


Research Article

On J -Cone Metric Spaces over a Banach Algebra and Some Fixed-Point Theorems

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In the present paper, we define J -cone metric spaces over a Banach algebra which is a generalization of G_{p_b} -metric space (G_{p_b} -MS) and cone metric space (CMS) over a Banach algebra. We give new fixed-point theorems assuring generalized contractive and expansive maps without continuity. Examples and an application are given at the end to support the usability of our results.

1. Introduction

The notion of a generalized partial b -metric (for short, a G_{p_b} -metric) space was introduced by Hussain et al. [1] in 2014 by generalizing the notions of a G -metric space (G -MS) and a partial b -metric space. They studied the related topological properties and provided fixed point theorems for some contractive maps.

Huang and Zhang [2] generalized the notion of metric spaces to a CMS. Later, the interesting concept of a cone metric space over a Banach algebra (for short, CMS over a BA) was proposed by Liu and Xu [3], by replacing a CMS with a CMS over a BA. Motivated by these ideas, many authors further considered a CMS over a BA (see [4, 5]). Also, in [6], the authors introduced the concept of cone b_2 -metric space over Banach algebras which generalizes the notions of b_2 -metric space and cone metric spaces over Banach algebra. There are some very recent references (such as [7, 8]), where it is shown that the fixed point theory continues to provide useful tools for studying problems of Mathematical Physics.

The present paper is organized as follows. In Section 2, we recall some definitions. In Section 3, we generalize the concepts of a G_{p_b} -metric space and a CMS over a Banach algebra by introducing a J -CMS over a Banach algebra (for short, J -CMS over a BA) with some examples. Section 4 is devoted to define generalized contractive and expansive maps. Finally, in Section 5 and Section 6, we prove some fixed point theorems for such certain contractive and expansive maps in the framework of a J -CMS over a BA. Our work generalizes and extends some interesting results of [9]. Two examples and an application are given to verify the strength of our main results.

2. Preliminaries

We start with some known concepts. Denote by Y a real Banach algebra (for short, BA). From now on, assume that there is a unit element $e \in Y$. An element $\omega \in Y$ is called invertible if there is $\pi \in Y$ so that $\omega\pi = \pi\omega = e$. We denote by ω^{-1} the inverse of ω . For more related details, one may check [10].

Proposition 1 [10]. Assume that the spectral radius $\mathfrak{S}\mathfrak{R}(\omega)$ of an element $\omega \in Y$ is less than 1, that is,

$$\mathfrak{S}\mathfrak{R}(\omega) = \lim_{n \rightarrow \infty} \|\omega^n\|^{1/n} = \inf \|\omega^n\|^{1/n} < 1, \quad (1)$$

then $e - \omega$ is invertible, where $e \in Y$ is unit. In addition,

$$(e - \omega)^{-1} = \sum_{i=0}^{\infty} \omega^i. \quad (2)$$

Remark 2 ([11]). If $\mathfrak{S}\mathfrak{R}(\omega) < 1$, then $\|\omega^n\| \rightarrow 0$ as $n \rightarrow \infty$.

A set $\mathfrak{C} \subset Y$ is said to be a cone if

- (1) $\mathfrak{C} \neq \emptyset$, closed and $\mathfrak{C} \neq \{\theta\}$
- (2) $\alpha\mathfrak{C} + \beta\mathfrak{C} \subset \mathfrak{C}$ for all $\alpha, \beta \geq 0$
- (3) $\mathfrak{C}^2 = \mathfrak{C}\mathfrak{C} \subset \mathfrak{C}$
- (4) $(-\mathfrak{C}) \cap \mathfrak{C} = \{\theta\}$,

where θ is the null of Y . The partial ordering on Y is given as $\omega \leq \pi$ iff $\pi - \omega \in \mathfrak{C}$. We write $\omega < \pi$ to indicate $\omega \leq \pi$, but $\omega \neq \pi$, while $\omega \ll \pi$ stands for $\pi - \omega \in \text{int } \mathfrak{C}$ (here, $\text{int } \mathfrak{C}$ is the interior of \mathfrak{C}).

Definition 3 ([2, 3]). Let \mathcal{M} be a nonempty set. If $d : \mathcal{M} \times \mathcal{M} \rightarrow Y$ verifies

- (1) $\theta \leq d(\omega, \pi)$ for all $\omega, \pi \in \mathcal{M}$ and $d(\omega, \pi) = \theta$ iff $\omega = \pi$
- (2) $d(\omega, \pi) = d(\pi, \omega)$ for all $\omega, \pi \in \mathcal{M}$
- (3) $d(\omega, \pi) \leq d(\omega, \bar{\omega}) + d(\bar{\omega}, \pi)$ for all $\omega, \pi, \bar{\omega} \in \mathcal{M}$

then d is named as a cone metric on \mathcal{M} , and (\mathcal{M}, d) is said to be a CMS over the BA Y .

Definition 4 ([1]). Let \mathcal{M} be a nonempty set and $s \geq 1$. Assume that the function $G_{p_b} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+$ is so that:

- (G_{p_b} 1) $\omega = \pi = \bar{\omega}$ if $G_{p_b}(\omega, \pi, \bar{\omega}) = G_{p_b}(\bar{\omega}, \bar{\omega}, \bar{\omega}) = G_{p_b}(\pi, \pi, \pi) = G_{p_b}(\omega, \omega, \omega)$
- (G_{p_b} 2) $G_{p_b}(\omega, \omega, \bar{\omega}) \leq G_{p_b}(\omega, \omega, \pi) \leq G_{p_b}(\omega, \pi, \bar{\omega})$ for all $\omega, \pi, \bar{\omega} \in \mathcal{M}$ with $\bar{\omega} \neq \pi$
- (G_{p_b} 3) $G_{p_b}(\omega, \pi, \bar{\omega}) = G_{p_b}(p\{\omega, \pi, \bar{\omega}\})$, where p is any permutation of ω, π and $\bar{\omega}$ (symmetry in all three variables)
- (G_{p_b} 4) $G_{p_b}(\omega, \pi, \bar{\omega}) \leq s[G_{p_b}(\omega, a, a) + G_{p_b}(a, \pi, \bar{\omega}) - G_{p_b}(a, a, a)] + ((1-s)/3)[G_{p_b}(\omega, \omega, \omega) + G_{p_b}(\pi, \pi, \pi) + G_{p_b}(\bar{\omega}, \bar{\omega}, \bar{\omega})]$ for all $\omega, \pi, \bar{\omega}, a \in \mathcal{M}$ (rectangle inequality).

Then, G_{p_b} is named as a G_{p_b} -metric and (\mathcal{M}, G_{p_b}) is called a G_{p_b} -metric space.

Using (G_{p_b} 4) and $s \geq 1$, we have

$$G_{p_b}(\omega, \pi, \bar{\omega}) \leq s[G_{p_b}(\omega, a, a) + G_{p_b}(a, \pi, \bar{\omega}) - G_{p_b}(a, a, a)]. \quad (3)$$

The G_{p_b} -metric space G_{p_b} is called symmetric if $G_{p_b}(\omega, \omega, \pi) = G_{p_b}(\omega, \pi, \pi)$ holds for all $\omega, \pi \in \mathcal{M}$. Otherwise, G_{p_b} is an asymmetric G_{p_b} -metric.

Example 5 [1]. Let $\mathcal{M} = [0, \infty)$ and let $G_{p_b} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+$ be given by $G_{p_b}(\omega, \pi, \bar{\omega}) = [\max\{\omega, \pi, \bar{\omega}\}]^p$ where $p > 1$.

