

Burr X Exponential – G Family of Distributions: Properties and Application

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Authors' contributions

This work was carried out in collaboration among all authors. Author AAS designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author SISD modified the derivatives and corrected the draft of the manuscript. Author IA managed the literature searches and managed the analyses of the study. Author YMB proofread the mathematical derivatives. All authors read and approved the final manuscript.

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Abstract

In this paper, we developed a new class of continuous distributions called Burr X Exponential-G Family. Also, we obtained sub-models of this family of distributions such as Burr X Exponential-Rayleigh (BXE-R) and Burr X Exponential Lomax (BXE-Lx) distributions; by showing their respective densities functions. Some structural properties of the proposed family of distributions were derived such as moment, moment generating function, probability weighted moment, renyi entropy and order statistics. We estimate the parameters of the model by using Maximum Likelihood methods. Finally, the results obtained are validated using two real data sets. The results show that BXE-Lx distribution provides better fit in the data sets than some other well known distributions. However, this new family of distributions will serve as an additional generator for developing new sub models to modeling positive real data sets.

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1 Introduction

Probability distributions have been used to model data in various disciplines such as engineering, medicine, Biological science, among other. However, at least one parameter is added to these probability distributions to boost their flexibility. There are methods of adding parameters to distributions proposed by several statisticians; this method of adding parameter to distribution is pioneered by Gupta et al. [1], who proposed the exponentiated-G class, by raising the cumulative distribution function (cdf) to a positive power parameter. Although, several methods for generating new families of distributions have been studied in the recent years; these methods were introduced by Alzaatreh et al. [2]. However, statisticians often define new families of probability distributions that extend well-known distributions in order to provide great flexibility in modeling data in practice. That is, several distributions have been constructed by extending common families of continuous distributions. These generalized distributions give more flexibility by adding at least one shape or scale parameter to the baseline model.

Moreover, statisticians aimed at developing families of distributions in order to produce distributions that can be skewed to the right, left, unimodal, J or reversed J shape or provide persistently better fits that can outfit other existing distributions with the same underlying model. In fact, so many families of continuous probability distributions have been developed with one or two parameters; for example marshall-olkin-G by Marshall and Olkin [3], beta generalized-G by Eugene et al. [4], exponentiated generalized-G by Cordeiro et al. [5], new weibull-G by Tahir et al. [6], burr X-G by Haitham et al. [7], generalized odd generalized exponential family by Alizadeh et al. [8], lomax-G by Cordeiro et al. [9], Type I general exponential class of distributions by Hamedani et al. [10], the transmuted weibull-G family of distributions by Morad et al. [11] and the transmuted transmuted-G family by Mansour et al. [12].

Also, most recently new families of continuous probability distributions have been introduced and developed with at least two parameters such as; transmuted exponentiated generalized-G by Yousof et al. [13], kumaraswamy marshall-olkin-G by Alizadeh et al. [14], zografos-balakrishnan odd log-logistic family by Gauss et al. [15], the gamma-weibull-G family of distributions by Broderick et al. [16], kumaraswamy transmuted-G by Afify et al. [17], the transmuted weibull G family by Alizadeh et al. [18], exponentiated transmuted family by Merovc et al. [19], the exponentiated generalized-G Poisson family by Aryal and Yousof [20], the exponentiated kumaraswamy-G class by Silva et al. [21], beta transmuted-H by Afify et al. [22], topp-leone odd log-logistic family by de Brito et al.[23], the extended weibull-G family of distributions by Korkmaz [24], the exponentiated generalized topp leone-G Family of distributions by Reyad, et al. [25], the topp leone exponentiated G family of distributions by Ibrahim, et al. [26], among others.

However, this paper developed; a new family of continuous distributions with additional two parameters; called the Burr X Exponential-G family (BXE-G) and studied some of its mathematical properties. The remaining sections of the paper; is organized as follows. The cdf and pdf of the new family of distributions are defined in section 2. Useful expansion of BXE-G is discussed in section 3. Some mathematical properties of the BXE-G are discussed in section 4. The maximum likelihood estimates and the observed information matrix obtained for the parameters of BXE-G is presented in section 5. Some special sub-models of the BXE-G family are introduced in section 6. Applications to two real data sets for the sub-models are shown in Section 7 and Section 8 concludes the paper.

2 The New Burr X Exponential Family of Distributions

In this section, we presented the method used to derive this new family of distributions called Burr X Exponential – G family and also defined its cdf, pdf and reliability analysis.

2.1 Burr X – G family of distributions

Given the baseline cdf $G(x; \phi)$ and pdf $g(x; \phi)$, with parameter vector ϕ , Haitham et al. [7] introduced and developed the Burr X – G (BX-G) family of distributions with cdf and pdf given by

$$F_{BX-G}(x; \theta, \phi) = \left[1 - \exp \left\{ - \left(\frac{G(x; \phi)}{1 - G(x; \phi)} \right)^2 \right\} \right]^\theta \tag{1}$$

and

$$f_{BX-G}(x; \theta, \phi) = \frac{2\theta g(x; \phi)G(x; \phi)}{(1 - G(x; \phi))^3} \exp \left\{ - \left(\frac{G(x; \phi)}{1 - G(x; \phi)} \right)^2 \right\} \left[1 - \exp \left\{ - \left(\frac{G(x; \phi)}{1 - G(x; \phi)} \right)^2 \right\} \right]^{\theta-1} \tag{2}$$

respectively for $x > 0, \theta > 0$, where $\bar{G}(x; \phi) = 1 - G(x; \phi)$ and ϕ is the vector of parameters for the baseline cdf.

