 6H_n - n, \\ 2C_n &= 2M_{n+1} - 2M_n + 4. \end{aligned}$$

We use these formulas and Lemma 6, we get Binet's formula of generalized Bigollo numbers as:

$$\begin{aligned} 2W_n &= (5W_0-4W_1+W_2)B_{n+2} - 2(9W_0-8W_1+2W_2)B_{n+1} + (17W_0-18W_1+5W_2)B_n \\ &= 2(3W_2-10W_1+7W_0)M_n - 2(W_2-4W_1+3W_0)M_{n+1} - 2(W_2-3W_1+2W_0)n + 2W_2 - 8W_1 + 8W_0 \\ &= 2(3W_2-8W_1+5W_0)H_{n+1} - 2(5W_2-14W_1+9W_0)H_n - 2(W_2-3W_1+2W_0)n + 2W_2 - 8W_1 + 8W_0. \end{aligned}$$

# 6 The Recurrence Relations of Generalized Bigollo Sequence

If we take $r = 4, s = -5, t = 2$ in [25, Theorem 2], we have the next Proposition.

**Proposition 9.** *For $n \in \mathbb{Z}$, generalized Bigollo numbers get the following equation:*

$$W_{-n} = 2^{-n}(W_{2n} - C_n W_n + \frac{1}{2}(C_n^2 - C_{2n})W_0).$$

Using the Corollary 6 and Proposition 9 in [25], we get the following corollary which presents the relation between the special cases of generalized Bigollo sequence for the negative and the positive index: for modified Bigollo, Bigollo-Lucas and Bigollo numbers: take $W_n = B_n$, $B_0 = 0, B_1 = 1, B_2 = 4$ and taking $W_n = C_n$ and also $C_0 = 3, C_1 = 4, C_2 = 6$.

**Corollary 10.** *For $n \in \mathbb{Z}$, it is written the next recurrence relations:*

**(a)** *Bigollo sequence:*

$$B_{-n} = 2^{-n}(B_{2n} - B_n C_n).$$

**(b)** *Bigollo-Lucas sequence:*

$$C_{-n} = 2^{-n-1}\left(C_n^2 - C_{2n}\right).$$

By using the equation $2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n$ (and Corollary 10 or Proposition 9 ),

$$B_{-n} = \frac{1}{2^{n+1}}(14B_n B_{n+1} - 5B_n B_{n+2} - 9B_n^2 + 2B_{2n})$$

is written. Note also that since $B_n = 2M_n - n$ and $M_{-n} = -\frac{1}{2^n}M_n = \frac{-2^n+1}{2^n}$, we get

$$B_{-n} = -2^{-n+1}M_n + n$$

and using $C_n = H_n + 1$ and $H_{-n} = \frac{1}{2^n}H_n = \frac{2^n+1}{2^n}$, then we obtain

$$C_{-n} = 2^{-n}H_n + 1.$$

# 7 Sum Formulas

In next Corollary we give sum formulas of Mersenne and Mersenne-Lucas numbers.

**Corollary 11.** *Let $n \geq 0$. For Mersenne and Mersenne-Lucas numbers, the following properties are true :*

*1.*

- **(a)** $\sum_{k=0}^{n} M_k = -(n-1)M_n + 2(n+1)M_{n-1} + 1$.
- **(b)** $\sum_{k=0}^{n} M_{2k} = \frac{1}{3}(-(n-3)M_{2n} + 4(n+1)M_{2n-2} + 3)$.
- **(c)** $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{3}(-(n-3)M_{2n+1} + 4(n+1)M_{2n-1} + 2)$.

*2.*

- **(a)** $\sum_{k=0}^{n} H_k = -(n-1)H_n + 2(n+1)H_{n-1} - 3$.
- **(b)** $\sum_{k=0}^{n} H_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} - 5)$.
- **(c)** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} - 6)$.