Obviously, (\mathcal{M}, G_{p_b}) is a symmetric G_{p_b} -metric space, which is not a G -metric space. In fact, if $\omega = \pi = \bar{\omega} > 0$, then $G_{p_b}(\omega, \pi, \bar{\omega}) = \omega^p > 0$. It is easy to see that G_{p_b} 1 - G_{p_b} 4 are satisfied.

3. A J -CMS over a Banach Algebra

Here, we introduce the notion of a J -CMS over the BA Y , as a generalization of a generalized partial b -CMS and a CMS over the BA Y .

Definition 6. Let \mathcal{M} be a nonempty set and $s \geq 1$. Suppose that $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ so that:

- (J_1) $\omega = \pi = \bar{\omega}$ if $J(\omega, \pi, \bar{\omega}) = J(\bar{\omega}, \bar{\omega}, \bar{\omega}) = J(\pi, \pi, \pi) = J(\omega, \omega, \omega)$
- (J_2) $J(\omega, \omega, \omega) \leq J(\omega, \omega, \pi) \leq J(\omega, \pi, \bar{\omega})$ for all $\omega, \pi, \bar{\omega} \in \mathcal{M}$ with $\bar{\omega} \neq \pi$
- (J_3) $J(\omega, \pi, \bar{\omega}) = J(p\{\omega, \pi, \bar{\omega}\})$, where p is any permutation of ω, π and $\bar{\omega}$ (symmetry in all three variables)
- (J_4) $J(\omega, \pi, \bar{\omega}) \leq s[J(\omega, a, a) + J(a, \pi, \bar{\omega}) - J(a, a, a)] + ((1-s)/3)[J(\omega, \omega, \omega) + J(\pi, \pi, \pi) + J(\bar{\omega}, \bar{\omega}, \bar{\omega})]$ for all $\omega, \pi, \bar{\omega}, a \in \mathcal{M}$ (rectangle inequality).

Then, J is called a J -cone metric and (\mathcal{M}, J) is called a J -CMS over the BA Y .

Since $s \geq 1$, from (J_4), we have

$$J(\omega, \pi, \bar{\omega}) \leq s[J(\omega, a, a) + J(a, \pi, \bar{\omega}) - J(a, a, a)]. \quad (4)$$

The J -CMS J is said to be symmetric if $J(\omega, \omega, \pi) = J(\omega, \pi, \pi)$ holds for all $\omega, \pi \in \mathcal{M}$. Otherwise, J is asymmetric.

We now present some examples.

Example 7. Let $Y = C_{\mathbb{R}}^1[0, \infty)$ which is endowed with the norm $\|\omega\| = \|\omega\|_{\infty} + \|\omega'\|_{\infty}$. Under the pointwise multiplication, Y is a real Banach algebra with unit $e = 1$. Let $\mathfrak{C} = \{(\omega, \tau) \in Y : \omega(t), \tau(t) \geq 0 \forall t \in [0, \infty)\}$. Moreover, P is not normal (see [12]).

Let $G_{p_b} : [0, \infty)^3 \rightarrow \mathbb{R}^+$ be given by $G_{p_b}(\omega, \pi, \bar{\omega}) = [\max\{\omega, \pi, \bar{\omega}\}]^p$ where $p \geq 1$. Let $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y^2$ be given by

$$J(\omega, \pi, \varpi)(t) = \left(G_{p_b}(\omega, \pi, \varpi), \alpha G_{p_b}(\omega, \pi, \varpi) \right) e^t. \quad (5)$$

Obviously, (\mathcal{M}, G_{p_b}) is not a G -metric space. But (\mathcal{M}, J) is a J -CMS over the BA Y . Indeed, if $\omega = \pi = \varpi > 0$, then $J(\omega, \pi, \varpi)(t) = (\omega^p, \alpha\omega^p)e^t > \theta$. Hence, $(J_1) - (J_3)$ are satisfied. Now, we show that (J_4) holds. For all $\omega, \pi, \varpi, a \in \mathcal{M}$, we have

$$[\max \{\omega, \pi, \varpi\}]^p e^t \leq s([\max \{\omega, a\}]^p + [\max \{a, \pi, \varpi\}]^p - a^p) e^t + \frac{(1-s)}{3}(\omega^p + \pi^p + \varpi^p) e^t, \quad (6)$$

(by Example 12 in [1]). Therefore,

$$J(\omega, \pi, \varpi) \leq s[J(\omega, a, a) + J(a, \pi, \varpi) - J(a, a, a)] + ((1-s)/3)[J(\omega, \omega, \omega) + J(\pi, \pi, \pi) + J(\varpi, \varpi, \varpi)], \quad (7)$$

for all $\omega, \pi, \varpi, a \in \mathcal{M}$ (rectangle inequality).

The following examples show that a J -CMS over a BA need not be a G_b -CMS ([13]).

Example 8. Let $\mathcal{M} = [0, 1]$ and Y be the set of all real-valued continuously differentiable functions on \mathcal{M} with the norm $\|\omega\| = \|\omega\|_\infty + \|\omega'\|_\infty$. Define the multiplication in the usual way. Let $\mathfrak{C} = \{\omega \in Y : \omega(t) \geq 0, t \in \mathcal{M}\}$. Clearly, \mathfrak{C} is a non-normal cone and Y is a BA with a unit $e = 1$. Consider $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ as

$$J(\omega, \pi, \varpi)(t) = ((\max \{\omega, \pi\})^2 + (\max \{\pi, \varpi\})^2 + (\max \{\varpi, \omega\})^2) e^t, \quad (8)$$

for all $\omega, \pi, \varpi \in \mathcal{M}$. Thus, (\mathcal{M}, J) is a J -CMS over the BA Y , but it is not a G_b -CMS since $J(\omega, \omega, \omega)(t) = 3\omega^2 e^t \neq \theta$.

Example 9. Let $Y = C_{\mathbb{R}}^1[0, 1]$ and consider a norm on Y as $\|\omega\| = \|\omega\|_\infty + \|\omega'\|_\infty$ for all $\omega \in Y$. Let the multiplication on Y be the pointwise multiplication. Then, Y is a real unit BA with unit $e = 1$. Set $\mathfrak{C} = \{\omega \in Y : \omega \geq 0\}$ which is a cone in Y . Let $\mathcal{M} = [0, \infty)$ and $a > 0$ be any constant. Consider $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ as

$$J(\omega, \pi, \varpi)(t) = ([\max \{\omega, \pi, \varpi\}]^2 + a) e^t, \quad (9)$$

for all $\omega, \pi, \varpi \in \mathcal{M}$. This (\mathcal{M}, J) is a J -CMS over the BA Y , but it is not a G_b -CMS since $J(\omega, \omega, \omega)(t) = (\omega^2 + a) e^t \neq \theta$.

Example 10. Let $Y = C[a, b]$ be the set of all continuous functions on $[a, b]$ with the norm $\|\omega\| = \|\omega\|_\infty + \|\omega'\|_\infty$. Taking the usual multiplication, then Y is a BA with a unit 1. Set $\mathfrak{C} = \{\omega \in Y : \omega(t) \geq 0, t \in [a, b]\}$ and $\mathcal{M} = \mathbb{R}^+$. Consider $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ by

$$J(\omega, \pi, \varpi)(t) = ([\max \{\omega, \pi, \varpi\}]^2 + |\omega - \pi|^2 + |\pi - \varpi|^2 + |\varpi - \omega|^2) e^t, \quad (10)$$

for all $\omega, \pi, \varpi \in \mathcal{M}$. Then, (\mathcal{M}, J) is a J -CMS over the BA Y . But it is not a G_b -CMS, because $J(\omega, \omega, \omega)(t) = \omega^2 e^t \neq \theta$.