2.2 Exponential – G family of distributions

Given the baseline cdf $G(x; \beta)$ and pdf $g(x; \beta)$, with parameter vector β , Fatou and Ibrahim[27] suggested the Exponential – G (E-G) family of distributions with cdf and pdf given by,

$$F_{E-G}(x; \lambda, \beta) = 1 - \exp \left\{ - \lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} \tag{3}$$

$$f_{E-G}(x; \lambda, \beta) = \frac{\lambda g(x; \beta)}{[\bar{G}(x; \beta)]^2} \exp \left\{ - \lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} \tag{4}$$

2.3 Burr X exponential – G family of distributions

The cdf and pdf of the Burr X Exponential – G family of distributions (BXE-G) are given by the following lemma:

$$F_{BXE-G}(x; \theta, \lambda, \beta) = \left[1 - \exp \left\{ - \left(\frac{F_{E-G}(x; \lambda, \beta)}{F_{E-G}(x; \lambda, \beta)} \right)^2 \right\} \right]^\theta \tag{5}$$

and

$$f_{BXE-G}(x; \theta, \lambda, \beta) = \frac{2\theta f_{E-G}(x, \lambda, \beta)}{(F_{E-G}(x; \lambda, \beta))^3} (F_{E-G}(x; \lambda, \beta)) \exp \left\{ - \left(\frac{F_{E-G}(x; \lambda, \beta)}{F_{E-G}(x; \lambda, \beta)} \right)^2 \right\} \times \left[1 - \exp \left\{ - \left(\frac{F_{E-G}(x; \lambda, \beta)}{F_{E-G}(x; \lambda, \beta)} \right)^2 \right\} \right]^{\theta-1} \tag{6}$$

respectively.

Proof:

$$F_{BXE-G}(x; \theta, \lambda, \beta) = \int_0^{T(x)} 2\theta f_{E-G}(t; \lambda, \beta) \frac{F_{E-G}(t; \lambda, \beta)}{[\overline{F}_{E-G}(t; \lambda, \beta)]^3} e^{-T(t)^2} (1 - e^{-T(t)^2})^{\theta-1} dt \tag{7}$$

where; $T(t) = \frac{F_{E-G}(t; \lambda, \beta)}{\overline{F}_{E-G}(t; \lambda, \beta)}$ and $\frac{dT(t)}{dt} = \frac{f_{E-G}(t; \lambda, \beta)}{[\overline{F}_{E-G}(t; \lambda, \beta)]^2}$

So $F_{BXE-G}(x; \theta, \lambda, \beta) = \int_0^{T(x)} 2\theta T(t) e^{-T(t)^2} [1 - e^{-T(t)^2}]^{\theta-1} dT(t)$

Let $\rho = 1 - e^{-T(t)^2}$ then, $\frac{d\rho}{dT(t)} = 2T(t) e^{-T(t)^2}$

If $T(t) = 0$, then $\rho = 0$. Also, if $T(t) = T(x)$, then $\rho = 1 - e^{-T(x)^2}$

There fore

$$F_{BXE-G}(x; \theta, \lambda, \beta) = \int_0^{1-e^{-T(x)^2}} \theta \rho^{\theta-1} d\rho = [\rho^\theta]_0^{1-e^{-T(x)^2}} = \left[1 - \exp \left\{ - \left(\frac{F_{E-G}(x; \lambda, \beta)}{\overline{F}_{E-G}(x; \lambda, \beta)} \right)^2 \right\} \right]^\theta$$

hence the proof.

From equation (7), the pdf of the BXE-G family is given as:

$$f_{BXE-G}(x; \theta, \lambda, \beta) = \frac{2\theta f_{E-G}(x; \lambda, \beta)}{[\overline{F}_{E-G}(x; \lambda, \beta)]^2} T(x) e^{-T(x)^2} (1 - e^{-T(x)^2})^{\theta-1} \tag{8}$$

where $T(x) = \frac{F_{E-G}(x; \lambda, \beta)}{\overline{F}_{E-G}(x; \lambda, \beta)} = \frac{1 - \exp \left\{ -\lambda \left(\frac{G(x, \beta)}{\overline{G}(x, \beta)} \right) \right\}}{\exp \left\{ -\lambda \left(\frac{G(x, \beta)}{\overline{G}(x, \beta)} \right) \right\}} = \exp \left\{ \lambda \left(\frac{G(x, \beta)}{\overline{G}(x, \beta)} \right) \right\} - 1$ (9)

From equation (8),

$$\begin{aligned} \frac{2\theta f_{E-G}(x; \lambda, \beta)}{[\overline{F}_{E-G}(x; \lambda, \beta)]^2} &= \frac{2\theta \lambda g(x, \beta)}{[\overline{G}(x, \beta)]^2} \exp \left\{ -\lambda \left(\frac{G(x, \beta)}{\overline{G}(x, \beta)} \right) \right\} \frac{1}{\left[\exp \left\{ -\lambda \left(\frac{G(x, \beta)}{\overline{G}(x, \beta)} \right) \right\} \right]^2} \\ &= \frac{2\theta \lambda g(x, \beta) \exp \left\{ \lambda \left(\frac{G(x, \beta)}{\overline{G}(x, \beta)} \right) \right\}}{[\overline{G}(x, \beta)]^2} \end{aligned} \tag{10}$$