Proof. It is given in [19, Corollary 25]. □

In the following Corollary we present sum formulas of Bigollo and Bigollo-Lucas numbers.

**Corollary 12.** *For $n \geq 0$, Bigollo and Bigollo-Lucas numbers get the following equalities:*

1.
   - **(a)** $\sum_{k=0}^{n} B_k = \frac{1}{2}(-4(n-1)M_n + 8(n+1)M_{n-1} - n^2 - n + 4)$.
   - **(b)** $\sum_{k=0}^{n} B_{2k} = \frac{1}{3}(-2(n-3)M_{2n} + 8(n+1)M_{2n-2} - 3(n-1)(n+2))$
   - **(c)** $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{3}(-2(n-3)M_{2n+1} + 8(n+1)M_{2n-1} + 4 - 3(n+1)^2)$.
2.
   - **(a)** $\sum_{k=0}^{n} C_k = -(n-1)H_n + 2(n+1)H_{n-1} + n - 2$.
   - **(b)** $\sum_{k=0}^{n} C_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} + 3n - 2)$.
   - **(c)** $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} + 3n - 3)$.

Proof. The proof is valid from the identities (2.4) and (2.5) and Corollary 11, i.e.,

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1. \quad \square \end{aligned}$$

# 8 Matrices Formulation of Generalized Bigollo Numbers

Matrix forms of $W_n$ is shown by

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{8.1}$$

$A$ square matrix is defined as:

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (2.1) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{8.2}$$

and using (8.1) or (8.2 )and induction we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = B$ in (8.2) we have

$$\begin{pmatrix} B_{n+2} \\ B_{n+1} \\ B_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_{n+1} \\ B_n \\ B_{n-1} \end{pmatrix}.$$

We also define

$$N_n = \begin{pmatrix} B_{n+1} & -5B_n + 2B_{n-1} & 2B_n \\ B_n & -5B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & -5B_{n-2} + 2B_{n-3} & 2B_{n-2} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & -5W_n + 2W_{n-1} & 2W_n \\ W_n & -5W_{n-1} + 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & -5W_{n-2} + 2W_{n-3} & 2W_{n-2} \end{pmatrix}$$

**Theorem 13.** *For every* $m, n \in \mathbb{Z}$,

**(a)** $N_n = A^n$, *i.e.,*

$$A^n = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & -5B_n + 2B_{n-1} & 2B_n \\ B_n & -5B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & -5B_{n-2} + 2B_{n-3} & 2B_{n-2} \end{pmatrix}.$$

**(b)** $U_1 A^n = A^n U_1$

**(c)** $U_{n+m} = U_n N_m = N_m U_n$.

Proof. If we take $r = 4, s = -5, t = 2$ in [23, Theorem 5.1.], we get these properties. □

For every integer $m$ and $n$, some characterizations of $A^n$ is written as

$$\begin{aligned} A^n &= 4A^{n-1} - 5A^{n-2} + 2A^{n-3}, \\ A^{n+m} &= A^n A^m = A^m A^n, \\ \det(A^n) &= 2^n, \end{aligned}$$

.

Using the above last Theorem and the identities

$$\begin{aligned} B_n &= 2M_n - n, \\ B_n &= 4H_{n+1} - 6H_n - n, \end{aligned}$$

we get next identities of Mersenne and Mersenne-Lucas numbers.

**Corollary 14.** *For every* $n \in \mathbb{Z}$, *we write the following identities for Mersenne and Mersenne-Lucas numbers.*

**(a)** *Mersenne Numbers.*

$$A^n = \begin{pmatrix} 2M_{n+1} - n - 1 & -2M_{n+1} - 4M_n + 3n + 2 & 4M_n - 2n \\ 2M_n - n & 2M_{n+1} - 8M_n + 3n - 1 & -2M_{n+1} + 6M_n - 2n + 2 \\ -M_{n+1} + 3M_n - n + 1 & 4M_{n+1} - 10M_n + 3n - 4 & -3M_{n+1} + 7M_n - 2n + 4 \end{pmatrix}.$$