In the following, (\mathcal{M}, J) is assumed to be a J -CMS over the BA Y .

Lemma 11. (a) If $J(\omega, \pi, \varpi) = \theta$, then $\omega = \pi = \varpi$.
(b) If $\omega \neq \pi$, then $J(\omega, \pi, \pi) > \theta$.

Definition 12. For an $\omega \in \mathcal{M}$ and $c > \theta$, the J -ball with center ω and radius $c > \theta$ is

$$B_J(\omega, c) = \{\pi \in \mathcal{M} : J(\omega, \omega, \pi) \ll J(\omega, \omega, \omega) + c\}. \quad (11)$$

The θ -Cauchyness and convergence are given as follows.

Definition 13. A sequence $\{\omega_n\}$ in (\mathcal{M}, J) converges to $\omega \in \mathcal{M}$, whenever for each $c \gg \theta$, there is an integer N so that $J(\omega_n, \omega, \omega) \ll c$ for all $n \geq N$.

Definition 14. A sequence $\{\omega_n\}$ in \mathcal{M} is called a θ -Cauchy sequence in (\mathcal{M}, J) if for every $c \gg \theta$ there is $n_0 \in \mathbb{N}$ such that $J(\omega_n, \omega_m, \omega_m) \ll c$ for all $n, m \geq n_0$.

Definition 15. \mathcal{M} is said to be θ -complete if every θ -Cauchy sequence $\{\omega_n\}$ in (\mathcal{M}, J) is convergent to $\omega \in \mathcal{M}$ so that $J(\omega, \omega, \omega) = \theta$.

Definition 16. Let (\mathcal{M}', J') and (\mathcal{M}, J) be two J -CMS over the BA Y . Then, a function $f : \mathcal{M} \rightarrow \mathcal{M}'$ is called continuous at $\omega \in \mathcal{M}$ iff for each convergent sequence $\{\omega_n\}$ to ω , we have $\{f\omega_n\}$ is convergent to $f(\omega)$.

4. Generalized Contractive and Expansive Maps

In this section, we consider contractive and expansive maps in a J -CMS over a BA. Some examples are also presented.

Definition 17. Let (\mathcal{M}, J) be a J -CMS over the BA Y and \mathfrak{C} be a cone in Y . A map $\chi : \mathcal{M} \rightarrow \mathcal{M}$ is named as a generalized contractive mapping if there is $k \in \mathfrak{C}$ (with $\mathfrak{C}\mathfrak{R}(k) < 1$) so that for all $\omega, \pi, \varpi \in \mathcal{M}$,

$$J(\chi\omega, \chi\pi, \chi\varpi) \leq kJ(\omega, \pi, \varpi). \quad (12)$$

Example 18. Let Y be a BA and \mathfrak{C} be a cone (as in Example 8) and let $\mathcal{M} = \mathbb{R}^+$. Define a mapping $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ by

$$J(\omega, \pi, \varpi)(t) = ([\max \{\omega, \pi, \varpi\}]^2 + |\omega - \pi|^2 + |\pi - \varpi|^2 + |\varpi - \omega|^2) e^t, \quad (13)$$

for all $\omega, \pi, \varpi \in \mathcal{M}$. Then, (\mathcal{M}, J) is a J -CMS over the BA Y . Take $\chi : \mathcal{M} \rightarrow \mathcal{M}$ by $\chi(\omega) = \ln(1 + (\omega/2))$. In view of $\ln(1 + u) \leq u$ for each $u \geq 0$, we have for all $\omega, \pi, \varpi \in \mathcal{M}$,

$$\begin{aligned}
J(\chi\omega, \chi\pi, \chi\bar{\omega})(t) &= \left(\left[\max \left\{ \ln \left(1 + \frac{\omega}{2} \right), \ln \left(1 + \frac{\pi}{2} \right), \ln \left(1 + \frac{\bar{\omega}}{2} \right) \right\} \right]^2 \right. \\
&\quad + \left| \ln \left(1 + \frac{\omega}{2} \right) - \ln \left(1 + \frac{\pi}{2} \right) \right|^2 + \left| \ln \left(1 + \frac{\pi}{2} \right) \right. \\
&\quad \left. - \ln \left(1 + \frac{\bar{\omega}}{2} \right) \right|^2 + \left| \ln \left(1 + \frac{\bar{\omega}}{2} \right) - \ln \left(1 + \frac{\omega}{2} \right) \right|^2 \right) e^t \\
&\leq \left(\left[\max \left\{ \frac{\omega}{2}, \frac{\pi}{2}, \frac{\bar{\omega}}{2} \right\} \right]^2 + \left| \frac{\omega}{2} - \frac{\pi}{2} \right|^2 + \left| \frac{\pi}{2} - \frac{\bar{\omega}}{2} \right|^2 + \left| \frac{\bar{\omega}}{2} - \frac{\omega}{2} \right|^2 \right) e^t \\
&\leq \frac{1}{4} ([\max \{\omega, \pi, \bar{\omega}\}]^2 + |\omega - \pi|^2 + |\pi - \bar{\omega}|^2 + |\bar{\omega} - \omega|^2) e^t \\
&= \frac{1}{4} J(\omega, \pi, \bar{\omega})(t),
\end{aligned} \tag{14}$$

where $k = 1/4$. Obviously, χ is a generalized contractive map on \mathcal{M} .

Definition 19. Let (\mathcal{M}, J) be a J -CMS over the BA Y and \mathfrak{C} be a cone in Y . A map $\chi : \mathcal{M} \rightarrow \mathcal{M}$ is said to be an expansive mapping if for all $\omega, \pi, \bar{\omega} \in \mathcal{M}$,

$$J(\chi\omega, \chi\pi, \chi\bar{\omega}) \geq kJ(\omega, \pi, \bar{\omega}), \tag{15}$$

where $k, k^{-1} \in \mathfrak{C}$ and are called generalized contractive constants with $\mathfrak{S}\mathfrak{R}(k^{-1}) < 1$.

Example 20. Let Y be a BA and \mathfrak{C} be a cone (as given in Example 9) and let $\mathcal{M} = \mathbb{R}^+$. As in Example 18, (\mathcal{M}, J) is a J -CMS over the BA Y . Define $\chi\omega = 3\omega + \omega^2$ for all ω .

