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Nonlinear Spinor Field Equation of the Bilinear Pauli-Fierz Invariant $I_v = S^2 + P^2$: Exact Spherical Symmetric Soliton-Like Solutions in General Relativity

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Abstract
Taking into account the own gravitational field of elementary particles, we have obtained exact static spherical symmetric solutions to the spinor field equation. The nonlinear terms L_N are arbitrary functions of bilinear Pauli-Fierz invariant $I_v = S^2 + P^2$. It characterizes the self-interaction of a spinor field. We have investigated in detail equations with power and polynomial nonlinearities. The spinor field equation with a power-law nonlinearity have regular solutions with a localized energy density and regular metric. In this case a soliton-like configuration has finite and negative total energy. As for equations with polynomial nonlinearity, the obtained solutions are regular with a localized energy density and regular metric but its total energy is finite and positive.
Keywords: proper gravitational field, elementary particles, symmetric metric

1. Introduction

The concept of soliton is present in many branches of pure science. In the elementary particles physics, the soliton as regular localized stable solutions of nonlinear differential field equations are used as the simplest models of extended particles (Perring J. K. and Skyrme T.H. R, 1962; Scott A.C., Chu F. Y. F. and McLaughlin D. W., 1973; Rybakov Yu.P., 1985). The nonlinearity of the field equations plays a crucial role in the obtaining of regular solutions. It describes the fields interactions. But let us emphasize that the choice of field equations is one of the principle problems in nonlinear theory (Marshak R.E. and Sudershan E.C.G., 1961). In many models elaborated in the pure science in order to describe the configuration of elementary particles, the gravitational field equation is absent. However the gravitational field equation is nonlinear by nature and the field itself is universal and unscreenable.
The present work, is a part II of all investigated initiatied in (Adomou A., Massou S. and Edou J.(2019) *International Journal of Applied Mathematics and Theoretical Physics*, 118-128 doi: 10.11648/j.ijamtp.20190504.14). Here, We have extended the results to the exact spherical symmetric solutions of equations with polynomial nonlinearity taking into account the gravitational field equation.

The paper is organized as follows. The section 2 deals with model and fields equations. In section 3 the general solutions are obtained. The section 4 adresses discussion of main results. In the section 5, we determined the total charge and total spin. Finally some conclusions of the work are given in the last section (section 6)

2. Model and Fields Equations

This section is devolted to establish the spinor and gravitational fields equations. To do so,let us consider the lagrangian density for the self-consistent system of spinor and gravitational fields under the following expression (Adomou A. and Shikin G.N., 1998):

$$\begin{aligned} L &= \frac{R}{2\chi} + L_{Sp} \\ &= \frac{R}{2\chi} + \frac{i}{2}(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi) - m\bar{\psi}\psi + L_N, \end{aligned} \tag{1}$$

where L_N is the nonlinear part of the spinor field lagrangian, R is the scalar curvature and $\chi = \frac{8\Pi G}{C^4}$ is Einstein's gravitational constant. $L_N = G(I_v)$ is an arbitrary function depending on the bilinear Pauli-Fierz invariant $I_v = S^2 + P^2 = (\bar{\psi}\gamma^{\mu}\psi)g_{\mu\nu}(\bar{\psi}\gamma^{\nu}\psi)$.

In the present analysis, the gravitational field is defined by the static spherical symmetric metric under the form (Adanhoumè A., Adomou A., Codo F.P. and Hounkonnou M.N, 2012):

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}d\xi^2 - e^{2\beta}[d\theta^2 + \sin^2(\theta)d\varphi^2]. \tag{2}$$

Here, its signature is (+1, -1, -1, -1) and $c = 1$ is the speed of light considered to be unity. The metric functions α, β and γ are some functions depending only on $\xi = \frac{1}{r}$ (Bronnikov K. A, 1973) where r stands for the radial component of the spherical symmetric metric. They satisfy the coordinate condition given by the following expression (Bronnikov K. A, 1973):

$$\alpha = 2\beta + \gamma. \tag{3}$$

The matricial form of the metric tensor $g_{\mu\nu}$ associated to the metric (2) is

$$[g_{\mu\nu}] = \begin{pmatrix} e^{2\gamma} & 0 & 0 & 0 \\ 0 & -e^{2\alpha} & 0 & 0 \\ 0 & 0 & -e^{2\beta} & 0 \\ 0 & 0 & 0 & -e^{2\beta}\sin^2\theta \end{pmatrix}. \tag{4}$$

Varying (1) with respect to the metric tensor $g_{\mu\nu}$, we obtain the Einstein's field equations in the metric (2) under the condition (3) having the form (D. Brill and J. Wheeler, 1957)

$$G_0^0 = e^{-2\alpha}(2\beta'' - 2\gamma'\beta' - \beta'^2) - e^{-2\beta} = -\chi T_0^0, \tag{5}$$

$$G_1^1 = e^{-2\alpha}(2\beta'\gamma' + \beta'^2) - e^{-2\beta} = -\chi T_1^1, \tag{6}$$

$$G_2^2 = e^{-2\alpha}(\beta'' + \gamma'' - 2\beta'\gamma' - \beta'^2) = -\chi T_2^2, \tag{7}$$

$$G_2^2 = G_3^3, \quad T_2^2 = T_3^3 \tag{8}$$

where prime denotes differentiation with respect to the spatial variable ξ and T_ν^μ is the energy-momentum tensor of the spinor field.