**(b)** *Mersenne-Lucas Numbers.*

$$A^n = \begin{pmatrix} 6H_{n+1} - 8H_n - n - 1 & -14H_{n+1} + 20H_n + 3n + 2 & 8H_{n+1} - 12H_n - 2n \\ 4H_{n+1} - 6H_n - n & -10H_{n+1} + 16H_n + 3n - 1 & 6H_{n+1} - 10H_n - 2n + 2 \\ 3H_{n+1} - 5H_n - n + 1 & -8H_{n+1} + 14H_n + 3n - 4 & 5H_{n+1} - 9H_n - 2n + 4 \end{pmatrix}.$$

**Theorem 15.** *For every* $m, n \in \mathbb{Z}$, *we get*

$$W_{n+m} = W_n B_{m+1} + (-5W_{n-1} + 2W_{n-2}) B_m + 2W_{n-1} B_{m-1} \tag{8.3}$$

Proof. Take $r = 4, s = -5, t = 2$ in [23, Theorem 5.2.]. □

Using Lemma 6, we have next equalities.

$$\begin{aligned} & (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_m \\ = \; & (2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+2} \\ & +(W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{m+1} - 2(-W_1^2 + W_0W_2)W_m \end{aligned}$$

so (8.3) the next can be written as

$$\begin{aligned} & (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 W_{n+m} \\ = \; & W_n((2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+3} \\ & +(W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{m+2} - 2(-W_1^2 + W_0W_2)W_{m+1}) \\ & + (-5W_{n-1} + 2W_{n-2})((2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+2} \\ & +(W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{m+1} - 2(-W_1^2 + W_0W_2)W_m) \\ & +2W_{n-1}((2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+1} \\ & +(W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_m - 2(-W_1^2 + W_0W_2)W_{m-1}). \end{aligned}$$

**Corollary 16.** *For every* $m, n \in \mathbb{Z}$, *we get*

$$\begin{aligned} B_{n+m} &= B_nB_{m+1} + (-5B_{n-1} + 2B_{n-2})B_m + 2B_{n-1}B_{m-1}, \\ C_{n+m} &= C_nB_{m+1} + (-5C_{n-1} + 2C_{n-2})B_m + 2C_{n-1}B_{m-1}. \end{aligned}$$

Taking $m = n$ in this corollary, we get the next equalities:

$$\begin{aligned} B_{2n} &= B_nB_{n+1} + (-5B_{n-1} + 2B_{n-2})B_n + 2B_{n-1}^2, \\ C_{2n} &= C_nB_{n+1} + (-5C_{n-1} + 2C_{n-2})B_n + 2C_{n-1}B_{n-1}. \end{aligned}$$

# 9 Conclusions

Recently, quite a lot of work has been done on the sequences of Horadam numbers and generalized third-order Pell numbers for example Fibonacci, Lucas, Jacobsthal and Pell numbers; third order Jacobsthal, third-order Pell, third-order Pell-Lucas,Narayana, Perrin, Padovan, Padovan-Perrin and third order Jacobsthal-Lucas numbers.The sequences of numbers have been frequently used in important fields, especially in engineering, physics, nature and architecture.

In our study, we introduced the generalized Bigollo sequence, which is a third order sequence, and the special case of this sequence, Bigollo and Bigollo-Lucas sequences. Also we give Binet's formulas, Simson formulas, generating functions, some identities, the sum formulas, matrices and recurrence relations of these sequences. We found significant relationships between Bigollo, Bigollo-Lucas numbers (which are third order linear recurences) and special second order linear recurences (numbers), namely Mersenne and Mersenne-Lucas numbers

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, We give some studies on the applications of second order sequences.