Putting (9) and (10) into (8), yields

$$f_{BXE-G}(x; \theta, \lambda, \beta) = \frac{2\theta\lambda g(x, \beta) \exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\}}{[\bar{G}(x, \beta)]^2} \left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1 \right) \times \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\} \left[1 - \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\}\right]^{\theta-1} \quad (11)$$

which is the density function of the BXE- G family of distributions. From equation (5) the corresponding cdf is

$$F_{BXE-G}(x; \theta, \lambda, \beta) = \left[1 - \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\}\right]^{\theta} \quad (12)$$

when $x \rightarrow 0$, $F_{BXE-G}(0; \theta, \lambda, \beta) = \left[1 - \exp\left\{-\left(\exp\{0\} - 1\right)^2\right\}\right]^{\theta} \Rightarrow 1 - 1 = 0$

when $x \rightarrow \infty$, $F_{BXE-G}(\infty; \theta, \lambda, \beta) = \left[1 - \exp\left\{-\left(\exp\{\infty\} - 1\right)^2\right\}\right]^{\theta} \Rightarrow [1 - \exp(-\infty)]^{\theta} \Rightarrow 1$

And differentiating $F_{BXE-G}(\infty; \theta, \lambda, \beta)$ with respect to x will yield equation (11), which is the probability density function of the BXE-G family of distributions.

Henceforth, a random variable X with density function given in equation (11) follows, $BXE - G(x, \phi)$ where $\phi = (\theta, \lambda, \beta)$ is a vector of parameters', the survival function $S(x, \phi)$, hazard function $h(x, \phi)$, inverse hazard function $\tau(x, \phi)$ and cumulative hazard function $H(x, \phi)$ for BXE - G family are given by

$$S(x; \phi) = 1 - \left[1 - \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\}\right]^{\theta} \quad x \in R \quad (13)$$

$$h(x; \phi) = \frac{\left(2\theta\lambda g(x; \beta) \left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right) \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\}\right)}{\left(\bar{G}(x; \beta)\right)^2 \left(\exp\left\{-\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\}\right) \left[1 - \left[1 - \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\}\right]^{\theta}\right)} \times \left[1 - \exp\left\{-\left(\exp\left\{\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)}\right)\right\} - 1\right)^2\right\}\right]^{\theta-1} \quad x \in R \quad (14)$$

$$\tau(x; \phi) = \frac{\left(2\theta\alpha g(x; \beta) \left(\exp \left\{ \lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} - 1 \right) \exp \left\{ - \left(\exp \left\{ \lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} - 1 \right)^2 \right\} \right)}{\left(\bar{G}(x; \beta) \right)^2 \left(\exp \left\{ -\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} \right)} \times \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} - 1 \right)^2 \right\} \right]^{-1} \quad x \in R \quad (15)$$

$$H(x; \phi) = -\ln \left[1 - \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} - 1 \right)^2 \right\} \right]^\theta \right] \quad (16)$$

3 Linear Representation

In this section, we introduced a useful representation for the BXE-G pdf and cdf. Using generalized binomial and Taylor series expansions in equation (6), thus if $|x| < 1$ and $k > 0$ is a real non integer, the power series holds:

$$(1-x)^{k-1} = \sum_{a=0}^{\infty} \frac{(-1)^a \Gamma(k)}{a! \Gamma(k-a)} x^a \quad (17)$$

Applying the idea of equation (17) on the last term in (6), this becomes;

$$f_{BXE-G}(x; \phi) = 2\theta \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(\theta)}{p! \Gamma(\theta-p)} \frac{f_{E-G}(x; \lambda, \beta)}{\left(\bar{F}_{E-G}(x; \lambda, \beta) \right)^3} \left[F_{E-G}(x; \lambda, \beta) \right] \times \exp \left\{ - (p+1) \left(\frac{F_{E-G}(x; \lambda, \beta)}{\bar{F}_{E-G}(x; \lambda, \beta)} \right)^2 \right\} \quad (18)$$

Applying the power series to the term $\exp \left\{ - (p+1) \left(\frac{F_{E-G}(x; \lambda, \beta)}{\bar{F}_{E-G}(x; \lambda, \beta)} \right)^2 \right\}$, equation (18) becomes:

$$= 2\theta \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q} (p+1)^q \Gamma(\theta)}{p! q! \Gamma(\theta-p)} \frac{f_{E-G}(x; \lambda, \beta) \left(F_{E-G}(x; \lambda, \beta) \right)^{2q+1}}{\left(\bar{F}_{E-G}(x; \lambda, \beta) \right)^{2q+3}} \quad (19)$$

Also consider equation (17) and apply it on equation (19) becomes

$$= 2\theta \lambda \sum_{p,q,r=0}^{\infty} \frac{(-1)^{p+q+r} (p+1)^q \Gamma(\theta) [(2q+1)!]}{p! q! r! \Gamma(\theta-p) [2q+1-r]!} \frac{g(x; \beta)}{\left(\bar{G}(x; \beta) \right)^2} \left(\exp \left\{ -\lambda \left(\frac{G(x, \beta)}{\bar{G}(x, \beta)} \right) \right\} \right)^{r-2(q+1)} \quad (20)$$

$$= 2\theta\lambda \sum_{p,q,r,s=0}^{\infty} \frac{(-1)^{p+q+r+s} (p+1)^q \Gamma(\theta) [2q+1]! [r-2(q+1)]^s \lambda^s g(x;\phi) (G(x;\beta))^s}{p!q!r!s!\Gamma(\theta-p)[2q+1-r]!} \frac{1}{(\overline{G}(x;\beta))^{2+s}} \quad (21)$$