From the lagrangian (1), applying the variational principle, we obtain the spinor field equations for the functions ψ and $\bar{\psi}$ as follows

$$i\gamma^\mu\nabla_\mu\psi - m\psi + 2S\frac{\partial G}{\partial I}\psi + 2iP\frac{\partial G}{\partial J}\gamma^5\psi = 0, \tag{9}$$

$$i\nabla_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} - 2S\frac{\partial G}{\partial I}\bar{\psi} - 2iP\frac{\partial G}{\partial J}\gamma^5\bar{\psi} = 0, \tag{10}$$

with $I = S^2$ and $J = P^2$.

The general form of the metric energy-momentum tensor of the spinor field is

$$T_\mu^\nu = \frac{i}{4}g^{\nu\rho}(\bar{\psi}\gamma_\mu\nabla_\nu\psi + \bar{\psi}\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\nu\psi - \nabla_\nu\bar{\psi}\gamma_\mu\psi) - \delta_\mu^\nu L_{Sp} \tag{11}$$

Using the spinor field equations for the functions ψ and $\bar{\psi}$, L_{Sp} becomes

$$L_{S_P} = \frac{1}{2}\bar{\psi}(i\gamma^\mu\nabla_\mu\psi - m\psi) - \frac{1}{2}(i\nabla_\mu\bar{\psi}\gamma^\mu + m\bar{\psi})\psi + G(S,P), \tag{12}$$

$$= -2I\frac{\partial G}{\partial I} - 2J\frac{\partial G}{\partial J} + G(S,P), \tag{13}$$

$$= -2I_v\frac{\partial G}{\partial I_v} + G(S,P). \tag{14}$$

Taking into account (14), the nonzero components of the tensor T_ν^μ are:

$$T_0^0 = T_2^2 = T_3^3 = -L_{Sp} = 2I_v\frac{\partial G(S,P)}{\partial I_v} - G(S,P). \tag{15}$$

$$T_1^1 = \frac{i}{2}(\bar{\psi}\gamma^1\nabla_1\psi - \nabla_1\bar{\psi}\gamma^1\psi) + 2I_v\frac{\partial G(S,P)}{\partial I_v} - G(S,P). \tag{16}$$

In flat space-time, the Dirac's matrices $\bar{\gamma}^a$ are determined by the following expressions

$$\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \bar{\gamma}^5 = \gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

where I is the two order unity matrice and σ^i are Pauli's matrices defined as follows

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In curved space-time, the Dirac's matrices γ^μ are defined in the following way.
Using the equalities

$$\begin{aligned} \gamma_\mu(\xi) &= e^a_\mu(\xi)\bar{\gamma}_a, \\ g_{\mu\nu}(\xi) &= e^a_\mu(\xi)e^b_\nu(\xi)\eta_{ab} \end{aligned} \tag{17}$$

where $\eta_{ab} = diag(1, -1, -1, -1)$ is Minkowski's metric, $e^a_\mu(\xi)$ are tetradic 4-vectors, we obtain

$$\gamma^0 = e^{-\gamma}\bar{\gamma}^0 \ , \quad \gamma^1(x) = e^{-\alpha}\bar{\gamma}^1 \ , \quad \gamma^2 = e^{-\beta}\bar{\gamma}^2 \ , \quad \gamma^3 = \frac{e^{-\beta}\bar{\gamma}^3}{\sin\theta}. \tag{18}$$

The general form of the spinor affine connection matrices is

$$\Gamma_\mu(\xi) = \frac{1}{4}g_{\rho\mu}(\partial_\mu e^b_\sigma e^\rho_a - \Gamma^\rho_{\mu\sigma})\gamma^\delta\gamma^\sigma, \tag{19}$$

The expression (19) leads to

$$\Gamma_0 = -\frac{1}{2}e^{-2\beta}\bar{\gamma}^0\bar{\gamma}^1\gamma', \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{2}e^{-\beta-\gamma}\bar{\gamma}^2\bar{\gamma}^1\beta', \quad \Gamma_3 = \frac{1}{2}(e^{-\beta-\gamma}\bar{\gamma}^3\bar{\gamma}^1\beta'\sin\theta + \bar{\gamma}^3\bar{\gamma}^2\cos\theta) \tag{20}$$

In (9)-(11), ∇_μ is the covariant derivative of spinor. ∇_μ is linked to the spinor affine connection matrices $\Gamma_\mu(\xi)$ by

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial\xi^\mu} - \Gamma_\mu\psi \quad or \quad \nabla_\mu\bar{\psi} = \frac{\partial\bar{\psi}}{\partial\xi^\mu} + \Gamma_\mu\bar{\psi} \tag{21}$$

From (18) and (20), the Einstein's sommation gives

$$\gamma^\mu\Gamma_\mu = -\frac{1}{2}(e^{-\alpha}\alpha'\bar{\gamma}^1 + \bar{\gamma}^2 e^{-\beta}\cot\theta) \tag{22}$$