- For some implements of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [26].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [27].
- For the adaptation of Jacobsthal numbers to special matrices, see [28].
- For the adaptation of generalized k-order Fibonacci numbers to hybrid quaternions, see [29].
- For some applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [30].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [31].

- For the adaptation of generalized Fibonacci numbers to binomial sums, see [32].
- For the adaptation of generalized Jacobsthal numbers to hyperbolic numbers, see [33].
- For the adaptation of generalized Fibonacci numbers to dual hyperbolic numbers, see [34].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial for Fibonacci numbers, see [35].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [36].
- For the application of higher order Jacobsthal numbers to quaternions, see [37].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [38].
- For the implements of Fibonacci numbers to lacunary statistical convergence, see [39].
- For the implements of Fibonacci numbers to lacunary statistical convergence in IFNLS, see [40].
- For the implements of Fibonacci numbers to ideal convergence in IFNLS, see [41].
- For some identities on k-Mersenne Numbers, see [42].

Now we give some other implements of third order sequences.

- For the implements of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [43] and [44].
- For the adaptation of Tribonacci numbers to special matrices, see [45].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [46] and [47], respectively.
- For the application of Pell-Padovan numbers to groups, see [48].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [49].
- For the adaptation of Gaussian Tribonacci numbers to various graphs, see [50].
- For the implements of third-order Jacobsthal numbers to hyperbolic numbers, see [51].
- For the implements of Narayan numbers to finite groups see [52].
- For the adaptation of generalized third-order Jacobsthal sequence to binomial transform, see [53].
- For the implements of generalized Generalized Padovan numbers to Binomial Transform, see [54].
- For the implements of generalized Tribonacci numbers to Gaussian numbers, see [55].
- For the implement of generalized Tribonacci numbers to Sedenions, see [56].
- For the adaptation of Tribonacci and Tribonacci-Lucas numbers to matrices, see [57].
- For the adaptation of generalized Tribonacci numbers to circulant matrix, see [58].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [59].

Next, we now list some implements of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [60].
- For the adaptation of generalized Tetranacci numbers to Gaussian numbers, see [61].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [62].
- For the application of generalized Tetranacci numbers to binomial transform, see [63].

Also, we give some applications of fifth order sequences.

- For the adaptation of Pentanacci numbers to matrices, see [64].
- For the adaptation of generalized Pentanacci numbers to quaternions, see [65].
- For the application of generalized Pentanacci numbers to binomial transform, see [66].
  We now mention some applications of second order sequences of polynomials.
- For the application of generalized Fibonacci Polynomials to the summation formulas, see [67].
- For some applications of generalized Fibonacci Polynomials, see [68].
  We now give some implements of third order sequences of polynomials.
- For some applications of generalized Tribonacci Polynomials, see [69].

# Competing Interests

Authors have declared that no competing interests exist.

# References

[1] Sloane NJA. The on-line encyclopedia of integer sequences.
Available:http://oeis.org/

[2] Bravo JJ, Gómez CA. Mersenne k-fibonacci numbers. Glasnik Matematički. 2016;51(2):307-319.

[3] Brillhart J. On the factors of certain mersenne numbers. Math. Comp. 1960;14(72):365-369.

[4] Brillhart J. On the factors of certain mersenne numbers, II. Math. Comp. 1964;18:87-92.

[5] Catarino P, Campos H, Vasco P. On the mersenne sequence. Annales Mathematicae et Informaticae. 2016;46:37-53.

[6] Chelgham M, Boussayoud A. On the k-mersenne-lucas numbers. Notes on Number Theory and Discrete Mathematics. 2021;27(1): 7-13.
DOI: 10.7546/nntdm.2021.27.1.7-13

[7] Daşdemir A. Mersene, Jacobsthal, and Jacobsthal-Lucas numbers with negative subscripts. Acta Mathematica Universitatis Comenianae. 2019;88(1):145-156.

[8] Ehrman JR. The number of prime divisors of certain mersenne numbers. Math. Comp. 1967;21(100):700-704.

