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# On Reflexivity of Certain Hyponormal Operators with Double Commutant Property


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***Author's contribution***

*The sole author designed, analysed, interpreted and prepared the manuscript.*

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## Abstract


Sarason did pioneer work on the reflexivity and purpose of this paper is to discuss the reflexivity of different class of contractions. Among contractions it is now known that $C_{11}$ contractions with finite defect indices, $C_{.o}$ contractions with unequal defect indices and $C_1$. contractions with at least one finite defect indices are reflexive. More over the characterization of reflexive operators among $c_o$ contractions and completely non unitary weak contractions with finite defect indices has been reduced to that of S (Φ), the compression of the shift on $H^2 \ominus \Phi H^2$, Φ is inner. The present work is mainly focused on the reflexivity of contractions whose characteristic function is constant. This class of operator include many other isometries, co-isometries and their direct sum. We shall also discuss the reflexivity of hyponormal contractions, reflexivity of $C_1$. contractions and weak contractions. It is already known that normal operators isometries, quasinormal and sub-normal operators are reflexive. We partially generalize these results by showing that certain hyponormal operators with double commutant property are reflexive. In addition, reflexivity of operators which are direct sum of a unitary operator and $C_{.o}$ contractions with unequal defect indices,is proved Each of this kind of operator is reflexive and satisfies the double commutant property with some restrictions.



*Keywords: Reflexivity; contractions; weak contractions; hyponormal operators; double commutant property; direct sum.*


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# 1 Introduction

A bounded linear operator T on a complex separable Hilbert space H is reflexive if Alg T = Alg Lat T, where Alg Lat T and Alg T denote respectively the weakly closed algebra of operators which leave invariant every invariant sub-space of T and the weakly closed algebra generated by T and I.

An operator T has double commutant property if {T}" = Alg T.

Let T be a $C_{.o}$ contraction with $m = d_T < n = d_{T*} \leq \infty$. Let T is defined on

$H = H_n^2 \ominus \Theta H_m^2$ by $Tf = P_H(e^{it})$ for $f \in H$, where $\Theta$ denotes the characteristic function of T and $P_H$ denotes the (orthogonal) projection onto H. Let J, defined on $H' = H_n^2\ \Theta\Omega H_m^2$ by $Jg = P_H(e^{it}\ g)$

for $g \in H'$, be its Jordan Model,

where

$$\Omega = \begin{pmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_m \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \Big\} n - m$$

is a n x m matrix valued inner function with $\varphi_j$ in $H^\infty$ satisfying $\varphi_{j+1} \mid \varphi_j$ for j = 1,2,…., m-1.

**1. Lemma [1]:** Let $A = U \oplus T$, where U is an absolutely continuous unitary operator and T is a $C_{.o}$ contraction with

$d_T < d_{T*} \leq \infty$, then Alg Lat A = Alg A = $\{\varphi(A): \varphi \in H^\infty\}$.

In particular, A is reflexive.

**2. THEOREM:** Let $A = U \oplus T$ where U is a unitary operator and T is a C.o contraction with

$d_T < d_T^* \leq \infty$.

Then A is reflexive.

**Proof:** $U = U_s \oplus U_a$ be the decomposition of U into direct sum of a singular unitary operator $U_s$ and an absolutely continuous unitary operator $U_a$ [2].

Now it suffices to show that

Alg $U_s$ = Alg $(U_a \oplus T)$ = Alg A.

Indeed, if this is the case then the reflexivity of A follows immediately from that of $U_s$ and $U_a \oplus T$.

To prove Alg $U_s \oplus$ Alg $(U_a \oplus T)$ = Alg A, let $V_1 \in$ Alg $U_s$ and $V_2 \in$ Alg $(U_a \oplus T)$, by

Lemma 1, $V_2 \in \varphi(U_a \oplus T)$ for some $\varphi \in H^\infty$. Let W be the (unique) minimal unitary dilation of T. When W is absolutely continuous and hence $\varphi(U_a \oplus W)$ is well defined.

Since $V_1 \oplus \varphi(U_a \oplus W) \in$ Alg $U_s \oplus$ Alg $(U_a \oplus W)$ = Alg $(U_s \oplus U_a \oplus W)$ there exist polynomials $\{p_\lambda\}$ such that $p\lambda(U_s \oplus U_a \oplus W) \to V_1 \oplus \varphi(U_a \oplus W)$ in the strong operator topology.

Compressing these operators onto the space on which A is acting,

we obtain $p_\lambda(A) \to V_1 \oplus \varphi(U_a \oplus T) = V_1 \oplus V_2$ strongly. This shows that $V_1 \oplus V_2$ is in Alg A.