$$\begin{aligned}
J(\chi\omega, \chi\pi, \chi\bar{\omega})(t) &= \left[\left(\left[\max \{3\omega + \omega^2, 3\pi + \pi^2, 3\bar{\omega} + \bar{\omega}^2\} \right]^2 \right. \right. \\
&\quad + \left| (3\omega + \omega^2) - (3\pi + \pi^2) \right|^2 + \left| (3\pi + \pi^2) - (3\bar{\omega} + \bar{\omega}^2) \right|^2 \\
&\quad \left. \left. + \left| (3\bar{\omega} + \bar{\omega}^2) - (3\omega + \omega^2) \right|^2 \right) \right] e^t \\
&= \left(\left[\max \{3\omega + \omega^2, 3\pi + \pi^2, 3\bar{\omega} + \bar{\omega}^2\} \right]^2 \right. \\
&\quad + \left| 3(\omega - \pi) + (\omega - \pi)(\omega + \pi) \right|^2 \\
&\quad + \left| 3(\pi - \bar{\omega}) + (\pi - \bar{\omega})(\pi + \bar{\omega}) \right|^2 \\
&\quad \left. + \left| 3(\bar{\omega} - \omega) + (\bar{\omega} - \omega)(\bar{\omega} + \omega) \right|^2 \right) e^t \\
&= \left(\left[\max \{3\omega + \omega^2, 3\pi + \pi^2, 3\bar{\omega} + \bar{\omega}^2\} \right]^2 \right. \\
&\quad + |\omega - \pi|^2 [3 + (\omega + \pi)]^2 + |\pi - \bar{\omega}|^2 [3 + (\pi + \bar{\omega})]^2 \\
&\quad \left. + |\bar{\omega} - \omega|^2 [3 + (\bar{\omega} + \omega)]^2 \right) e^t \\
&\geq \left(\left[\max \{3\omega, 3\pi, 3\bar{\omega}\} \right]^2 + 9|\omega - \pi|^2 \right. \\
&\quad \left. + 9|\pi - \bar{\omega}|^2 + 9|\bar{\omega} - \omega|^2 \right) e^t \\
&\geq 9 \left(\left[\max \{\omega, \pi, \bar{\omega}\} \right]^2 + |\omega - \pi|^2 + |\pi - \bar{\omega}|^2 + |\bar{\omega} - \omega|^2 \right) e^t \\
&= 9J(\omega, \pi, \bar{\omega})(t),
\end{aligned} \tag{16}$$

where $k = 9$. Clearly, χ is an expansive map in \mathcal{M} .

Definition 21 ([14]). Let \mathfrak{C} be a solid cone in a Banach space \mathcal{E} . A sequence $\{u_n\} \subset \mathfrak{C}$ is said to be a c -sequence if for every $c \gg \theta$ there is an integer N so that $u_n \ll c$ for all $n > N$.

Lemma 22 ([5]). If \mathcal{E} is a real Banach space with a solid cone \mathfrak{C} , then $\{u_n\}$ is a c -sequence whenever $\{u_n\} \subset \mathfrak{C}$ be a sequence with $\|u_n\| \rightarrow 0(\infty)$.

Lemma 23 ([10]). Let Y be a BA with a unit e and $k \in Y$. Then, $\lim \|k^n\|^{1/n}$ exists, and the spectral radius $\mathfrak{S}\mathfrak{R}(k)$ verifies

$$\mathfrak{S}\mathfrak{R}(k) = \lim \|k^n\|^{1/n} = \inf_n \|k^n\|^{1/n}. \tag{17}$$

Then, $(\zeta e - k)$ is invertible in Y provided that $\mathfrak{S}\mathfrak{R}(k) < |\zeta|$. Moreover,

$$(\zeta e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\zeta^{i+1}} \tag{18}$$

where the constant ζ is complex.

Lemma 24 ([10]). Let Y be a BA with a unit e and $\vartheta, \bar{\omega} \in Y$. Then,

$$\mathfrak{S}\mathfrak{R}(\vartheta + \bar{\omega}) \leq \mathfrak{S}\mathfrak{R}(\vartheta) + \mathfrak{S}\mathfrak{R}(\bar{\omega}), \quad \mathfrak{S}\mathfrak{R}(\vartheta\bar{\omega}) \leq \mathfrak{S}\mathfrak{R}(\vartheta)\mathfrak{S}\mathfrak{R}(\bar{\omega}), \tag{19}$$

provided that ϑ commutes with $\bar{\omega}$.

Lemma 25 ([12]). Let \mathcal{E} be a real Banach space with a solid cone \mathfrak{C} . Then,

- (1) $\vartheta \ll \kappa$, if $\vartheta \leq \bar{\omega} \ll \kappa$
- (2) $\vartheta = \theta$, if $\vartheta \ll \kappa$ for every $\kappa \gg \theta$

Lemma 26 ([11]). Let \mathfrak{C} be a solid cone in the BA Y and suppose that $k \in \mathfrak{C}$. Then, $\{ku_n\}$ is a c -sequence provided that $\{u_n\}$ be a c -sequence in \mathfrak{C} .

Lemma 27 ([5]). Let Y be a BA with a unit e and let $k \in Y$. Then,

$$\mathfrak{S}\mathfrak{R} \left((\zeta e - k)^{-1} \right) \leq \frac{1}{|\zeta| - \mathfrak{S}\mathfrak{R}(k)}, \tag{20}$$

where ζ is a complex constant with $\mathfrak{S}\mathfrak{R}(k) < |\zeta|$.

Lemma 28 ([5]). Let \mathfrak{C} be a solid cone in the BA Y with a unit e and let $\vartheta, k, l \in \mathfrak{C}$ so that $l \leq k$ and $\vartheta \leq l\vartheta$. Then, $\vartheta = \theta$ provided that $\mathfrak{S}\mathfrak{R}(k) < 1$.

In this paper, we prove some fixed-point theorems for generalized contractive and expansive maps in the setting of a J -CMS over a BA.

5. Main Results

Theorem 29. Let (\mathcal{M}, J) be a θ -complete symmetric J -CMS over the BA Y and $\chi : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping so that for all $\omega, \pi \in \mathcal{M}$

$$J(\chi\omega, \chi\pi, \chi\pi) \leq kJ(\omega, \pi, \pi), \quad (21)$$

where $\mathfrak{R}(k) < 1/s$. Then, χ admits a unique fixed point.

Proof. Let ω_0 be an arbitrary point in \mathcal{M} . Define a sequence $\{\omega_n\}$ in \mathcal{M} by $\omega_n = \chi^n \omega_0$. By (21), we have

$$J(\omega_n, \omega_{n+1}, \omega_{n+1}) = J(\chi\omega_{n-1}, \chi\omega_n, \chi\omega_n) \leq kJ(\omega_{n-1}, \omega_n, \omega_n). \quad (22)$$

Continuing in the same argument, we will get

$$J(\omega_n, \omega_{n+1}, \omega_{n+1}) \leq k^n J(\omega_0, \omega_1, \omega_1). \quad (23)$$

Moreover, for all $n, m \in N; n < m$, we have

$$\begin{aligned} J(\omega_n, \omega_m, \omega_m) &\leq s[J(\omega_n, \omega_{n+1}, \omega_{n+1}) + J(\omega_{n+1}, \omega_m, \omega_m) - J(\omega_{n+1}, \omega_{n+1}, \omega_{n+1})] \\ &\leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + sJ(\omega_{n+1}, \omega_m, \omega_m) \\ &\leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + s^2[J(\omega_{n+1}, \omega_{n+2}, \omega_{n+2}) \\ &\quad + J(\omega_{n+2}, \omega_m, \omega_m) - J(\omega_{n+2}, \omega_{n+2}, \omega_{n+2})] \\ &\leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + s^2J(\omega_{n+1}, \omega_{n+2}, \omega_{n+2}) + s^2J(\omega_{n+2}, \omega_m, \omega_m) \\ &\vdots \leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + s^2J(\omega_{n+1}, \omega_{n+2}, \omega_{n+2}) + \dots \\ &\quad + s^{m-n}J(\omega_{m-1}, \omega_m, \omega_m) \leq sk^n J(\omega_0, \omega_1, \omega_1) \\ &\quad + s^2k^{n+1}J(\omega_0, \omega_1, \omega_1) + \dots + s^{m-n}k^{m-1}J(\omega_0, \omega_1, \omega_1) \\ &= sk^n [e + sk + (sk)^2 + \dots + (sk)^{m-n-1}] J(\omega_0, \omega_1, \omega_1) \\ &\leq sk^n \left(\sum_{i=0}^{\infty} (sk)^i \right) J(\omega_0, \omega_1, \omega_1) = (e - sk)^{-1} sk^n J(\omega_0, \omega_1, \omega_1). \end{aligned} \quad (24)$$