Taking into account (21) and (22), the spinor field equations (9) and (10) lead to the following expressions

$$ie^{-\alpha}\bar{\gamma}^1(\partial_\xi + \frac{1}{2}\alpha')\psi + \frac{i}{2}\bar{\gamma}^2 e^{-\beta}\psi\cot\theta - (m - D)\psi + iE(S,P)\gamma^5\psi = 0, \tag{23}$$

$$ie^{-\alpha}\bar{\gamma}^1(\partial_\xi + \frac{1}{2}\alpha')\bar{\psi} + \frac{i}{2}\bar{\gamma}^2 e^{-\beta}\bar{\psi}\cot\theta + (m - D)\bar{\psi} - iE(S,P)\gamma^5\bar{\psi} = 0, \tag{24}$$

where

$$D(S,P) = 2S\frac{dG}{dI_v}, \quad E(S,P) = 2P\frac{dG}{dI_v} \tag{25}$$

From (23), we obtain the following system of equations

$$V_4' + \frac{1}{2}\alpha' V_4 - \frac{i}{2}e^{\alpha-\beta}V_4\cot\theta + ie^\alpha(m - D)V_1 - Ee^\alpha V_3 = 0, \tag{26}$$

$$V_3' + \frac{1}{2}\alpha' V_3 + \frac{i}{2}e^{\alpha-\beta}V_3\cot\theta + ie^\alpha(m - D)V_2 - Ee^\alpha V_4 = 0, \tag{27}$$

$$V_2' + \frac{1}{2}\alpha' V_2 - \frac{i}{2}e^{\alpha-\beta}V_2\cot\theta - ie^\alpha(m - D)V_3 + Ee^\alpha V_1 = 0, \tag{28}$$

$$V_1' + \frac{1}{2}\alpha' V_1 + \frac{i}{2}e^{\alpha-\beta}V_1\cot\theta - ie^\alpha(m - D)V_4 + Ee^\alpha V_2 = 0, \tag{29}$$

where $\psi(\xi) = V_\delta(\xi)$ with $\delta = 1, 2, 3, 4$. Let us remark that in order to solve the set of equations (26)-(29), we must determine D(S,P) and E(S,P) and then S and P as functions of $e^{\alpha(\xi)}$.

3. General Solutions

In the preceding section we derived the fundamental equations for nonlinear spinor fields and metric functions. This part consists of solving the fundamental fields equations. By doing so, from (26)-(29), we write the equations for the functions $S = \bar{\psi}\psi$, $P = i\bar{\psi}\gamma^5\psi$ and $R = \bar{\psi}\bar{\gamma}^5\bar{\gamma}^1\psi$ as follows

$$S' + \alpha' S + 2Ee^{\alpha}R = 0 \tag{30}$$

$$R' + \alpha' R + 2(m-D)e^{\alpha}P + 2Ee^{\alpha}S = 0 \tag{31}$$

$$P' + \alpha' P + 2(m-D)e^{\alpha}R = 0 \tag{32}$$

It follows that the first integral for the system (30)-(32) is

$$P^2 - R^2 + S^2 = Ae^{-2\alpha(\xi)} = -\frac{A}{g_{11}}, \qquad A = const. \tag{33}$$

The expression (33) comes to confirm that the spinor field of elementary particles and the own gravitational are linked by nature. Also, the same relation proves that the consideration of the proper gravitational fied is very important in purpose to obtain solutions having the interest physics properties.

Let us now study the system of the invariant functions (30)- (32) considering massless elementary particles (m=0) and setting

$$P_0(\xi) = e^{\alpha}P(\xi); \qquad S_0(\xi) = e^{\alpha}S(\xi); \qquad R_0(\xi) = e^{\alpha}R(\xi). \tag{34}$$

Inserting (34) into (30)-(32) we get the following system in P_0; S_0 and R_0:

$$S_0' + 2Ee^{\alpha}R_0 = 0 \tag{35}$$

$$R_0' - 2De^{\alpha}P_0 + 2Ee^{\alpha}S_0 = 0 \tag{36}$$

$$P_0' - 2De^{\alpha}R_0 = 0 \tag{37}$$

The previous system leads to the differential equation:

$$P_0P_0' - R_0R_0' + S_0S_0' = 0. \tag{38}$$

The first integral of the equation (38) is

$$P_0^2 - R_0^2 + S_0^2 = A_0, \tag{39}$$

with A_0 being constant.
Then multiplying (35) by D(S,P) and (37) by E(S,P) and combining the results, we obtain

$$DS_0' + EP_0' = 0 \tag{40}$$

Taking into account the expressions for D(S,P) and E(S,P) (25) we find

$$2S\frac{dG}{dI_\nu}S_0' + 2P\frac{dG}{dI_\nu}P_0' = 0 \tag{41}$$

The equation (41) implies that

$$SS_0' + PP_0' = 0. \tag{42}$$

When we multiply (42) by $e^{\alpha(\xi)}$ and taking into account (18), we get the equation

$$S_0S_0' + P_0P_0' = 0 \tag{43}$$

which has the first integral given by the following relation

$$P_0^2 + S_0^2 = A_1^2 \tag{44}$$

where A_1 being integration constant.

Since $P_0 = e^{\alpha} P$ and $S_0 = e^{\alpha} S$, from (34), we deduce

$$I_v(\xi) = P^2 + S^2 = A_1^2 e^{-2\alpha(\xi)}. \tag{45}$$

By substituting (44) into (33), we obtain

$$R_0 = \sqrt{A_1^2 - A}, \quad where \quad A_1^2 - A > 0. \tag{46}$$

It is follows that

$$R(\xi) = A_2 e^{-\alpha(\xi)}, \quad with \quad A_2^2 = A_1^2 - A. \tag{47}$$

Taking into account (25) and $P_0 = \sqrt{A_1^2 - S_0^2}$ due to (44), the expression (35) becomes

$$\frac{dS_0}{d\xi} + 4\sqrt{A_1^2 - S_0^2}.A_2 \frac{dG}{dI_v} = 0 \tag{48}$$

The equation (48) has the first integral

$$S_0(\xi) = -A_1 \sin\Omega(\xi), \quad S(\xi) = -A_1 e^{-\alpha(\xi)} \sin\Omega(\xi) \tag{49}$$

where

$$\Omega(\xi) = 4A_2 \int \frac{dG}{dI_v} + A_3, \quad A_3 = const. \tag{50}$$

Introducing (49) into (44), we get

$$P_0(\xi) = A_1 \cos\Omega(\xi), \quad P(\xi) = A_1 e^{-\alpha(\xi)} \cos\Omega(\xi). \tag{51}$$