This completes the proof.

The next result generalizes DEDDEN'S result that isometies are reflexive.

**3. Corollary:** Any hyponormal contraction T with $d_T < \infty$ is reflexive.

**Proof:** Let $T = T_1 \oplus T_2$ be the decomposition of T into direct sum of its unitary part $T_1$ and c.n.u. part $T_2$ (c.f. [3] P - 9). Then $T_2$, being a c.n.u. hyponormal contraction, is of class $C_{.o}$ [4].

Moreover, $d_{T_2} = d_T < \infty$ and $d_{T_2} \leq d_{T_2^*}$, we have two cases to consider:

1. If $d_{T_2} = d_{T_2^*} < \infty$ then $T_2$ is a $C_o$ (N) contraction [cf [3], p.266].

   Hyponormal contraction is normal hence reflexive from theorem that normal operators are reflexive.

2. If $d_{T_2} = d_{T_2^*} < \infty$, then the reflexivity of T follows from theorem 2.

**4. Lemma:** Let T be a c.n.u. $C_{1.}$ contraction with $d_T < \infty$ and let $T_2 = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{.1} & * \\ 0 & C_{.0} \end{bmatrix}$. Then T is reflexive if and only if $T_1 \oplus T_2$.

**Proof:** Let $d_T \neq d_{T^*}$. Otherwise $T = T_1$ is itself of class $C_{11}$. Assume that $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ is acting on

$H = H_1 \oplus H_2$. Since $T_1$ is $C_{11}$

contraction with finite defect indices, we have $T \sim T_1 \oplus T_2$ (c.f. [5] Theorem 2.1). Moreover, there are quasi-affinities $X : H \to H_1 \oplus H_2$ and $Y : H_1 \oplus H_2 \to H$ which intertwine T and $T_1 \oplus T_2$ and such that $XY = \delta(T_1 \oplus T_2)$ and $YX = \delta(T)$ for some outer function $\delta$. Let $T_1 \oplus T_2$ is reflexive and $W \in$ Alg Lat T. Since any invariant sub space for $T_1 \oplus T_2$ is of the form $\overline{XK}$ where $K$ is some invariant subspace for T (c.f. [6] Corollary 2.2)[7]. Since

$$WK \subseteq K$$

we have

$$\overline{XWYXK} = \overline{YX\delta(T)K} = \overline{XWK} \subseteq \overline{XK}$$

We use the fact that $\delta(T \mid K)$ is quasi affinity for outer $\delta$.

This implies that $XWY \in$ Alg Lat $(T_1 \oplus T_2)$ = Alg $(T_1 \oplus T_2)$. Hence $XWY \in \varphi(T_1 \oplus T_2)$ for some $\varphi \in H^\infty$ [6, Th.=3.13]. Pre multiplying and post multiplying by Y and X from the left and right of above equation, we obtain

$$YXWYX = Y\varphi(T_1 \oplus T_2)X = YX\varphi(T)$$

It follows that $W\delta(T) = \varphi(T)$. For any $V \in \{T\}'$, we have

$$WV\delta(T) = W\delta(T)V = \varphi(T)V = V\varphi(T) = VW\delta(T),$$

Hence WV = VW. We conclude that $W \in \{T\}''$ = Alg T (c.f. [6]. theorem 3.13). Hence T is reflexive as asserted. Similarly, we can prove the converse.

**5. Lemma** [8]: A c.n.u. $C_1$. contraction T with $d_T < \infty$, is reflexive.

**6. Lemma** [8]: Let $T = U_s \oplus U_a \oplus T'$ be a contraction, where $U_s$ and $U_a$ are singular and absolutely continuous unitary operators and T' is c.n.u.,

then

$$\text{Alg } T = \text{Alg } U_s \oplus \text{Alg } (U_a \oplus T').$$

**7. THEOREM:** A $C_1$. contraction T with $d_T < \infty$ is reflexive.

**Proof:** Let $T = U_s \oplus U_a \oplus T'$ be as in Lemma 6 then

$$\text{Alg } T = \text{Alg } U_s \oplus \text{Alg } (U_a \oplus T').$$

implies that

$$\text{Alg Lat } T = \text{Alg Lat } U_s \oplus \text{Alg Lat } (U_a \oplus T').$$

(c.f. [8] Proposition 1.3). Since the unitary operator $U_s$ is reflexive and to complete the proof it suffices to show that $U_a \oplus T'$ is reflexive. We may assume that T' is not of class $C_{11}$, otherwise T will also be of class $C_{11}$. Hence reflexive.