Consider the series expansion

$$(1-x)^{-k} = \sum_{b=0}^{\infty} \frac{\Gamma(k+b)}{b! \Gamma(k)} x^b, \quad |x| < 1, b > 0 \quad (22)$$

$$= 2\theta\lambda^{s+1} \sum_{p,q,r,s,t=0}^{\infty} \frac{(-1)^{p+q+r+s} (p+1)^q \Gamma(\theta) [2q+1]! [r-2(q+1)]^s \Gamma(2+s+t) [s+t+1]}{p!q!r!s!t!\Gamma(\theta-p)\Gamma(2+s)[2q+1-r]! [s+t+1]} g(x;\beta) (G(x;\beta))^{s+t}$$

$$f(x) = \sum_{p,q,r,s,t=0}^{\infty} \delta_{p,q,r,s,t} \psi_{s+t+1}(x) \quad (23)$$

Where

$$\delta_{p,q,r,s,t} = 2\theta\lambda^{s+1} \frac{(-1)^{p+q+r+s} (p+1)^q \Gamma(\theta) [2q+1]! [r-2(q+1)]^s \Gamma(2+s+t)}{p!q!r!s!t!\Gamma(\theta-p)\Gamma(2+s)[2q+1-r]! [s+t+1]}$$

and $\psi_{s+t+1}(x) = (s+t+1)g(x;\beta)G(x;\beta)^{s+t}$ is the exponentiated-G distribution with power parameter $s+t+1$. From (23), the cdf of the BXE-G family can also be expressed as a mixture of exp-G cdfs given by with respect to x , we have:

$$F(x) = \sum_{p,q,r,s,t=0}^{\infty} \delta_{p,q,r,s,t} \psi_{s+t+1}(x) \quad (24)$$

Where $\psi_{s+t+1}(x) = G(x)^{s+t+1}$

4 Mathematical Properties

In this section, we presented some mathematical properties of the BXE-G family such as incomplete moments, moment generating function, Rényi entropy, order statistics and probability weighted moment.

4.1 Raw, incomplete moments and moment generating function

Suppose X is a random variable with BXE-G distribution, then the raw moment, say μ'_n , is given by

$$\begin{aligned} \mu'_n &= E(x^n) = \int_{-\infty}^{\infty} x^n f_{BXE-G}(x) dx \\ &= \sum_{p,q,r,s,t=0}^{\infty} \delta_{p,q,r,s,t} \int_{-\infty}^{\infty} x^n \psi_{s+t+1}(x) dx \end{aligned} \quad (25)$$

$$= \sum_{p,q,r,s,t=0}^{\infty} \delta_{p,q,r,s,t}^* \psi_{n,s+t} \tag{26}$$

Where:

$$\delta_{p,q,r,s,t}^* = (s+t+1) \delta_{p,q,r,s,t}$$

and

$\psi_{n,s+t} = \int_{-\infty}^{\infty} x^n g(x) G(x)^{s+t} dx$ is the probability weighted moment of the baseline distribution. For integer values of n and $\mu = \mu'_1 = E(x)$, we can obtain the BXE – G central moment of the BXE-G distribution, say μ_n to be

$$\mu_n = E(X - \mu)^n = \sum_{i=0}^n \binom{n}{i} \mu'_i (-\mu)^{n-i} \tag{27}$$

From (27), the measures of skewness and kurtosis of the BXE-G distribution can be obtained as

$$\text{Skewness (X)} = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1^3}{(\mu'_2 - \mu_1^2)^{3/2}} \quad \text{and} \quad \text{Kurtosis (X)} = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6\mu_1^2\mu'_3 - 3\mu_1^4}{\mu'_2 - \mu_1^2}$$

The r th incomplete moment of X denoted by $m_r(\varpi)$ is

$$m_n = \int_{-\infty}^{\infty} x^n f_{BXE-G}(x) dx = \sum_{p,q,r,s,t=0}^{\infty} \delta_{p,q,r,s,t}^* \psi_{n,s+t}$$

The moment generating function, say $M_x(t)$ of the BXE-G distribution can be obtained as follows:

$$M_x(t) = E(e^{tx}) = \sum_{p,q,r,s,t,v=0}^{\infty} \frac{t^v}{v!} \delta_{p,q,r,s,t}^* \psi_{n,s+t} \tag{28}$$

4.2 Renyi entropy

It plays an essential role in information theory. It has been used in applied statistics, queuing theory and reliability theory. It is used as indices of diversity and quantifies the uncertainty or randomness of a system. The Renyi Entropy for BXE-G family of distribution is defined by

$$I_R(v) = (1-v)^{-1} \log \int_{-\infty}^{\infty} f^v(x) dx \quad \text{for } v > 0 \text{ and } v \neq 1$$

From equation (17) we obtain

$$f^v(x) = \sum_{p,q,r,s,t=0}^{\infty} \xi_{p,q,r,s,t} g(x)^v G(x)^{2(s+v)}$$

Where

$$\xi_{p,q,r,s,t} = \frac{(2\theta\lambda^{s+1})^v (-1)^{p+q+r+s} (p+v)^q (v\theta-v)!(2q+v)! [\lambda(r-2\{q+v\})]^s \Gamma(s+2v+t)}{p!q!r!s!t!(v\theta-v-p)!(2q+v-c)\Gamma(s+2v)}$$

Therefore the Renyi Entropy for the BXE-G family is defined as

$$I_R(v) = (1-v)^{-1} \log \left\{ f^v(x) = \sum_{p,q,r,s,t=0}^{\infty} \xi_{p,q,r,s,t} \int_{-\infty}^{\infty} g(x;\beta)^v G(x;\beta)^{2(s+v)} dx \right\} \tag{29}$$

4.3 Order statistics

Order statistics plays a very significant role in statistics. Let $X_{1;n} \leq X_{2;n} \leq \dots \leq X_{n;n}$ be the ordered sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{k;n}$, the k th order statistics is given by

$$f_{i;n} = \frac{f(x)}{\beta(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-1}{j} F^{j+i-1}(x)$$