In view of Remark 2, $\|sk^n J(\omega_0, \omega_1, \omega_1)\| \leq \|k^n\| \|J(\omega_0, \omega_1, \omega_1)\| \rightarrow 0(\infty)$, by Lemma 22, we have $\{sk^n J(\omega_0, \omega_1, \omega_1)\}$ is a c -sequence. Using Lemma 25 and Lemma 26, $\{\omega_n\}$ is a θ -Cauchy sequence in \mathcal{M} . By the θ -completeness of \mathcal{M} , there is $\rho \in \mathcal{M}$ so that

$$\lim J(\omega_n, \rho, \rho) = \lim_{n, m \rightarrow \infty} J(\omega_n, \omega_m, \omega_m) = J(\rho, \rho, \rho) = \theta. \quad (25)$$

Furthermore, one has

$$\begin{aligned} J(\rho, \chi\rho, \chi\rho) &\leq sJ(\rho, \chi\omega_n, \chi\omega_n) + sJ(\chi\omega_n, \chi\rho, \chi\rho) - sJ(\chi\omega_n, \chi\omega_n, \chi\omega_n) \\ &\leq sJ(\rho, \chi\omega_n, \chi\omega_n) + sJ(\chi\omega_n, \chi\rho, \chi\rho) \\ &\leq sJ(\rho, \omega_{n+1}, \omega_{n+1}) + skJ(\omega_n, \omega_{n+1}, \rho). \end{aligned} \quad (26)$$

Hence, $\{J(\rho, \omega_{n+1}, \omega_{n+1})\}$ and $\{J(\omega_n, \omega_{n+1}, \rho)\}$ are c -sequences, then by using Lemma 25 and Lemma 26, we get that $\chi\rho = \rho$. Thus, ρ is a fixed point of χ . If $\chi\zeta = \zeta$, then one writes

$$J(\rho, \rho, \zeta) = J(\chi\rho, \chi\rho, \chi\zeta) = kJ(\rho, \rho, \zeta). \quad (27)$$

That is,

$$(e - k)J(\rho, \rho, \zeta) \leq \theta. \quad (28)$$

The multiplication by

$$(e - k)^{-1} = \sum_{i=0}^{\infty} k^i \geq 0, \quad (29)$$

yields that $J(\rho, \rho, \zeta) \leq \theta$. Thus, $J(\rho, \rho, \zeta) = \theta$, so $\rho = \zeta$, a contradiction. Hence, the fixed point is unique.

Corollary 30. Let (\mathcal{M}, J) be a θ -complete symmetric J -CMS over the BA Y and $\chi : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping so that for all $\omega, \pi, \bar{\omega} \in \mathcal{M}$

$$J(\chi\omega, \chi\pi, \chi\bar{\omega}) \leq aJ(\omega, \chi\omega, \bar{\omega}) + bJ(\omega, \chi\omega, \pi), \quad (30)$$

where $\mathfrak{R}(a + b) < 1/s$. Then, χ has a unique fixed point.

Proof. In view of the proof of Theorem 29, we get the required result.

Theorem 31. Let (\mathcal{M}, J) be a θ -complete symmetric J -CMS over the BA Y and $\chi : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping so that for all $\omega, \pi \in \mathcal{M}$,

$$\begin{aligned} J(\chi\omega, \chi\pi, \chi^2\pi) &\leq aJ(\omega, \chi\omega, \chi^2\omega) + bJ(\pi, \chi\pi, \chi^2\pi) \\ &\quad + cJ(\omega, \chi\omega, \chi\pi) + dJ(\pi, \chi\pi, \chi^3\omega), \end{aligned} \quad (31)$$

where $s\mathfrak{R}(a + c) + \mathfrak{R}(b + d) < 1$ and $s^2\mathfrak{R}(c + d) + \mathfrak{R}(bs) < 1$. Then, χ admits a unique fixed point.

Proof. Take $\omega_0 \in \mathcal{M}$. We construct $\{\omega_n\}$ in \mathcal{M} so that

$$\omega_{n+1} = \chi\omega_n, \text{ for all } n = 0, 1, 2, \dots \quad (32)$$

Notice that if $\omega_{n'} = \omega_{n'+1}$ for some integer n' , then there is a fixed point. Suppose that $\omega_n \neq \omega_{n+1}$ for all $n \in N$. That is,

$$J(\omega_n, \omega_{n+1}, \omega_{n+2}) > \theta. \quad (33)$$

From (31), with $\omega = \omega_{n-1}$ and $\pi = \omega_n$, we have

$$\begin{aligned} J(\chi\omega_{n-1}, \chi\omega_n, \chi^2\omega_n) &\leq aJ(\omega_{n-1}, \chi\omega_{n-1}, \chi^2\omega_{n-1}) + bJ(\omega_n, \chi\omega_n, \chi^2\omega_n) \\ &\quad + cJ(\omega_{n-1}, \chi\omega_{n-1}, \chi\omega_n) + dJ(\omega_n, \chi\omega_n, \chi^3\omega_{n-1}), \end{aligned} \quad (34)$$

which implies that

$$\begin{aligned} J(\omega_n, \omega_{n+1}, \omega_{n+2}) &\leq aJ(\omega_{n-1}, \omega_n, \omega_{n+1}) + bJ(\omega_n, \omega_{n+1}, \omega_{n+2}) \\ &\quad + cJ(\omega_{n-1}, \omega_n, \omega_{n+1}) + dJ(\omega_n, \omega_{n+1}, \omega_{n+2}), \\ &= (a + c)J(\omega_{n-1}, \omega_n, \omega_{n+1}) + (b + d)J(\omega_n, \omega_{n+1}, \omega_{n+2}). \end{aligned} \quad (35)$$

Thus,

$$(e - b - d)J(\omega_n, \omega_{n+1}, \omega_{n+2}) \leq (a + c)J(\omega_{n-1}, \omega_n, \omega_{n+1}). \quad (36)$$

Since $s\mathfrak{E}\mathfrak{R}(a+c) + \mathfrak{E}\mathfrak{R}(b+d) < 1$ leads to $\mathfrak{E}\mathfrak{R}(b+d) < 1$, by Lemma 23, $(e-b-d)$ is invertible. So

$$J(\omega_n, \omega_{n+1}, \omega_{n+2}) \leq (e-b-d)^{-1}(a+c)J(\omega_{n-1}, \omega_n, \omega_{n+1}). \quad (37)$$

Put $k = (e-b-d)^{-1}(a+c)$. Hence,

$$J(\omega_n, \omega_{n+1}, \omega_{n+2}) \leq k J(\omega_{n-1}, \omega_n, \omega_{n+1}) \leq \dots \leq k^n J(\omega_0, \omega_1, \omega_2). \quad (38)$$

By Lemma 24 and Lemma 27, we have

$$\begin{aligned} \mathfrak{E}\mathfrak{R}(k) &= \mathfrak{E}\mathfrak{R}[(e-b-d)^{-1}(a+c)] \\ &\leq \mathfrak{E}\mathfrak{R}((e-b-d)^{-1})\mathfrak{E}\mathfrak{R}(a+c) \\ &\leq \frac{\mathfrak{E}\mathfrak{R}(a+c)}{1-\mathfrak{E}\mathfrak{R}(b+d)} < \frac{1}{s}. \end{aligned} \quad (39)$$