Let $R \in$ Alg Lat $(U_a \oplus T')$, then $R = R_1 \oplus R_2$,

Where $R_1 \in$ Alg Lat $U_a$ and $R_2 \in$ Alg Lat T ' = Alg T ' by lemma 5.

Hence there exist $\eta_1 \in L^\infty$ and $\eta_2 \in H^\infty$ such that $R_1 = \eta_1 (U_a)$ and $R_2 = \eta_2 (T')$ (c.f. [6] Theorem 3.13). We assume that $U_a \oplus T'$ is acting on

$H_a \oplus H'$. Let $T' = \begin{bmatrix} T_1' & * \\ 0 & T_2' \end{bmatrix}$ on $H' = H'_1 \oplus H'_2$ be the triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$.

Then $T' \sim T'_1 \oplus T'_2$. As before, let $U' = M_{E1} \oplus \ldots \oplus M_{Ep}$

on K be a unitary operator quasi-similar to $T'_1$ and let $S_{m-n}$ on $H^2_{m-n}$ be such that

$S_{m-n} \overset{ci}{\prec} T'_2 \prec S_{m-n^*}$ Also, let $U'_a = M_{F1} \oplus \ldots \oplus M_{Fq}$ on $H_a$ be the unitary equivalent to $U_a$. We deduce, that there are operators

$X : H_a \oplus H' \to H_a \oplus K \oplus H^2_{m-n}$ and $Y : H_a \oplus K \oplus H^2_{m-n} \to H_a \oplus H'$ which intertwine $U_a \oplus T'$

and

$$U'_a \oplus U' \oplus S_{m-n}$$

Satisfy $XY = \delta (U'_a \oplus U' \oplus S_{m-n})$, $YX = \delta (U_a \oplus T')$ and $XRY = (\eta_1\delta) (U'_a) \oplus (\eta_2\delta) (U' \oplus S_{m-n})$

for some $\delta \in H^\infty$.

Now let us consider the invariant subspace.

$$M = \{\chi_{F1} f \oplus \cdots \oplus \chi_{Fq} f\, \chi_{E1} f \oplus \cdots \oplus \chi_{Fp} f \oplus \underbrace{f \oplus \cdots \oplus f}_{m-n} : f \in H^2\}$$

for $U'_a \oplus U' \oplus S_{m-n.}$ We deduce that $\eta_1 = \eta_2$ a.e.$F_1$.

Hence $R_2 = \eta_2(U_a \oplus T') \in$ Alg $(U_a \oplus T')$ which shows that $U_a \oplus T'$, whence T is reflexive.

**8. Definition:** A contraction T is a weak contraction if

(1) Its spectrum $\sigma(T)$ does not fill the open unit disc and
(2) $I - T^*T$ is of finite trace.

**9. Lemma [8,9,10]:** Let T be a c.n.u. weak contraction with finite defect indices and let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 2} \end{bmatrix}$. Assume that $\theta_T(e^{it})$ is not isometric for almost all t.

Then T is reflexive if and only if $T_1 \oplus T_2$.

The proof of above lemma is analogous to lemma 4.

**10. Lemma [8]:** Let T be a c.n.u. weak contraction with finite defect indices. If $\theta_T(e^{it})$ is not isometric for almost all t, then T is reflexive.

**11. THEOREM:** Let T be a c.n.u. weak contraction with finite defect indices and let $E_1 = \{e^{it} : \theta_T(e^{it})$ not isometric$\}$. Then the following statements are equivalent:

(1) T is reflexive $\Leftrightarrow$ either $E_1 = T$ or $E_1 \neq T$ a.e. and the $C_o$ part of T is reflexive.

Here we are using the convention that if the $C_o$ part of T is acting on {0} then it is reflexive.

**Proof:** By Lemma 10 it will be sufficient to show that if $E_1 \neq T$ a.e. then T is reflexive if and only if its $C_o$ part is.

Let $T_o$ and $T_1$ be the $C_o$ and $C_{11}$ parts of T. Assume that T, $T_o$ and $T_1$ are acting on H, $H_o$ and $H_1$ respectively.
Let us assume that T is reflexive. Let $V_o \in$ Alg Lat $T_o$ and $S \in \{T\}''$ be such that

$H_o = \overline{SH}$ (c.f.[11] theorem 1) [12].

Since $E_1 \neq T$ a.e., we have $\{T\}''$ = Alg T (c.f. [11] Theorem 3)[12]. Hence $S \in$ Alg T. The reflexivity of $T_o$, follows from theorem 8 and the fact that

Alg Lat $T_o \cap \{T_o\}'$ = Alg $T_o$. [c.f.[5] Theorem 3.3).