[9] Ford K, Luca F, Shparlinski IE. On the largest prime factor of the mersenne numbers. Bull. Austr. Math. Soc. 2009;79(3):455-463.

[10] Goy T. On new identities for mersenne numbers. Applied Mathematics E-Notes. 2018;18:100-105.

[11] Granger R, Moss A. Generalized mersenne numbers revisited. Mathematics of Computation. 2013;82(284):2389-2420.

[12] Jaroma JH, Reddy KN. Classical and alternative approaches to the mersenne and fermat numbers. American Mathematical Monthly. 2007;114(8):677-687.
DOI: 10.1080/00029890.2007.11920459

[13] Murata L, Pomerance C. On the largest prime factor of a mersenne number. Number Theory. 2004;36:209-218.

[14] Ochalik P, Włoch A. On generalized mersenne numbers. Their Interpretations and Matrix Generators, Ann. Univ. Mariae Curie-Skłodowska Sect. A. 2018;72(1):69-76.
DOI: 10.17951/a.2018.72.1.69

[15] Pomerance C. On primitive divisors of mersenne numbers. Acta Arith. 1986;46:355-367.

[16] Samuel S, Wagstaff Jr. Divisors of mersenne numbers. Math. Comp. 1983;40:385-397.

[17] Schinzel A. On primitive prime factors of $a^n - b^n$. Proc. Cambridge Philos. Soc. 1962;58(4):555-562.

[18] Solinas JA. Generalized mersenne numbers. Technical report CORR-39, Dept. of C&O, University of Waterloo; 1999.
Available from http://www.cacr.math.uwaterloo.ca

[19] Soykan Y. A study on generalized mersenne numbers. Journal of Progressive Research in Mathematics. 2021;18(3):90-112.

[20] Soykan Y. On generalized p-mersenne numbers. Earthline Journal of Mathematical Sciences. 2022;8(1):83-120.
DOI:https://doi.org/10.34198/ejms.8122.83120

[21] Stewart CL. The greatest prime factor of $a^n - b^n$. Acta Arith. 1975;26:427-433.

[22] Zatorsky R, Goy T. Parapermanents of triangular matrices and some general theorems on number sequences. Journal of Integer Sequences. 2016;19( 16.2.2).

[23] Soykan Y. A study on generalized (r,s,t)-numbers. Math LAB Journal. 2020;7;101-129.

[24] Soykan Y. Simson identity of generalized m-step fibonacci numbers. International Journal of Advances in Applied Mathematics and Mechanics. 2019;7(2):45-56.

[25] Soykan Y. On the recurrence properties of generalized tribonacci sequence. Earthline Journal of Mathematical Sciences. 2021;6(2):253-269.
DOI: https://doi.org/10.34198/ejms.6221.253269

[26] Azak AZ. Pauli gaussian fibonacci and pauli gaussian lucas quaternions. Mathematics. 2022:10(4655).
DOI:https://doi.org/10.3390/math10244655

[27] Büyük H, Taşkara N. On the solutions of three-dimensional difference equation systems via pell numbers. European Journal of Science and Technology. 2022;34(Special Issue):433-440.

[28] Vasanthi S, Sivakumar B. Jacobsthal matrices and their properties. Indian Journal of Science and Technology. 2022;15(5):207-215.
DOI: https://doi.org/10.17485/IJST/v15i5.1948

[29] Gül K. Generalized k-order fibonacci hybrid quaternions. Erzincan University Journal of Science and Technology. 2022;15(2):670-683.
DOI: 10.18185/erzifbed.1132164

[30] Yılmaz N. Split complex Bi-periodic fibonacci and lucas numbers. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 2022;71(1):153-164.
DOI:10.31801/cfsuasmas.704435

[31] Yılmaz N. The generalized bivariate fibonacci and lucas matrix polynomials. Mathematica Montisnigri. 2022;LIII:33-44.
DOI: 10.20948/mathmontis-2022-53-5