From equation pdf and cdf above we have

$$f(x) F^{j+i-1}(x) = \sum_{p,q,r,s,t,u=0}^{\infty} \eta_{p,q,r,s,t,u} v_{t+u+1}$$

Consequently for the BXE-G Family we have

$$f_{i;n} = \sum_{j=0}^{n-i} \sum_{p,q,r,s,t,u=0}^{\infty} \eta_{j,p,q,r,s,t,u} v_{t+u+1} \tag{30}$$

Where

$$\eta_{j,p,q,r,s,t,u} = \frac{2\theta\lambda^{s+1} (-1)^{j+p+q+r+s+t} (p+q+1)^r \Gamma(\theta) [\theta(j+i-1)]! (2r+1)! (2+u)! [s-2(1+r)]^t (n-i)!}{j! p! q! r! s! t! u! 2! (n-i-j)! \beta(i, n-i+1) \Gamma(\theta-p) [\theta(j+i-1)-q]! (2r+1-s)! (t+u+1)}$$

and

$$v_{t+u+1} = (t+u+1) g(x) G(x)^{t+u}$$

4.4 Probability weighted moment

The probability weighted moment (PWM) is a useful approach for estimating the model parameters of that distribution whose inverse form can or cannot be expressed in explicit form.

The $(j+k)^{th}$ PWM of X has the BXE-G distribution, say $M_{j,k}$ is given by

$$M_{j,k} = E[X^j F^k(x)] = \int_{-\infty}^{\infty} x^j F(x)^k f(x) dx = \int_{-\infty}^{\infty} x^j \frac{2\theta\lambda}{(\overline{F}_{E-G}(x; \lambda, \beta))^2} \frac{g(x; \beta)}{(\overline{G}(x; \beta))^2} [F_{E-G}(x; \lambda, \beta)] \exp\left\{-\left(\frac{F_{E-G}(x; \lambda, \beta)}{\overline{F}_{E-G}(x; \lambda, \beta)}\right)^2\right\} \times \left[1 - \exp\left\{-\left(\frac{F_{E-G}(x; \lambda, \beta)}{\overline{F}_{E-G}(x; \lambda, \beta)}\right)^2\right\}\right]^{\theta(k+1)-1} dx$$

After series of mathematical manipulation we obtained

$$M_{j,k} = E[X^j F^k(x)] = \sum_{p,q,r,s,t=0}^{\infty} \pi_{p,q,r,s,t} \zeta_{j,s+t} \tag{31}$$

Where

$$\pi_{p,q,r,s,t} = 2\theta\lambda^{s+1} \frac{(-1)^{p+q+r+s} (p+1)^q \Gamma(\theta(k+1)) [2q+1]! [r-2(q+1)]^s \Gamma(2+s+t) [s+t+1]}{p!q!r!s!t! \Gamma(\theta(k+1)-p) \Gamma(2+s) [2q+1-r]! [s+t+1]}$$

and

$$\zeta_{j,s+t}(x) = \int_{-\infty}^{\infty} x^j g(x; \beta) G(x; \beta)^{s+t} dx$$

5 Parameter Estimation

In the literature, parameter estimation is proposed with several approaches, but the maximum likelihood method is the most commonly used among others. Therefore, the maximum likelihood estimators of the unknown parameters of the BXE-G family from complete samples are determined. Let X_1, \dots, X_n be observed values from BXE – G distributions with vector of parameters ϕ . The Log-likelihood function can be expressed as

$$l(\phi) = n \log(2) + n \log(\theta) + n \log(\lambda) + \lambda \sum_{i=1}^n m(x_i; \beta) - 2 \sum_{i=1}^n \log(\overline{G}(x_i; \beta)) + \sum_{i=1}^n \log(g(x_i; \beta)) + \sum_{i=1}^n \log(-1 + \exp\{\lambda m(x_i; \beta)\}) - \sum_{i=1}^n (H(x_i; \beta))^2 + (\theta - 1) \sum_{i=1}^n \log(1 - \exp\{-(H(x_i; \beta))^2\}) \tag{32}$$

The component score vector

$$U(\phi) = \frac{\partial l}{\partial \phi} = \left(\frac{\partial l}{\partial \theta}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \beta} \right)^T$$

are

$$U_{\theta} = \frac{n}{\theta} + \sum_{i=1}^n \log \left[1 - \exp \left\{ - (H(x; \beta))^2 \right\} \right]$$

$$U_{\lambda} = \frac{n}{\lambda} + \sum_{i=1}^n m(x; \beta) + \frac{\sum_{i=1}^n m(x; \beta)}{H(x; \beta) \exp \left\{ - (H(x; \beta))^2 \right\}} - \frac{2 \sum_{i=1}^n H(x; \beta) m(x; \beta)}{\exp \left\{ - \lambda m(x; \beta) \right\}}$$

$$+ \frac{2(\theta - 1) \sum_{i=1}^n \exp \left\{ - (H(x; \beta))^2 \right\} H(x; \beta) m(x; \beta)}{\left(1 - \exp \left\{ - (H(x; \beta))^2 \right\} \right) \exp \left\{ - \lambda m(x; \beta) \right\}}$$

and

$$U_{\beta} = \lambda \sum_{i=1}^n m'(x; \beta) - 2 \sum_{i=1}^n \frac{\bar{G}'(x_i; \beta)}{\bar{G}(x_i; \beta)} + \sum_{i=1}^n \frac{g'(x_i; \beta)}{g(x_i; \beta)} + \sum_{i=1}^n \frac{H'(x; \beta)}{H(x; \beta)} - 2 \sum_{i=1}^n H(x; \beta) H'(x; \beta)$$

$$+ (\theta - 1) \sum_{i=1}^n \frac{\left[1 - \exp \left\{ - (H(x; \beta))^2 \right\} \right]'}{1 - \exp \left\{ - (H(x; \beta))^2 \right\}}$$

where

$$H(x; \beta) = \left(\exp \left\{ \lambda (m(x; \beta)) \right\} - 1 \right); \text{ thus, } m(x; \beta) = \frac{G(x; \beta)}{G(x; \beta)}$$

6 The BXE-G Sub Models

Two special sub-models of the BXE-G family, together with the plots of their respective probability densities and hazard functions are introduced in this section.