Conversely, if $T_o$ is reflexive, let $V \in$ Alg Lat T then $V H_o \subseteq H_o$ and $VH_1 \subseteq H_1$.

Let $V_o = V \mid H_o$ and $V_1 = V \mid H_1$. We have $V_o$ Alg Lat $T_o$ = Alg $T_o$ and $V_1 \in$ Alg Lat $T_1$ = Alg $T_1$, since $T_1$, being of class $C_{11}$, is reflexive. Hence $V_oT_o = T_oV_o$ and $V_1T_1 = T_1V_1$. It follows that

$VT = TV$ on $H_o \vee H_1 = H$ (c.f. [3] P. 332)[13,14]. So $V \in$ Alg Lat $T \cap \{T\}'$ = Alg T. (c.f.[5], Theorem 3).

This proves the reflexivity of T.

**12. Theorem [5]:** Let $T_1$ and $T_2$ be $C_o(N)$ Contractions on $H_1$ and $H_2$ respectively. Assume that $T_1$ is quasi-similar to $T_2$. Then $T_1$ is reflexive if and only if $T_2$ is.

**13. Corollary:** Quasi-similarity preserves the reflexivity for completely non unitary weak contractions with finite defect indices that is if $T_1$ and $T_2$ be a c.n.u. weak contractions with finite defect indices and $T_1$ is quasi-similar to $T_2$. Then $T_1$ is reflexive if and only if $T_2$ is.

**Proof:** Since $T_1$ and $T_2$ are quasi-similar and the quasi-similarity of $T_1$ and $T_2$ implies that of $C_o$ parts (c.f. [8] Corollary 1)[12,15]. The conclusion now follows from theorem 11 and Theorem 12.

**14. THEOREM:** Let T be a c.n.u. contraction on H and let $T = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ be a triangulation on

$$H = H_1 \oplus H_2.$$

If the characteristic function of T admits a right outer scalar multiple $\delta(\lambda)$.

Then $T \sim T_1 \oplus T_2$. Moreover there are quasi affinities $Y : H \to H_1 \oplus H_2$ and $Z : H_1 \oplus H_2 \to H$ intertwining T and $T_1 \oplus T_2$ and such that $YZ = \delta\,(T_1 \oplus T_2)$ and $ZY = \delta\,(T)$.

**15. THEOREM:** Let T be a contraction with at least one finite defect index. Assume that the outer factor of the characteristic function of T admits a right outer scalar multiple. If T is not a weak contraction, the T is reflexive.

**Proof:** By lemma 6 we may assume that T has no singular unitary summand. Let $T = U \oplus \widetilde{T}$ .

$$H = H_o \oplus \widetilde{H}$$

and

$$\widetilde{T} = \begin{bmatrix} T_1 & \widetilde{X} \\ 0 & T_2 \end{bmatrix}$$

on $\widetilde{H} = H_1 \oplus H_2$ be the canonical triangulation of type $\begin{bmatrix} C_{.1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ where U is absolutely continuous unitary operator and $\widetilde{T}$ be c.n.u. Then $\widetilde{T} = T_1 \oplus T_2$ and there are quasi-affinities $\widetilde{Y}$ and $\widetilde{Z}$ intertwining $\widetilde{T}$ and $T_1 \oplus T_2$ and such that $\widetilde{Y}\widetilde{Z} = \delta\,(T_1 \oplus T_2)$ and $\widetilde{Y}\widetilde{Z} = \delta\,(\widetilde{T})$ for some outer function $\delta$.

Let $Y = \delta\,(U) \oplus \widetilde{Y}$ and $Z = I_{H_0} \oplus \widetilde{Z}$, then Y and Z are quasi-affinities intertwining T and $M \equiv U \oplus T_1 \oplus T_2$ and satisfying $ZY = \delta(M)$ and $ZY = \delta(T)$ c.f. [16] theorem 2.1)[17]. For $K \in$ Lat T. The mappings $K \to \overline{YK}$ and $L \to \overline{ZL}$ preserve the lattice operations in Lat T and Lat M and are inverse to each other. Hence invariant subspaces of T and M are of the forms $\overline{ZL}$ and $\overline{YK}$, where $L \in$ Lat M and $K \in$ Lat T. Doing the same way as in Lemma 6 by using these facts, we may show that T is reflexive if and only if M is. Next we make further reduction. Let

$$T = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix},$$

On $H_1 = H_3 \oplus H_4$ be the canonical triangulation of type $\begin{bmatrix} C_{01} & * \\ 0 & C_{11} \end{bmatrix}$ (c.f. [6, Lemma 3.2]).