Using the spinor field equation in the form (23) and the conjugate one in the form (24), we obtain the following expression for T_1^1 from (16)

$$T_1^1 = mS - G(I_v) \tag{52}$$

In the following paragraph, we shall solve the Einstein equations by determining the expressions of the metric functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$. In view of $T_0^0 = T_2^2$ for (15), the difference of Einstein equations (5) and (7) implies

$$\beta'' - \gamma'' = e^{2\beta+2\gamma}. \tag{53}$$

The equation (53) may be transformed to Liouville equation type (G.N. Shikin, 1995). Then, the equation (53) has the solution

$$\beta(\xi) = \frac{H}{4}(1 + \frac{2}{C}) \ln \frac{H}{CT^2(h, \xi + \xi_1)} = (1 + \frac{2}{C})\gamma(\xi). \tag{54}$$

$$\gamma(\xi) = \frac{H}{4} \ln \frac{H}{CT^2(h, \xi + \xi_1)}. \tag{55}$$

H and C being integration constants.

The function T has the form

$$T(h, \xi + \xi_1) = \begin{cases} \frac{1}{h}\sinh[h(\xi + \xi_1)], h > 0 \\ (\xi + \xi_1), h = 0 \\ \frac{1}{h} sin[h(\xi + \xi_1)], h < 0 \end{cases} \tag{56}$$

h being an integration constant and ξ_1 another nonzero integration constant.
Taking into account (55), (56) and (3) we obtain the following expressions between the metric functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$

$$\alpha(\xi) = \frac{H}{2}(\frac{3}{2} + \frac{2}{C}) \ln \frac{H}{CT^2(h, \xi + \xi_1)}, \tag{57}$$

$$\beta(\xi) = \frac{2 + C}{4 + 3C}\alpha(\xi); \quad \gamma(\xi) = \frac{C}{4 + 3C}\alpha(\xi). \tag{58}$$

The equation (6) is the first integral of the equations (5) and (7). It is also a first order differential equation. Substituting (52) and (58) into (6), we get a following expression

$$(\alpha')^2 = \frac{(4+3C)^2}{(3C^2+8C+4)} e^{2\alpha} \left[e^{\frac{4+2C}{4+3C}(-\alpha)} - \chi(mS - G(I_v)) \right] \tag{59}$$

For a concrete analytique form of the function $G(I_v)$, we can define the metric function $\alpha(\xi)$ from (59). Considering massless elementary particles, i.e. $m = 0$, without losing the generality (Heisenberg W., 1966), the solution of the equation (59) is

$$\int \frac{d\alpha}{\frac{4+3C}{\sqrt{3C^2+8C+4}} e^{\alpha} \sqrt{e^{\frac{4+2C}{4+3C}(-\alpha)} + \chi G(I_v)}} = \pm(\xi + \xi_0), \qquad \xi_0 = const. \tag{60}$$

Let us now pass from $\alpha(\xi)$ to $I_v(\xi)$. Taking into account (45), we have

$$e^{\alpha(\xi)} = \frac{A_1}{\sqrt{I_v}}, \qquad d\alpha = -\frac{1}{2}\frac{dI_v}{I_v}. \tag{61}$$

Inserting (61) into (60), one gets the following solution

$$\int \frac{dI_v}{\frac{2A_1(4+3C)}{\sqrt{3C^2+8C+4}} \sqrt{I_v} \sqrt{(\frac{I_v}{A_1^2})^{\frac{2+C}{4+3C}} + \chi G(I_v)}} = \pm 2(\xi + \xi_0), \qquad \xi_0 = const. \tag{62}$$

Let us remark that, knowing the analytical form of $G(I_v)$, we can determine the analytic explicite form of the invariant function $I_v(\xi)$. Furthermore, we can determine the metric functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$ from the equation(45) and (58) respectively as well as the functions S, P, D(S,P) and E(S,P).
From (49) and (51), we define the functions D(S,P) and E(S,P) by the following relations

$$D(S,P) = 2S\frac{dG(I_v)}{dI_v} = -2C_1 \sin\Omega(\xi).e^{-\alpha(\xi)}\frac{dG(I_v)}{dI_v}, \tag{63}$$

$$E(S,P) = 2P\frac{dG(I_v)}{dI_v} = 2C_1 \cos\Omega(\xi).e^{-\alpha(\xi)}\frac{dG(I_v)}{dI_v}. \tag{64}$$

Introducing (63) and (64) into (26)-(29) and considering $m = 0$ and $W_\delta(\Omega(\xi)) = V_\delta(\xi)e^{\frac{\alpha}{2}}$, $\delta = 1, 2, 3, 4$ where $\Omega(\xi)$ is defined by (50), we get

$$\begin{aligned}
W_4'(\Omega) - \Phi(I_v)W_4 + ia\sin\Omega(\xi)W_1 - a\cos\Omega(\xi)W_3 &= 0, &(65)\\
W_3'(\Omega) + \Phi(I_v)W_3 + ia\sin\Omega(\xi)W_2 - a\cos\Omega(\xi)W_4 &= 0, &(66)\\
W_2'(\Omega) - \Phi(I_v)W_2 - ia\sin\Omega(\xi)W_3 + a\cos\Omega(\xi)W_1 &= 0, &(67)\\
W_1'(\Omega) + \Phi(I_v)W_1 - ia\sin\Omega(\xi)W_4 + a\cos\Omega(\xi)W_2 &= 0 &(68)
\end{aligned}$$

with $\Phi(I_v) = \frac{i}{8C_2\frac{dG}{dI_v}}\cot\theta$, $a = \frac{C_1}{2C_2}$ and $W_\delta'(\Omega) = \frac{dW_\delta}{d\Omega}$.
Here for simplicity, let us pass to the new functions $U_\delta(\Omega)$ in the system (65)-(68):

$$\begin{aligned}
U_1 &= W_1 + W_2 + W_3 + W_4, &(69)\\
U_2 &= W_1 + W_2 - W_3 - W_4, &(70)\\
U_3 &= W_1 - W_2 + W_3 - W_4, &(71)\\
U_4 &= W_1 - W_2 - W_3 + W_4. &(72)
\end{aligned}$$