[32] Ulutaş YT, Toy D. Some equalities and binomial sums about the generalized fibonacci number $u_n$. Notes on Number Theory and Discrete Mathematics. 2022;28(2):252-260.
DOI: 10.7546/nntdm.2022.28.2.252-260

[33] Soykan Y, Taşdemir E. A study on hyperbolic numbers with generalized jacobsthal numbers components. International Journal of Nonlinear Analysis and Applications. 2022;13(2):1965-1981.
DOI: http://dx.doi.org/10.22075/ijnaa.2021.22113.2328

[34] Soykan Y. On dual hyperbolic generalized fibonacci numbers. Indian J Pure Appl Math; 2021.
DOI: https://doi.org/10.1007/s13226-021-00128-2

[35] Deveci Ö, Shannon AG. On recurrence results from matrix transforms. Notes on Number Theory and Discrete Mathematics. 2022;28(4):589-592.
DOI: 10.7546/nntdm.2022.28.4.589-592

[36] Prasad K, Mahato H. Cryptography using generalized fibonacci matrices with affine-hill cipher. Journal of Discrete Mathematical Sciences & Cryptography. 2022;25(8-A):2341-2352.
DOI : 10.1080/09720529.2020.1838744

[37] Özkan E, Uysal M. On quaternions with higher order jacobsthal numbers components. Gazi University Journal of Science. 2023;36(1):336-347.
DOI: 10.35378/gujs. 1002454

[38] Goy T, Shattuck M. Fibonacci and lucas identities from toeplitz-hessenberg matrices. Appl. Appl. Math. 2019;14(2):699-715.

[39] Bilgin NG, Bozma G. Fibonacci lacunary statistical convergence of order $\gamma$ in IFNLS. International Journal of Advances in Applied Mathematics and Mechanics. 2021;8(4):28-36.

[40] Kişi, Ö, Tuzcuoglu I. Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces. Journal of Progressive Research in Mathematics. 2020;16(3):3001-3007.

[41] Kişi Ö, Debnathb P. Fibonacci ideal convergence on intuitionistic fuzzy normed linear spaces. Fuzzy Information and Engineering. 2022;1-13.
DOI: https://doi.org/10.1080/16168658.2022.2160226

[42] Uslu K, Deniz V. Some identities of k-mersenne numbers. Advances and Applications in Discrete Mathematics. 2017;18(4):413-423.
DOI: http://dx.doi.org/10.17654/AADMOct2017_413_423

[43] Cerda-Morales G. identities for third order jacobsthal quaternions. Advances in Applied Clifford Algebras. 2017;27(2):1043-1053.

[44] Cerda-Morales G. On a generalization of tribonacci quaternions. Mediterranean Journal of Mathematics. 2017;14(239):1-12.

[45] Yilmaz N, Taskara N. Tribonacci and tribonacci-lucas numbers via the determinants of special Matrices. Applied Mathematical Sciences. 2014;8(39):1947-1955.

[46] Shtayat J, Al-Kateeb A. An encoding-decoding algorithm based on padovan numbers; 2019.
arXiv:1907.02007

[47] Basu M, Das M. Tribonacci matrices and a new coding theory. Discrete Mathematics, Algorithms and Applications. 2014;6(1):1450008(17 pages).

[48] Deveci, Ö, Shannon AG. Pell–padovan-circulant sequences and their applications. Notes on Number Theory and Discrete Mathematics. 2017;23(3):100-114.

[49] Göcen M. The exact solutions of some difference equations associated with adjusted jacobsthal-padovan numbers. Kırklareli University Journal of Engineering and Science. 2022;8(1):1-14.
DOI: 10.34186/klujes.1078836

[50] Sunitha K, Sheriba M. Gaussian tribonacci r-graceful labeling of some tree related graphs. Ratio Mathematica. 2022;44:188-196.