Thus, $(e-sk)$ is invertible, so $\|(sk)^n\| \rightarrow 0(\infty)$. Due to (J_2) ,

$$J(\omega_n, \omega_n, \omega_{n+1}) \leq J(\omega_n, \omega_{n+1}, \omega_{n+2}), \quad (40)$$

with $\omega_n \neq \omega_{n+1}$, and by the symmetry property

$$J(\omega_{n+1}, \omega_{n+1}, \omega_n) = J(\omega_{n+1}, \omega_n, \omega_n) = J(\omega_n, \omega_n, \omega_{n+1}). \quad (41)$$

Then, by (38) and (41), we have $J(\omega_{n+1}, \omega_{n+1}, \omega_n) \leq k^n J(\omega_0, \omega_1, \omega_2)$. Moreover, for all $n, m \in N; n < m$, we have

$$\begin{aligned} J(\omega_n, \omega_m, \omega_m) &\leq s[J(\omega_n, \omega_{n+1}, \omega_{n+1}) + J(\omega_{n+1}, \omega_m, \omega_m) - J(\omega_{n+1}, \omega_{n+1}, \omega_{n+1})] \\ &\leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + sJ(\omega_{n+1}, \omega_m, \omega_m) \\ &\leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + s^2[J(\omega_{n+1}, \omega_{n+2}, \omega_{n+2}) + J(\omega_{n+2}, \omega_m, \omega_m) \\ &\quad - J(\omega_{n+2}, \omega_{n+2}, \omega_{n+2})] \leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) \\ &\quad + s^2J(\omega_{n+1}, \omega_{n+2}, \omega_{n+2}) + s^2J(\omega_{n+2}, \omega_m, \omega_m) \\ &\vdots \leq sJ(\omega_n, \omega_{n+1}, \omega_{n+1}) + s^2J(\omega_{n+1}, \omega_{n+2}, \omega_{n+2}) + \dots + s^{m-n}J(\omega_{m-1}, \omega_m, \omega_m) \\ &\leq sk^n J(\omega_0, \omega_1, \omega_2) + s^2k^{n+1}J(\omega_0, \omega_1, \omega_2) + \dots + s^{m-n}k^{m-1}J(\omega_0, \omega_1, \omega_2) \\ &= sk^n [e + sk + (sk)^2 + \dots + (sk)^{m-n-1}] J(\omega_0, \omega_1, \omega_2) \\ &\leq sk^n \left(\sum_{i=0}^{\infty} (sk)^i \right) J(\omega_0, \omega_1, \omega_2) = sk^n (e-sk)^{-1} J(\omega_0, \omega_1, \omega_2). \end{aligned} \quad (42)$$

In view of Remark 2, $\|sk^n J(\omega_0, \omega_1, \omega_2)\| \leq \|sk^n\| \|J(\omega_0, \omega_1, \omega_2)\| \rightarrow 0(\infty)$; by Lemma 22, we have $\{sk^n J(\omega_0, \omega_1, \omega_2)\}$ is a c -sequence. Next, by using Lemma 25 and Lemma 26, ω_n is a θ -Cauchy sequence in \mathcal{M} . By the θ -completeness of \mathcal{M} , there is $\bar{\omega} \in \mathcal{M}$ so that

$$\lim J(\omega_n, \bar{\omega}, \bar{\omega}) = \lim_{n, m \rightarrow \infty} J(\omega_n, \omega_m, \omega_m) = J(\bar{\omega}, \bar{\omega}, \bar{\omega}) = \theta. \quad (43)$$

To see that $\bar{\omega}$ is a fixed point of χ , we have

$$\begin{aligned} J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) &\leq s[J(\bar{\omega}, \chi\omega_n, \chi\omega_n) + J(\chi\omega_n, \chi\bar{\omega}, \chi^2\bar{\omega}) - J(\chi\omega_n, \chi\omega_n, \chi\omega_n)] \\ &= sJ(\bar{\omega}, \chi\omega_n, \chi\omega_n) + sJ(\chi\omega_n, \chi\bar{\omega}, \chi^2\bar{\omega}) \\ &\leq sJ(\bar{\omega}, \chi\omega_n, \chi\omega_n) + s[aJ(\omega_n, \chi\omega_n, \chi^2\omega_n) + bJ(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) \\ &\quad + cJ(\omega_n, \chi\omega_n, \chi\bar{\omega}) + dJ(\bar{\omega}, \chi\bar{\omega}, \chi^3\omega_n)]. \end{aligned} \quad (44)$$

Then,

$$\begin{aligned} J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) &\leq sJ(\bar{\omega}, \omega_{n+1}, \omega_{n+1}) + s[aJ(\omega_n, \omega_{n+1}, \omega_{n+2}) \\ &\quad + bJ(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) + cJ(\omega_n, \omega_{n+1}, \chi\bar{\omega}) \\ &\quad + dJ(\bar{\omega}, \chi\bar{\omega}, \omega_{n+3})] \\ &\leq sbJ(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) + csJ(\bar{\omega}, \bar{\omega}, \chi\bar{\omega}) \\ &\quad + sdJ(\bar{\omega}, \chi\bar{\omega}, \bar{\omega}), \end{aligned} \quad (45)$$

which implies that

$$(e-bs)J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) \leq s(c+d)J(\bar{\omega}, \bar{\omega}, \chi\bar{\omega}). \quad (46)$$

Since $s^2\mathfrak{E}\mathfrak{R}(c+d) + \mathfrak{E}\mathfrak{R}(bs) < 1$ leads to $\mathfrak{E}\mathfrak{R}(bs) < 1$, it follows by Lemma 23 that $(e-bs)$ is invertible. So, $J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) \leq (e-bs)^{-1}s(c+d)J(\bar{\omega}, \bar{\omega}, \chi\bar{\omega})$. Put $h = (e-bs)^{-1}s(c+d)$. Then

$$J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) \leq hJ(\bar{\omega}, \bar{\omega}, \chi\bar{\omega}). \quad (47)$$

Note that by Lemma 24 and Lemma 27,

$$\begin{aligned} \mathfrak{E}\mathfrak{R}(h) &= \mathfrak{E}\mathfrak{R}[(e-bs)^{-1}(sc+sd)] \\ &\leq \mathfrak{E}\mathfrak{R}((e-bs)^{-1})\mathfrak{E}\mathfrak{R}(sc+sd) \\ &\leq \frac{\mathfrak{E}\mathfrak{R}(sc+sd)}{1-\mathfrak{E}\mathfrak{R}(bs)} < \frac{1}{s}. \end{aligned} \quad (48)$$

In view of (47), there is arise 2 cases.

Case I. If $\chi\bar{\omega} = \chi^2\bar{\omega}$, then

$$\begin{aligned} J(\bar{\omega}, \chi\bar{\omega}, \chi\bar{\omega}) &\leq hJ(\bar{\omega}, \bar{\omega}, \chi\bar{\omega}) \\ &\leq hJ(\bar{\omega}, \chi\bar{\omega}, \chi\bar{\omega}) \quad (\text{by symmetry}) \end{aligned} \quad (49)$$

$$(e-h)J(\bar{\omega}, \chi\bar{\omega}, \chi\bar{\omega}) \leq \theta.$$

We multiply by

$$(e-h)^{-1} = \sum_{i=0}^{\infty} (h^i) \geq 0, \quad (50)$$

to get $J(\bar{\omega}, \chi\bar{\omega}, \chi\bar{\omega}) = \theta$, which implies that $\bar{\omega} = \chi\bar{\omega}$. Hence, $\bar{\omega}$ is the fixed point of χ .