Inserting (69)-(72) into (65)-(68) and summing the results, we obtain the following set equations:

$$\begin{aligned}
U_4'' + iU_4' - [a^2 + \phi^2(I_v)]U_4 + i\phi(I_v)U_2 &= 0 &(73)\\
U_3'' + iU_3' - [a^2 + \phi^2(I_v)]U_3 - i\phi(I_v)U_1 &= 0 &(74)\\
U_2'' + iU_2' - [a^2 + \phi^2(I_v)]U_2 + i\phi(I_v)U_4 &= 0 &(75)\\
U_1'' + iU_1' - [a^2 + \phi^2(I_v)]U_1 - i\phi(I_v)U_3 &= 0 &(76)
\end{aligned}$$

Summing (73) and (75) and setting $X(\Omega) = U_2 + U_4$, we have the differential equation

$$X'' + iX' - [a^2 + \phi^2(I_v) - i\phi(I_v)]X = 0 \tag{77}$$

yields the solution

$$U_2 + U_4 = K_2 e^{r_2\Omega(\xi)} + K_4 e^{r_4\Omega(\xi)}, \quad K_2, \quad K_4 = const \tag{78}$$

with

$$r_{2,4} = -\frac{i}{2} \pm \sqrt{a^2 - \frac{1}{4}\left(1 + \frac{\cot\theta}{4C_2\frac{dG}{dI_v}}\right)^2}. \tag{79}$$

Substracting equations (73) and (75) and using $Y(\Omega) = U_2 - U_4$, we obtain the equation

$$Y'' + iY' - [a^2 + \phi^2(I_v) + i\phi(I_v)]Y = 0. \tag{80}$$

The solution of the equation (80) is given by the following expression

$$U_2 - U_4 = K_2' e^{r_2'\Omega(\xi)} + K_4' e^{r_4'\Omega(\xi)}, \quad K_2', \quad K_4' = const. \tag{81}$$

where

$$r_{2,4}' = -\frac{i}{2} \pm \sqrt{a^2 - \frac{1}{4}\left(1 - \frac{\cot\theta}{4C_2\frac{dG}{dI_v}}\right)^2}. \tag{82}$$

Taking the realtions (78) and (81) into account, we deduce the expressions of the functions U_2 and U_4 as follows

$$U_2(\Omega) = \frac{1}{2}\left[K_2 e^{r_2\Omega(\xi)} + K_4 e^{r_4\Omega(\xi)} + K_2' e^{r_2'\Omega(\xi)} + K_4' e^{r_4'\Omega(\xi)}\right] \tag{83}$$

$$U_4(\Omega) = \frac{1}{2}\left[K_2 e^{r_2\Omega(\xi)} + K_4 e^{r_4\Omega(\xi)} - K_2' e^{r_2'\Omega(\xi)} - K_4' e^{r_4'\Omega(\xi)}\right] \tag{84}$$

We also obtain the differential equation of the function Z setting $Z(\Omega) = U_1 + U_3$ and combining the equations (74) and (76)

$$Z'' - iZ' - [a^2 + \phi^2(I_v) + i\phi(I_v)]Z = 0. \tag{85}$$

The equation (85) has solution

$$U_1 + U_3 = K_1 e^{r_1\Omega(\xi)} + K_3 e^{r_3\Omega(\xi)}, \quad K_1, \quad K_3 = const \tag{86}$$

with

$$r_{1,3} = \frac{i}{2} \pm \sqrt{a^2 - \frac{1}{4}\left(1 - \frac{\cot\theta}{4C_2\frac{dG}{dI_v}}\right)^2}. \tag{87}$$

Choosing $M(\Omega) = U_1 - U_3$, the difference of the equations (74) and (76) leads to

$$M'' - iM' - [a^2 + \phi^2(I_v) - i\phi(I_v)]M = 0. \tag{88}$$

Its solution is

$$U_1 - U_3 = K_1' e^{r_1'\Omega(\xi)} + K_3' e^{r_3'\Omega(\xi)}, \quad K_1', \quad K_3' = const \tag{89}$$

where

$$r_{1,3}' = \frac{i}{2} \pm \sqrt{a^2 - \frac{1}{4}\left(1 + \frac{\cot\theta}{4C_2\frac{dG}{dI_v}}\right)^2}. \tag{90}$$

The expressions of the functions $U_1(\Omega)$ and $U_3\Omega)$ are defined from (86) and (88) as follows

$$U_1(\Omega) = \frac{1}{2}\left[K_1 e^{r_1\Omega(\xi)} + K_3 e^{r_3\Omega(\xi)} + K_1' e^{r_1'\Omega(\xi)} + K_3' e^{r_3'\Omega(\xi)}\right] \tag{91}$$

$$U_3(\Omega) = \frac{1}{2}\left[K_1 e^{r_1\Omega(\xi)} + K_3 e^{r_3\Omega(\xi)} - K_1' e^{r_1'\Omega(\xi)} - K_3' e^{r_3'\Omega(\xi)}\right] \tag{92}$$

Let us remark that as the functions $U_\delta(\Omega)$ are known, we can pass to the functions $W_\delta(\Omega)$ and then to the functions $V_\delta(\xi)$. Thus, we have found the general solutions to the set equations (26)-(29) for $m = 0$ containing eight integration constants K_1, K_2, K_3, K_4, K_1', K_2', K_3' and K_4' and an arbitrary function $G(I_v)$.