[51] Dikmen CM, Altınsoy M. On third order hyperbolic jacobsthal numbers. Konuralp Journal of Mathematics. 2022;10(1):118-126.

[52] Kuloğlu B, Özkan E, Shannon AG. The narayana sequence in finite groups. Fibonacci Quarterly. 2022;60(5):212-221.

[53] Soykan Y, Taşdemir E, Göcen M. Binomial transform of the generalized third-order jacobsthal sequence. Asian-European Journal of Mathematics. 2022;15(12).
DOI: https://doi.org/10.1142/S1793557122502242

[54] Soykan Y, Taşdemir E, Okumuş İ. A study on binomial transform of the generalized padovan Sequence. Journal of Science and Arts. 2022;22(1):63-90.
DOI: https://doi.org/10.46939/J.Sci.Arts-22.1-a06

[55] Soykan Y, Taşdemir E, Okumuş İ, Göcen M. Gaussian generalized tribonacci numbers. Journal of Progressive Research in Mathematics (JPRM). 2018;14(2):2373-2387.

[56] Soykan Y, Okumuş İ, Taşdemir E. On generalized tribonacci sedenions. Sarajevo Journal of Mathematics. 2020;16(1):103-122.
ISSN 2233-1964
DOI: 10.5644/SJM.16.01.08

[57] Soykan Y. Matrix sequences of tribonacci and tribonacci-lucas numbers. Communications in Mathematics and Applications. 2020;11(2):281-295.
DOI: 10.26713/cma.v11i2.1102

[58] Soykan Y. Explicit euclidean norm, eigenvalues, spectral norm and determinant of circulant matrix with the generalized tribonacci numbers. Earthline Journal of Mathematical Sciences. 2021;6(1):131-151.
DOI: https://doi.org/10.34198/ejms.6121.131151

[59] Taşyurdu Y, Polat YE. Tribonacci and tribonacci-lucas hybrinomials. Journal of Mathematics Research. 2021;13(5).

[60] Soykan Y. Tetranacci and tetranacci-lucas quaternions. Asian Research Journal of Mathematics. 2019;15(1): 1-24. Article no.ARJOM.50749

[61] Soykan Y. Gaussian generalized tetranacci numbers. Journal of Advances in Mathematics and Computer Science. 2019;31(3):1-21.
Article no.JAMCS.48063

[62] Soykan Y. Matrix sequences of tetranacci and tetranacci-lucas numbers. Int. J. Adv. Appl. Math. and Mech. 2019;7(2):57-69. ISSN: 2347-2529

[63] Soykan Y. On binomial transform of the generalized tetranacci sequence. International Journal of Advances in Applied Mathematics and Mechanics. 2021;9(2):8-27.

[64] Sivakumar B, James V. A notes on matrix sequence of pentanacci numbers and pentanacci cubes. Communications in Mathematics and Applications. 2022;13(2):603-611.
DOI: 10.26713/cma.v13i2.1725

[65] Soykan Y, Özmen N, Göcen M. On generalized pentanacci quaternions. Tbilisi Mathematical Journal. 2020;13(4):169-181.

[66] Soykan Y. Binomial transform of the generalized pentanacci sequence. Asian Research Journal of Current Science. 2021;3(1):209-231.

[67] Soykan Y. A study on generalized fibonacci polynomials: Sum formulas. International Journal of Advances in Applied Mathematics and Mechanics. 2022;10(1):39-118. ISSN: 2347-2529

[68] Soykan Y. On generalized fibonacci polynomials: Horadam polynomials. Earthline Journal of Mathematical Sciences. 2023;11(1):23-114. E-ISSN: 2581-8147
DOI: https://doi.org/10.34198/ejms.11123.23114

[69] Soykan Y. Generalized tribonacci polynomials. Earthline Journal of Mathematical Sciences. 2023;13(1):1-120.
DOI: https://doi.org/10.34198/ejms.13123.1120


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