Since Theorem 15 is applicable to $T_1^*$, we may argue as above to show that M is reflexive if and only if $N \equiv U \oplus T_3 \oplus T_4 \oplus T_2$ is. It is to be noted here that $T_4$ is a c.n.u. $C_{11}$ contraction with finite defect indices. Hence $T_4$ is quasi-similar to an absolutely continuous unitary operator, say N, and the quasi-similarity is implemented by quasi-affinities P and Q satisfying $PQ = \eta(N)$ and $QP = \eta(T_4)$ for some outer function $\eta$ (c.f. [18] lemma 2.1) [19]. As above we infer that N is reflexive if and only if $K \equiv U \oplus N \oplus T_3 \oplus T_2$ is.

Next we show the reflexivity of K. For simplicity, let $W = U \oplus N$. Since $C_{\cdot 0}$ contraction with unequal defect indices and $C_1$. Contraction with at least one finite defect indices are known to be reflexive [Theorem 7], we have to show the reflexivity of the following direct sums whose summands are non-trivial.

(i) $W \oplus T_2$ :- Since this is a direct sum of an absolutely continuous unitary operator and a $C_o$. contraction with unequal defect indices and its reflexivity has been proved in lemma 1.
(ii) $T_3 \oplus T_2$:- If $d_{T_2} = d_{T_2^*}$ then $T_2$ is a $C_{00}$ contraction. Hence $T_3 \oplus T_2$, being a $C_{.o}$ contraction with unequal defect indices, is reflexive. Thus we may assume that $d_{T_2} \neq d_{T_2^*}$.

Let $R \in$ Alg Lat $(T_3 \oplus T_2)$.

Then $R \in R_2 \oplus R_3$,

where

$R_j \in$ Alg Lat $T_j$, j = 2,3

There is $\varphi_j$ in $H^\infty$ such that $R_j \in \varphi_j(T_j)$, j = 2,3 (c.f. [20], Theorem 2). For any operator $J : H_2 \to H_3$ satisfying $JT_2 = T_3J$, let us consider the (closed) subspace $G = \{J x \oplus x : x \in H_2\}$ in Lat $(T_3 \oplus T_2)$ we infer from $RG \subseteq G$ that, for any $x \in H_2$, $\varphi_3(T_3) Jx \oplus \varphi_2(T_2) x = Jy \oplus y$ for some $y \in H_2$. It follows that $\varphi_3(T_3) J = J\varphi_2(T_2) = \varphi_2(T_3) J$. However $T_2 \overset{cd}{\prec} T_3$ (c.f. [16] lemma 3.4) [21]. Now we can conclude that $\varphi_3(T_3) = \varphi_2(T_3)$ whence $\varphi_3 = \varphi_2$ a.e.

This shows that $R = \varphi_2(T_3 \oplus T_2) \in$ Alg $(T_3 \oplus T_2)$ and the reflexivity of $T_3 \oplus T_2$ follows.

(iii) $K \in W \oplus T_3 \oplus T_2$. If $d_{T_2} = d_{T_2^*}$, then as in (ii), $T_3 \oplus T_2$ is a $C_0$. contraction with unequal defect indices, the reflexivity of K follows as in (i), Next we consider the case $d_{T_2} \neq d_{T_2^*}$

By Lemma 1, $W \oplus T_j$ is reflexive and Alg $(W \oplus T_j) = \{\varphi(W \oplus T_j): \varphi \in H^\infty\}$, j = 2,3.

Let $R \in$ Alg Lat K, then $R = R_o \oplus R_3 \oplus R_2$ with $R_o \oplus R_j \in$ Alg Lat $(W \oplus T_j)$, j = 2,3.

Hence $R_o \oplus R_j \in \varphi_j(W \oplus T_j)$ for some $\varphi_j \in H^\infty$, we infer from $R_0 = \varphi_2(W) = \varphi_3(W)$ that $\varphi_2 = \varphi_3$ a.e.. Thus $R_0 = \varphi_2(K) \in$ Alg K. This shows that K is reflexive and this completes the proof.

## 2 Conclusion

Direct sum of two reflexive operators is reflexive in the special case when one summand is unitary and the other $C_o$ with unequal finite defect indices. In general, this is an open question whether direct sum of two reflexive operators is reflexive? We partially generalize that certain hyponormal operators with double commutant property, are reflexive. Even larger class of operators which are direct sum of a unitary operator and $C_{.o}$ contractions with unequal defect indices. This kind of operator is reflexive and satisfies the double commutant property with some restrictions.

## Competing Interests

Author has declared that no competing interests exist.

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