In the following section, we analyze the equation (62) in details given the concrete form of nonlinear terms in spinor lagrangian.

4. Discussion

In the present section, we have studied in detail two distinct cases namely equations with power and polynomial nonlinearities.

4.1 Equations With Power Nonlinearity

This subsection is intended to study equations with power nonlinearity. By doing so, we consider the nonlinear terms in the form (Adomou A., R. Alvarado and G. N. Shikin1995):

$$G(I_v) = \lambda I_v^n, \qquad n \geq 2 \tag{93}$$

where λ and n are the parameter of nonlinearity and power of nonlinearity respectively. It is convenient to separately analyze the two cases $n = 2$ and $n > 2$.

• For n=2, we have Heisenberg-Ivanenko type nonlinear spinor field equation given by the following expression

$$ie^{-\alpha}\bar{\gamma}^1(\partial_\xi + \frac{1}{2}\alpha')\psi + \frac{i}{2}\bar{\gamma}^2 e^{-\beta}\psi \cot\theta - \left(m - 4\lambda I_v \bar{\psi}\psi\right)\psi - 4\lambda I_v(\bar{\psi}\psi)\psi = 0, \tag{94}$$

Substituting $G(I_v) = \lambda I_v^2$ into (62) with $\frac{2+C}{4+3C} = 1$, we obtain

$$I_v(\xi) = -\frac{1}{(8+6C)\sqrt{\frac{1+\chi\lambda A_1^2}{3C^2+8C+4}}(\xi+\xi_0)} \tag{95}$$

In this case, the energy density is given by the expression

$$T_0^0(\xi) = \frac{3\lambda(3C^2+8C+4)}{(8+6C)^2(1+\chi\lambda A_1^2)(\xi+\xi_0)} \tag{96}$$

From (96), the distribution of the spinor field energy density per unit invariant volume is

$$\begin{aligned} f(\xi) &= T_0^0(\xi)\sqrt{-3_g} \\ &= T_0^0(\xi)e^{\alpha(\xi)+2\beta(\xi)} \\ &= \frac{3\lambda(3C^2+8C+4)}{(8+6C)^2(1+\chi\lambda A_1^2)(\xi+\xi_0)}\sin\theta\exp\left[-\frac{H}{4}\ln\frac{H}{CT^2(h,\xi+\xi_1)}\right] \end{aligned} \tag{97}$$

Note that the set equations (26)-(29) possesses soliton-like solution when $G(I_v) = \lambda I_v^2$. Indeed $I_v(\xi)$ is a continuous and bounded function when $\xi \in [0, \xi_c]$, the quantities g_{00}, g_{11}, g_{22} and g_{33} are regular, the spinor field energy density is localized and the total energy $E = \int_0^{\xi_c} T_0^0(\xi)\sqrt{-3_g}d\xi$ is finite.

• Then, for $n > 2$, $L_N = G(I_v) = \lambda I_v^n$ we have

$$I_v(\xi) = \left\{\frac{1}{\sqrt{\chi\lambda A_1^2}\sinh\left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{2}{n-1}}; n > 2. \tag{98}$$

As for the energy densit, it is defined by the following expression

$$T_0^0(\xi) = \lambda(2n-1)\left\{\frac{1}{\sqrt{\chi\lambda A_1^2}\sinh\left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{2n}{n-1}}; n > 2. \tag{99}$$

From (98) and (99), when $\xi \longrightarrow 0$, $\xi_0 = 0$, we note that $I_v(\xi) \longrightarrow \infty$ and $T_0^0(\xi) \longrightarrow \infty$. This means that $T_0^0(\xi)$ has an infinite value when $\xi \longrightarrow 0$, $\xi_0 = 0$ and the initial set of equations has no solution with localized energy density. It should be noted that this result is in agreement with that found in (Adomou A. and Shikin G.N., 1998). It would be now interesting to consider in the sequel of this work the case where $n > 2$, $L_N = G(I_v) = \lambda I_v^n$ and $\lambda = -\Lambda^2 < 0$.

• For $n > 2$, $L_N = G(I_v) = \lambda I_v^n$ and $\lambda = -\Lambda^2 < 0$, we obtain

$$I_v(\xi) = \left\{ \frac{1}{\chi \Lambda^2 A_1^2 \cosh^2 \left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}} (n-1)(\xi + \xi_0) \right]} \right\}^{\frac{1}{n-1}} ; n > 2. \tag{100}$$

We remark that $I_v(\xi)$ is a continuous and bounded function when $\xi \in [0, \xi_c]$.
Then, the energy density is defined by the following expression

$$T_0^0(\xi) = -\Lambda^2(2n-1) \left\{ \frac{1}{\chi \Lambda^2 A_1^2 \cosh^2 \left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}} (n-1)(\xi + \xi_0) \right]} \right\}^{\frac{n}{n-1}} ; n > 2. \tag{101}$$

Let us consider the energy density per unit volume invariant:

$$\begin{aligned} \Gamma(\xi) &= T_0^0(\xi)\sqrt{-3_g} \\ &= -\Lambda^2(2n-1)\zeta(\xi) \left\{ \frac{1}{\chi \Lambda^2 A_1^2 \cosh^2 \left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}} (n-1)(\xi + \xi_0) \right]} \right\}^{\frac{n}{n-1}} \sin\theta \end{aligned} \tag{102}$$

where $\zeta(\xi) = \left[\frac{A_1}{\left\{ \frac{1}{\chi \Lambda^2 A_1^2 \cosh^2 \left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}} (n-1)(\xi+\xi_0) \right]} \right\}^{\frac{1}{n-1}}} \right]^{\frac{8+5C}{4+3C}}$.