6.1 BXE-Rayleigh (BXE-R) distribution

The cdf and pdf of the Rayleigh distribution are $G(x) = 1 - \exp \left\{ - \frac{\mathcal{G}}{2} x^2 \right\}$ and $g(x) = \mathcal{G}x \exp \left\{ - \frac{\mathcal{G}}{2} x^2 \right\}$

$\mathcal{G}, x > 0$, respectively. Then the cdf and pdf of BXE-R are respectively given by

$$f_{BXE-R}(x) = \frac{2\theta\lambda\mathcal{G}x}{\exp \left\{ - \frac{\mathcal{G}}{2} x^2 \right\}} \left[\exp \left\{ \lambda \left(\exp \left\{ \frac{\mathcal{G}x^2}{2} \right\} - 1 \right) \right\} - 1 \right] \left[\exp \left\{ \lambda \left(\exp \left\{ \frac{\mathcal{G}}{2} x^2 \right\} - 1 \right) \right\} \right]$$

$$\times \exp \left\{ - \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{\mathcal{G}}{2} x^2 \right\} - 1 \right) \right\} - 1 \right)^2 \right\} \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{\mathcal{G}}{2} x^2 \right\} - 1 \right) \right\} - 1 \right)^2 \right\} \right]^{\theta-1} \tag{33}$$

$$F_{BXE-R}(x) = \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\exp \left\{ \frac{\varrho}{2} x^2 \right\} - 1 \right) \right\} - 1 \right)^2 \right\} \right]^\theta \quad x > 0 \quad (34)$$

6.2 BXE-Lomax (BXE-Lx) distribution

The cdf and pdf of the Lomax distribution are $G(x) = 1 - \left[1 + \left(\frac{x}{\alpha} \right) \right]^{-\varphi}$ and $g(x) = \left(\frac{\varphi}{\alpha} \right) \left[1 + \frac{x}{\alpha} \right]^{-\varphi-1}$ respectively. Then, the cdf and pdf of BXE-Lomax are, respectively, given by:

$$F_{BXE-Lx}(x) = \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\left[1 + \left(\frac{x}{\alpha} \right) \right]^\varphi - 1 \right) \right\} - 1 \right)^2 \right\} \right]^\theta \quad x > 0 \quad (35)$$

and

$$f_{BXE-Lx}(x) = \frac{2\theta\lambda\varphi}{\alpha \left[1 + \left(\frac{x}{\alpha} \right) \right]^{\varphi+1}} \left(\exp \left\{ \lambda \left(\left[1 + \left(\frac{x}{\alpha} \right) \right]^\varphi - 1 \right) \right\} - 1 \right) \left(\exp \left\{ \lambda \left(\left[1 + \left(\frac{x}{\alpha} \right) \right]^\varphi - 1 \right) \right\} \right) \times \exp \left\{ - \left(\exp \left\{ \lambda \left(\left[1 + \left(\frac{x}{\alpha} \right) \right]^\varphi - 1 \right) \right\} - 1 \right)^2 \right\} \left[1 - \exp \left\{ - \left(\exp \left\{ \lambda \left(\left[1 + \left(\frac{x}{\alpha} \right) \right]^\varphi - 1 \right) \right\} - 1 \right)^2 \right\} \right]^{\theta-1} \quad (36)$$