Case II. If $\chi\bar{\omega} \neq \chi^2\bar{\omega}$, then

$$\begin{aligned} J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) &\leq h J(\bar{\omega}, \bar{\omega}, \chi\bar{\omega}) \\ &\leq h J(\bar{\omega}, \chi\bar{\omega}, \chi\bar{\omega}) \quad (\text{by symmetry}) \\ &\leq J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) \quad (\text{by } J_2) \\ (e-h)J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) &\leq \theta. \end{aligned} \tag{51}$$

Again, we multiply by

$$(e-h)^{-1} = \sum_{i=0}^{\infty} (h^i) \geq 0, \tag{52}$$

to have $J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) = \theta$. So, $\bar{\omega} = \chi\bar{\omega} = \chi^2\bar{\omega}$. Hence, $\bar{\omega}$ is a fixed point of χ . Suppose that $w(\neq \bar{\omega})$ is such that $\chi w = w$. One writes

$$\begin{aligned} J(\bar{\omega}, \bar{\omega}, \bar{\omega}) &= J(\chi\bar{\omega}, \chi\bar{\omega}, \chi\bar{\omega}) \\ &= J(\chi\bar{\omega}, \chi w, \chi w) (\text{by symmetry}) \\ &= J(\chi\bar{\omega}, \chi w, \chi^2 w) \\ &\leq a J(\bar{\omega}, \chi\bar{\omega}, \chi^2\bar{\omega}) + b J(w, \chi w, \chi^2 w) + c J(\bar{\omega}, \chi\bar{\omega}, \chi w) + d J(w, \chi w, \chi^3\bar{\omega}) \\ &\leq a J(\bar{\omega}, \bar{\omega}, \bar{\omega}) + b J(w, w, w) + c J(\bar{\omega}, \bar{\omega}, w) + d J(w, w, \bar{\omega}) \\ &\leq a J(\bar{\omega}, \bar{\omega}, \bar{\omega}) + b J(w, w, w) + c J(\bar{\omega}, \bar{\omega}, w) + d J(w, w, \bar{\omega}) [\text{by } (J_2)]. \end{aligned} \tag{53}$$

Hence,

$$J(\bar{\omega}, \bar{\omega}, w) \leq (b+c+d)J(\bar{\omega}, \bar{\omega}, w). \tag{54}$$

Since $\mathfrak{E}\mathfrak{R}(b+c+d) \leq s^2\mathfrak{E}\mathfrak{R}(c+d) + \mathfrak{E}\mathfrak{R}(bs) < 1$, by Lemma 28, we acquire that $J(\bar{\omega}, \bar{\omega}, w) = \theta$, that is, $\bar{\omega} = w$.

6. Fixed-Point Results of Expansive Maps

Here, the case of expansive mappings in the setting of a J -CMS over a BA is studied.

Theorem 32. Let (\mathcal{M}, J) be a θ -complete J -CMS over the BA Y and $\chi : \mathcal{M} \rightarrow \mathcal{M}$ be an onto mapping so that for all $w, \pi \in \mathcal{M}$,

$$J(\chi w, \chi^2 w, \chi\pi) \geq \alpha J(w, \chi w, \pi), \tag{55}$$

where $\mathfrak{E}\mathfrak{R}(\alpha^{-1}) < 1/s$. Then, χ possesses a unique fixed point.

Proof. Let $\omega_0 \in \mathcal{M}$. Since χ is onto, there is $\omega_1 \in \mathcal{M}$ so that $\omega_0 = \chi\omega_1$. By continuing this process, we get $\omega_n = \chi\omega_{n+1}$, for all $n \in N \cup \{0\}$. If $\omega_{n_0} = \omega_{n_0+1}$ for some $n_0 \in N \cup \{0\}$, then ω_{n_0} is a fixed point of χ . Assume that $\omega_n \neq \omega_{n+1}$ for all $n \in N$. From (55), with $\omega = \omega_{n+1}$ and $\pi = \omega_n$, we have

$$\begin{aligned} J(\omega_n, \omega_{n-1}, \omega_{n-1}) &= J(\chi\omega_{n+1}, \chi^2\omega_{n+1}, \chi\omega_n) \\ &\geq \alpha J(\omega_{n+1}, \chi\omega_{n+1}, \omega_n) \\ &= \alpha J(\omega_{n+1}, \omega_n, \omega_n) \end{aligned} \tag{56}$$

which implies that

$$J(\omega_{n+1}, \omega_n, \omega_n) \leq h J(\omega_n, \omega_{n-1}, \omega_{n-1}) \leq \dots \leq h^n J(\omega_0, \omega_1, \omega_1), \tag{57}$$

where $h = \alpha^{-1}$. Note that

$$\mathfrak{E}\mathfrak{R}(h) = \mathfrak{E}\mathfrak{R}(\alpha^{-1}) < \frac{1}{s}, \tag{58}$$

so $(e-sh)^{-1} = \sum_{i=0}^{\infty} (sh^i)$ and $\|sh^n\| \rightarrow 0(\infty)$. Following the lines of the proof of Theorem 29, we derive that $\{\omega_n\}$ is a θ -Cauchy sequence. Since (\mathcal{M}, J) is θ -complete, there is $\bar{\omega} \in \mathcal{M}$ so that $\omega_n \rightarrow \bar{\omega}$, that is,

$$\lim J(\omega_n, \bar{\omega}, \bar{\omega}) = \lim_{n,m \rightarrow \infty} J(\omega_n, \omega_m, \omega_m) = J(\bar{\omega}, \bar{\omega}, \bar{\omega}) = \theta. \tag{59}$$

Since χ is onto, there is $w \in \mathcal{M}$ so that $\bar{\omega} = \chi w$. From (55) with $\omega = \omega_{n+1}$ and $\pi = w$, we have

$$\begin{aligned} J(\omega_n, \omega_{n-1}, \bar{\omega}) &= J(\chi\omega_{n+1}, \chi^2\omega_{n+1}, \chi w) \geq \alpha J(\omega_{n+1}, \chi\omega_{n+1}, w) \\ &= \alpha J(\omega_{n+1}, \omega_n, w). \end{aligned} \tag{60}$$

So, $J(\omega_{n+1}, \omega_n, w) \leq h J(\omega_n, \omega_{n-1}, \bar{\omega})$.

Now, $\{J(\omega_n, \omega_{n-1}, \bar{\omega})\}$ is a c -sequence; then, by using Lemma 25 and Lemma 26, it follows that $J(\bar{\omega}, \bar{\omega}, w) = \theta$. That is, $\bar{\omega} = w$. Then, $\bar{\omega} = \chi w = \chi\bar{\omega}$. To prove the uniqueness, assume that $\bar{\omega} \neq u$ so that $\chi u = u$ and $\chi\bar{\omega} = \bar{\omega}$. Now, by

$$\begin{aligned} J(\bar{\omega}, \bar{\omega}, u) &= J(\chi\bar{\omega}, \chi^2\bar{\omega}, \chi u) \geq \alpha J(\bar{\omega}, \chi\bar{\omega}, u) \\ &\geq \alpha J(\bar{\omega}, \bar{\omega}, u) > J(\bar{\omega}, \bar{\omega}, u), \end{aligned} \tag{61}$$

which is a contradiction. Hence, $\bar{\omega} = u$.

Theorem 33. Let (\mathcal{M}, J) be a θ -complete J -CMS over the BA Y and $\chi : \mathcal{M} \rightarrow \mathcal{M}$ be so that for all $w, \pi \in \mathcal{M}$,

$$J(\chi w, \chi\pi, \chi^2\pi) \geq \alpha J(w, \chi w, \chi^2 w), \tag{62}$$

where $\mathfrak{E}\mathfrak{R}(\alpha^{-1}) < 1/s$. Then, χ admits a unique fixed point.