Let us emphasize that the spinor field energy density per unit invariant volume $\Gamma(\xi)$ is localized and the total energy $E = \int_0^{\xi_c} \Gamma(\xi) d\xi$ has a finite quantity and negative in space when the nonlinearity parameter is negative (A. Adomou, R. Alvarado and G. N. Shikin, 1995) .
Let us find the explicit form of the functions $V_\delta(\xi)$, $\delta = 1, 2, 3, 4$. To this end, we must determine an explit form of $U_\delta(\xi)$, then $W_\delta(\xi)$ and subsequently $V_\delta(\xi) = W_\delta(\xi) e^{-\frac{1}{2}\alpha(\xi)}$. We obtain:

$$V_1(\xi) = \frac{1}{4\sqrt{A_1}} \left[K_1 e^{r_1 \Omega(\xi)} + K_2 e^{r_2 \Omega(\xi)} + K_3 e^{r_3 \Omega(\xi)} + K_4 e^{r_4 \Omega(\xi)} \right] \upsilon(\xi) \tag{103}$$

$$V_2(\xi) = \frac{1}{4\sqrt{A_1}} \left[K_1' e^{r_1' \Omega(\xi)} + K_2' e^{r_2' \Omega(\xi)} + K_3' e^{r_3' \Omega(\xi)} + K_4' e^{r_4' \Omega(\xi)} \right] \upsilon(\xi) \tag{104}$$

$$V_3(\xi) = \frac{1}{4\sqrt{A_1}} \left[K_1 e^{r_1 \Omega(\xi)} + K_3 e^{r_3 \Omega(\xi)} - K_2 e^{r_2 \Omega(\xi)} - K_4 e^{r_4 \Omega(\xi)} \right] \upsilon(\xi) \tag{105}$$

$$V_3(\xi) = \frac{1}{4\sqrt{A_1}} \left[K_1' e^{r_1' \Omega(\xi)} + K_3' e^{r_3' \Omega(\xi)} - K_2' e^{r_2' \Omega(\xi)} - K_4' e^{r_4' \Omega(\xi)} \right] \upsilon(\xi) \tag{106}$$

where

$$\upsilon(\xi) = \left\{ \frac{1}{\sqrt{\chi \Lambda^2 A_1^2} \cosh \left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}} (n-1)(\xi + \xi_0) \right]} \right\}^{\frac{1}{2n-2}} \tag{107}$$

and

$$\Omega(\xi) = -\frac{4nA_2\sqrt{3C^2+8C+4}}{\chi A_1^2 (4+3C)(n-1)} \tanh \left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}} (n-1)(\xi + \xi_0) \right] + A_3. \tag{108}$$

The expression of the function $\Omega(\xi)$ is obtained by substituting $G(I_v) = -\Lambda^2 I_v^n$ in (50).

4.2 Equations With Polynomial Nonlinearity

This subsection aims to analyze the nonlinear spinor field equation (60), when $\frac{1}{A_1} \longrightarrow 0$ and $G(I_v)$ is under the polynomial nonlinearities

$$G(I_v) = \lambda\sqrt{I_v}\left(\sqrt{\frac{I_v}{\omega_0^2}} - 1\right)^2\left(2 - \sqrt{\frac{I_v}{\omega_0^2}}\right). \tag{109}$$

The function $G(I_v)$, in the equality (109), admits three roots $I_v = 0$, $I_v = \omega_0^2$ and $I_v = 4\omega_0^2$. By Substituting (109) into (60), we obtain the expression of the invariant function $I_v(\xi)$ as follows:

$$I_v(\xi) = \omega_0\left[1 + \frac{1}{\cosh(\zeta_0(\xi+\xi_0))}\right]^2 \tag{110}$$

Using the relation (110), for the metric function $-g_{11} = e^{2\alpha(\xi)}$, we find the following expression:

$$e^{2\alpha(\xi)} = \frac{A_1}{I_v} = \frac{A_1\cosh^2(\zeta_0(\xi+\xi_0))}{\omega_0\left[1+\cosh(\zeta_0(\xi+\xi_0))\right]^2} \tag{111}$$

From (111), one notes that for $\xi = \xi_0 = 0$, $e^{2\alpha(\xi)} = \frac{A_1}{4\omega_0}$ and $e^{2\alpha(\xi)} = \frac{A_1\cosh^2\zeta_0\xi_C}{\omega_0[1+\cosh\zeta_0\xi_C]^2}$ for $\xi = \xi_C$ and $\xi_0 = 0$. The function g_{11} is regular and stationary. In reason of the equality (58) traducing the relation between $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$, the functions g_{22} and g_{33} are also regular and stationary. Therefore, the metric is also regular and stationary for $\xi \in [0, \xi_C]$.