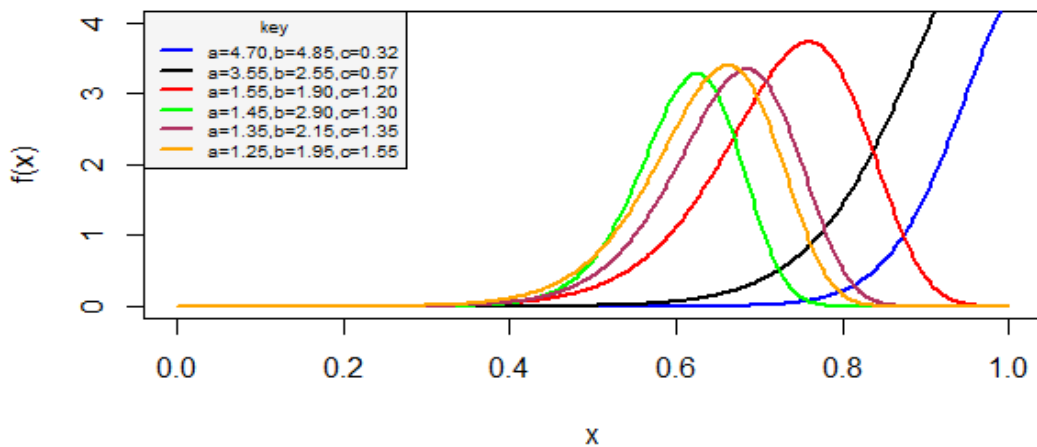


Fig. 1. Plot of the BXE-R density function for some parameters

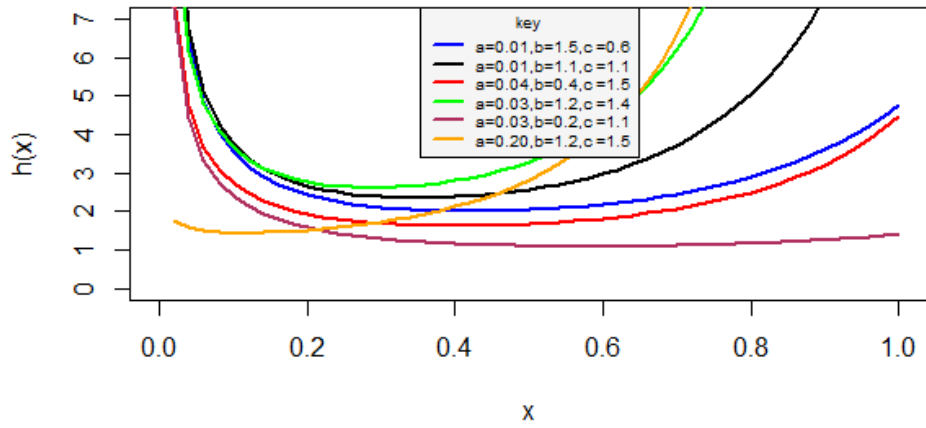


Fig. 2. Plot of the BXE-R hazard function for some parameters

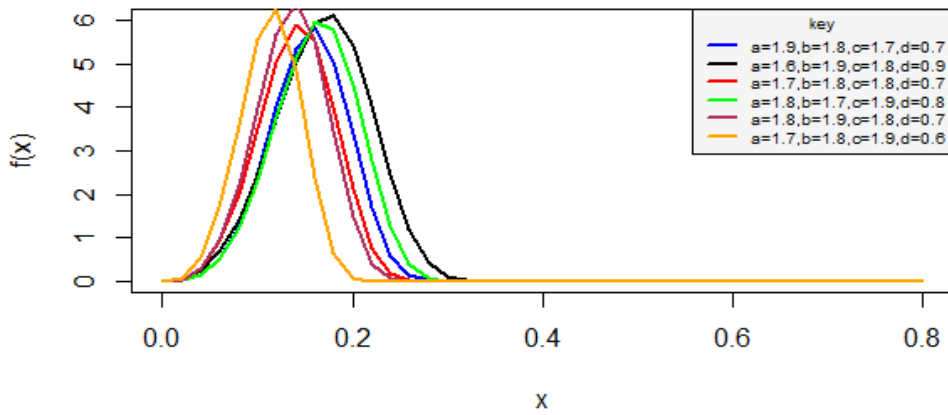


Fig. 3. Plot of the BXE-Lomax density function for some parameters

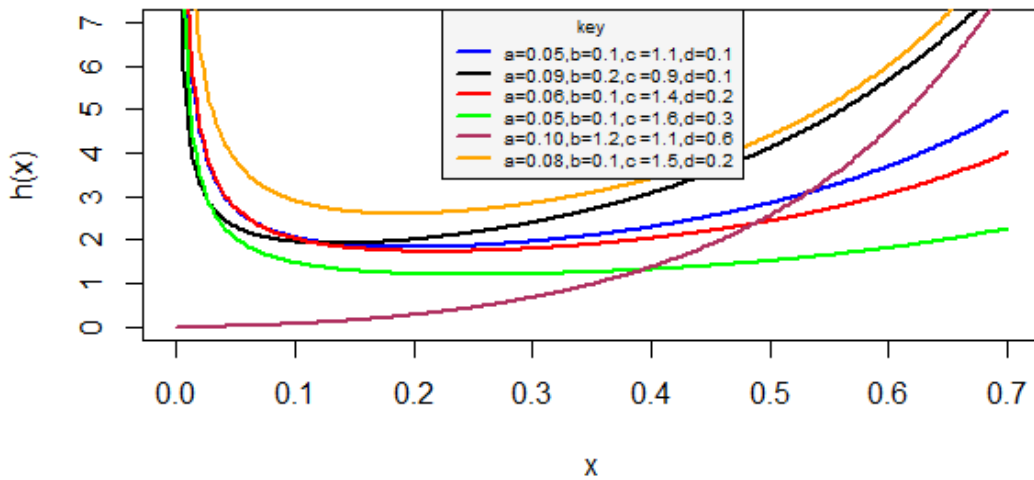


Fig. 4. Plot of the BXE-Lomax hazard function for some parameters

7 Application to Data Sets

In this section we consider BXE-Lx to illustrate its flexibility in application to real life data sets by comparing its performance with other well known distributions. The goodness-of-fit statistic and the MLE’s for the models’ parameters are presented in Tables 1 to 4 respectively. To compare the fitted models, the paper used some goodness-of-fit measures which include Akaike information criterion (*AIC*), consistent Akaike information criterion (*CAIC*), Bayesian information criterion (*BIC*), Hannan-Quinn information criterion (*HQIC*).

We compared the fits of the new BXE-Lx distribution with other competitive models like Top Leone Odd Lindley Lomax (TOLT-Lx), Burr X Lomax (BX-Lx), Exponential Lomax (E-Lx) and Lomax (Lx) Distributions. Their PDFs are available in literature.

Application 1: Modeling remission times (in months) of bladder cancer patients. This data set represents the remission times (in months) of a random sample of 128 bladder cancer patients by Lee and Wang [28]. This data is given as:

0.08, 5.85, 8.26, 11.98, 19.13, 1.76, 10.34, 14.83, 3.88, 5.32, 7.39, 3.25, 4.50, 2.09, 3.48, 4.87, 0.81, 2.62, 11.64, 17.36, 1.40, 3.02, 4.34, 34.26, 0.90, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.33, 5.49, 7.66, 11.25, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 17.14, 79.05, 1.35, 2.87, 12.07, 21.73, 2.07, 3.36, 6.93, 5.62, 7.87, 3.82, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 14.77, 32.15, 2.64, 5.71, 7.28, 9.74, 14.76, 26.31, 5.32, 7.32, 10.06, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 7.93, 11.79, 18.10, 1.46, 4.40, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 8.65, 12.63, 22.69.