Now let us get to the energy density, the distribution energy density per unit invariant volume and the field total energy . Taking into account the relations (15) and (110) and the usual algebras manipulations, the energy density of the spinor field is defined by:

$$T_0^0(\xi) = \lambda\sqrt{\omega_0}\left(\frac{I_v}{\omega_0}\right)\left(\sqrt{\frac{I_v}{\omega_0}} - 1\right)\left(5 - 3\sqrt{\frac{I_v}{\omega_0}}\right) \tag{112}$$

that is

$$T_0^0(\xi) = \frac{\lambda\sqrt{\omega_0}}{\cosh(\zeta_0(\xi+\xi_0))}\left(1 + \frac{1}{\cosh(\zeta_0(\xi+\xi_0))}\right)^2\left(2 - \frac{3}{\cosh(\zeta_0(\xi+\xi_0))}\right) \tag{113}$$

The analysis of the expression (113) shows that the energy density of the spinor field T_0^0 is negative, localized and alterning. Therefore the energy density per unit invariant volume is regular localized function. Thus total energy $E = \int_0^{\xi_c} T_0^0(\xi)\sqrt{-3_g}d\xi$ is finite as the integrand is positive.

Note that our solution describes a nonlinear spinor field configuration with regular localized energy density T_0^0, positive energy E and regular metric.

5. Total Charge and Total Spin

The following paragraph addresses to the total charge and the total spin. To this end, according to the expressions (103)-(106), let us write the components of the spinor current vector $j^\mu = \bar{\psi}\gamma^\mu\psi$ (Zhelnorovich V. A., 1982):

$$j^0 = \bar{F}(\Omega).e^{-\alpha-\gamma} \tag{114}$$

Since the configuration is static, the another components j^1, j^2 and j^3 are nulle. The component j^0 determines the charge density of the spinor field given by the expression:

$$\varrho(\xi) = (j_0 j^0)^{\frac{1}{2}} = \bar{F}(\Omega).e^{-\alpha} \tag{115}$$

From (61), (95) and (109), we have

$$\varrho(\xi) = \bar{F}(\Omega)A_1^{-1}\left\{\frac{1}{\chi\Lambda^2A_1^2\cosh^2\left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{1}{2(n-1)}} \tag{116}$$

The charge density per unit invariant volume of the spinor field is defined by:

$$\begin{aligned} q(\xi) &= \varrho(\xi)\sqrt{-3_g} \\ &= \bar{F}(\Omega)A_1^{-1}\left\{\frac{1}{\chi\Lambda^2A_1^2\cosh^2\left[\frac{(4+3C)}{\sqrt{3C^2+8C+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{1}{2(n-1)}}\zeta(\xi)\sin\theta \end{aligned} \tag{117}$$

The total charge of the spinor field is:

$$Q(\xi) = \int_0^{\xi_c} q(\xi)d\xi \tag{118}$$

ξ_C being the center of the field configuration.
It follows that from (117) and (118) the charge density and the charge density per unit invariant volume are localized because the integrands $\varrho(\xi)$ and $q(\xi)$ are continuous and limited functions when $\xi \in [0, \xi_c]$. Moreover, the total charge is a finite quantity when the nonlinear term of the spinor field is choosen under the form $L_N = -\Lambda^2 I_v^n$.
Then, the general expression of the spin tensor of the spinor field reads (N.N Bogoliubov and Shirkov D.V., 1976)

$$S^{\mu\nu,\varepsilon} = \frac{1}{4}\bar{\psi}\left\{\gamma^\varepsilon\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\varepsilon\right\}\psi \tag{119}$$

where $\sigma^{\mu\nu} = \left(\frac{i}{2}\right)\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right]$. The spatial density of the spin vector $S^{ik,0}$, i, k=1, 2, 3, is given by the following expression

$$S^{ik,0} = \frac{1}{4}\bar{\psi}\left\{\gamma^0\sigma^{ik} + \sigma^{ik}\gamma^0\right\}\psi = \frac{1}{2}\bar{\psi}\gamma^0\sigma^{ik}\psi \tag{120}$$

From (115), we find the components of the spin tensor of the spinor field as follows

$$S^{12,0} = 0, \qquad S^{13,0} = 0, \qquad S^{23,0} = \frac{1}{2}\bar{V}_\delta\gamma^0\sigma^{23}V_\delta.e^{-\alpha} \tag{121}$$

We remark that the only nontrivial component of the spin tensor is $S^{23,0}$. It defines the chronometric invariant spin tensor $S_{chI}^{23,0}$ and the projection of spin vector S_1 on ξ axis having the forms

$$S_{chI}^{23,0} = \left(S_{23,0}.S^{23,0}\right)^{\frac{1}{2}} = \bar{F}(\Omega)e^{-\alpha} \tag{122}$$

$$S_1 = \int_0^{\xi_c}\bar{F}(\Omega)e^{-\alpha}\sqrt{3_{-}g}d\xi \tag{123}$$

Let us remark that the integrands in (118) and (123) coincide. Thus, the total spin is also limited quantity as the total charge. As result, the geometry of the metric, the nonlinearity of the spinor field and the own gravitaional field play an important role in order to obtain a soliton-like solutions with limited total charge and total spin. Theses results are compatible with experimentaly results obtained in the accelerator particles.

6. Concluding Remarks

In this paper, we have obtained analytics spherical symmetric solutions to the spinor and gravitational fields equations which are regular with a localized energy density and finite total energy. Equations with power and polynomial nonlinearities are thoroughly scrutinized. Our solutions describe a nonlinear spinor field configuration with localized energy density T_0^0, postive total energy E in the equations with polynomial nonlinearity case and negative in the equations with power nonlinearity case. In the forthcoming paper, we will present the numerical solutions of the solutions obtained here in graphical form.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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