Table 1. The goodness-of-fit test for the bladder cancer patients’ data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BXE-Lx	595.4936	603.4937	603.8189	603.9224	598.0832
TOL-Lx	820.2928	828.2929	828.6181	828.7216	822.8825
BX-Lx	822.2918	828.2918	828.4853	828.6134	824.2340
EXP-Lx	827.679	833.6791	833.8726	834.0006	829.6212
Lx	827.670	831.6707	831.7667	831.8850	828.9654

Table 2. MLEs and their standard errors (in parentheses) for the bladder cancer patients’ data

Models	Estimate			
BXE-Lx	$\theta=0.2414$ (0.0334)	$\lambda=15.86$ (5.9800)	$\varphi=0.0059$ (0.0009)	$\alpha=0.7427$ (0.1379)
TOL-Lx	$\phi=1.6112$ (0.2875)	$\psi=5.8717$ (8.1437)	$\varphi=0.2626$ (0.3164)	$\alpha=13.7238$ (4.0404)
BX-Lx	$\theta=0.9338$ (0.2457)	$\varphi=0.2983$ (0.0500)	$\alpha=1.0202$ (0.6496)	
EXP-Lx	$\lambda=154.1321$ (3.0534)	$\varphi=0.0822$ (0.0081)	$\alpha=110.1296$ (4.6004)	
Lx	$\varphi=15.093$ (1.326)	$\alpha=131.769$ (4.196)		

Application 2: Modeling observations of the strengths of 1.5 cm glass fibers. This data has previously been used by Reyad and Othman [29]. This was obtained by workers at the UK National Physical Laboratory study.

0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11,1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50,1.50, 1.51, 1.52, 1.53,1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62,1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68,1.68, 1.69, 1.70, 1.70, 1.73, 1.76,1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

Table 3. The goodness-of-fit criteria for the strengths of 1.5 cm glass fiber’s data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
BXE-Lx	-29.6621	-21.6621	-21.2958	-21.1922	-27.0313
TOL-Lx	28.364	36.2363	36.9260	35.5614	30.2773
BX-Lx	35.9748	41.9748	42.3816	41.3728	37.5055
EXP-Lx	28.541	34.5409	34.9477	33.9390	30.0717
Lx	178.1588	182.1589	182.3589	181.7575	179.1793

Table 4. MLEs and their standard errors (in parentheses) for the strengths of 1.5 cm glass fiber’s data

Models	Estimate			
BXE-Lx	$\theta=2.7670$ (0.5645)	$\lambda =164.1$ (13.60)	$\varphi =0.0195$ (0.0031)	$\alpha =5.3840$ (0.8712)
TOL-Lx	$\phi=0.9879$ (0.4198)	$\psi =0.0217$ (0.0252)	$\varphi =38.5934$ (2.6865)	$\alpha =14.5160$ (2.6584)
BX-Lx	$\theta=3.0442$ (0.5481)	$\varphi =62.7647$ (2.6301)	$\alpha =114.6458$ (1.3717)	
EXP-Lx	$\lambda =0.0002$ (0.0015)	$\varphi =12.0077$ (6.6802)	$\alpha =1.6418$ (2.9149)	
Lx	$\varphi =120.712$ (5.932)	$\alpha =180.980$ (2.966)		

The MLE’s are computed using R codes and the log – likelihood function evaluated. The goodness of fit measures, AIC, CAIC, BIC and HQIC are computed. The lower the value of the criteria, the better the fit. The value of BXE-Lx distribution is compared with those of TLOL-Lx, BX-Lx, E-Lx and Lx. Some goodness of fit criteria and MLE’s of the models for the first data set and second data set are presented in Tables 1-4, respectively.

The values in Tables (1-4) indicate that the BXE-Lx model has the lowest values for AIC, CAIC, BIC and HQIC among all fitted models (for the two real data sets). Therefore, it is concluded that, the BXE-Lx model could be chosen as the best model that improve the flexibility on the real data sets.

The pdfs and hazard functions’ plots are displayed in Figs. 1-4. It is clear from Figs. (1-4), that the densities of the new distributions that is; BXE-R and BXE-Lx; can have left skewed, J shape and unimodal, that is right skewed with flat tail respectively, while the hazard functions’ (rate) shapes for the two distributions remain upward bathtub shapes, that is; their hazard rate exhibit increasing shape.

8 Summary and Conclusions

The idea of generating new extended models from families of distributions has been of great interest among statisticians in the past decade. We developed a new family of distributions called Burr X Exponential–G (BXE-G) family of distributions. Moreover, new distributions were developed as special cases of the proposed family. We provided some mathematical properties of this new family of distributions. We demonstrated the flexibility and usefulness of one of the new distributions by studying the remission times (in months) of some

random sample of 128 bladder cancer patients and 63 observations of the strengths of 1.5 cm glass fibers obtained by workers at the UK National Physical Laboratory study. It is shown that; the 128 bladder cancer patients and the 63 observations of the strengths of 1.5 cm glass fibers sets of data can be best modeled using BXE-Lx distribution. Thus, we expect the usefulness of these two proposed distributions (BXE-R & BXE-Lx) in different fields especially in lifetime and reliability studies when the hazard rate is increasing.

Competing Interests

Authors have declared that no competing interests exist.

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