Proof. Let $\omega_0 \in \mathcal{M}$. Since χ is onto, there is $\omega_1 \in \mathcal{M}$ so that $\omega_0 = \chi\omega_1$. Let $\omega_n = \chi\omega_{n+1}$, for all $n \in N \cup \{0\}$. In case $\omega_{n_0} = \omega_{n_0+1}$ for some $n_0 \in N \cup \{0\}$, then ω_{n_0} is a fixed point of χ . Suppose that $\omega_n \neq \omega_{n+1}$ for all $n \in N$. From (62) with $\omega = \omega_{n+1}$ and $\pi = \omega_n$, we have

$$J(\chi\omega_{n+1}, \chi\omega_n, \chi^2\omega_n) \geq \alpha J(\omega_{n+1}, \chi\omega_{n+1}, \chi^2\omega_{n+1}), \tag{63}$$

which implies that

$$J(\omega_n, \omega_{n-1}, \omega_{n-2}) \geq \alpha J(\omega_{n+1}, \omega_n, \omega_{n-1}), \quad (64)$$

or

$$J(\omega_{n+1}, \omega_n, \omega_{n-1}) \leq \alpha^{-1} J(\omega_n, \omega_{n-1}, \omega_{n-2}). \quad (65)$$

Put $h = \alpha^{-1}$. It is evident that

$$J(\omega_{n+1}, \omega_n, \omega_{n-1}) \leq h J(\omega_n, \omega_{n-1}, \omega_{n-2}), \quad (66)$$

where $\mathfrak{C}\mathfrak{R}(h) < 1/s$. By the mimic of the proof of Theorem 36, we can show that ω_n is a θ -Cauchy sequence. Since (\mathcal{M}, J) is a θ -complete J -CMS over a BA, there is $\bar{\omega} \in \mathcal{M}$ such that

$$\lim J(\omega_n, \bar{\omega}, \bar{\omega}) = \lim_{n,m \rightarrow \infty} J(\omega_n, \omega_m, \omega_m) = J(\bar{\omega}, \bar{\omega}, \bar{\omega}) = \theta. \quad (67)$$

Since χ is onto, there is $w \in \mathcal{M}$ so that $\bar{\omega} = \chi w$. From (62), we have for $\omega = w$ and $\pi = \omega_{n+1}$,

$$J(\bar{\omega}, \omega_n, \omega_{n-1}) = J(\chi w, \chi \omega_{n+1}, \chi^2 \omega_{n+1}) \geq \alpha J(w, \chi w, \chi^2 w). \quad (68)$$

So, $J(w, \chi w, \chi^2 w) \leq h J(\bar{\omega}, \omega_n, \omega_{n-1})$,

where $h = \alpha^{-1}$. Since $\{J(\bar{\omega}, \omega_n, \omega_{n-1})\}$ is a c -sequence, by using Lemma 25 and Lemma 26, it follows that $J(\bar{\omega}, \chi w, \chi^2 w) = \theta$. That is, $w = \chi w = \chi^2 w$.

7. Examples

The following examples demonstrate the results obtained in a J -CMS over Banach algebra Y .

Example 34. Take on $Y = C_{\mathbb{R}}^1[0, 1]$ the norm $\|\omega\| = \|\omega\|_{\infty} + \|\omega'\|_{\infty}$ for $\omega \in Y$. Consider in Y a just pointwise multiplication. Then, Y is a real unit BA with unit $e = 1$. Set the cone $\mathfrak{C} = \{\omega \in Y : \omega \geq 0\}$ in Y . Moreover, \mathfrak{C} is not normal (see [12]). Let $\mathcal{M} = [0, \infty)$. Take $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ as

$$J(\omega, \pi, \bar{\omega})(t) = \begin{cases} \theta & , \text{ if } \omega = \pi = \bar{\omega} \\ (\max\{\omega, \pi, \bar{\omega}\})^2 e^t & , \text{ otherwise} \end{cases} = \begin{cases} \theta & , \text{ if } \omega = \pi = \bar{\omega} \\ (\max\{\omega^2, \pi^2, \bar{\omega}^2\}) e^t & , \text{ otherwise} \end{cases} \quad (69)$$

be a J -CMS over a BA on \mathcal{M} . Define $\chi : \mathcal{M} \rightarrow \mathcal{M}$ by $\chi \omega = (\omega/5) \sin(\omega/5)$. We have

$$\begin{aligned} J(\chi \omega, \chi \pi, \chi \bar{\omega})(t) &= J\left(\frac{\omega}{5} \sin \frac{\omega}{5}, \frac{\pi}{5} \sin \frac{\pi}{5}, \frac{\bar{\omega}}{5} \sin \frac{\bar{\omega}}{5}\right)(t) \\ &\leq \max \left\{ \left(\frac{\omega}{5}\right)^2, \left(\frac{\pi}{5}\right)^2, \left(\frac{\bar{\omega}}{5}\right)^2 \right\} e^t = \frac{1}{25} \max\{\omega^2, \pi^2, \bar{\omega}^2\} e^t. \end{aligned} \quad (70)$$

Therefore, $J(\chi \omega, \chi \pi, \chi \bar{\omega})(t) \leq (1/25) J(\omega, \pi, \bar{\omega})(t)$. Then, the contractive condition (21) holds, and θ is the unique fixed point of χ . That is, the conditions of Theorem 36 hold.

Example 35. Let Y be a BA and \mathfrak{C} be a cone (the same ones as those in Example 8), and let $\mathcal{M} = \mathbb{R}^+$. Consider $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow Y$ as in Example 8. Then, (\mathcal{M}, J) is a J -CMS over the BA Y . Define $\chi : \mathcal{M} \rightarrow \mathcal{M}$ by $\chi \omega = 2\omega$. Choose $\alpha = 4$, then all the conditions of Theorem 6 hold and θ is the unique fixed point of χ . Indeed,

$$\begin{aligned} J(\chi \omega, \chi^2 \omega, \chi \pi)(t) &= \left((\max\{\chi \omega, \chi^2 \omega\})^2 + (\max\{\chi^2 \omega, \chi \pi\})^2 \right) \\ &\quad + (\max\{\chi \pi, \chi \omega\})^2 e^t \\ &= \left((\max\{2\omega, 4\omega\})^2 + (\max\{4\omega, 2\pi\})^2 \right) \\ &\quad + (\max\{2\pi, 2\omega\})^2 e^t \\ &= (16\omega^2 + 4(\max\{2\omega, \pi\})^2 + 4(\max\{\omega, \pi\})^2) e^t \\ &= 4(4\omega^2 + (\max\{2\omega, \pi\})^2 + (\max\{\omega, \pi\})^2) e^t, \end{aligned}$$

$$\begin{aligned} J(\omega, \chi \omega, \pi)(t) &= \left((\max\{\omega, \chi \omega\})^2 + (\max\{\chi \omega, \pi\})^2 \right) \\ &\quad + (\max\{\pi, \omega\})^2 e^t \\ &= \left((\max\{\omega, 2\omega\})^2 + (\max\{2\omega, \pi\})^2 \right) \\ &\quad + (\max\{\pi, \omega\})^2 e^t \\ &= (4\omega^2 + (\max\{2\omega, \pi\})^2 + (\max\{\omega, \pi\})^2) e^t, \end{aligned} \quad (71)$$

and so, $J(\chi \omega, \chi^2 \omega, \chi \pi)(t) \geq 4J(\omega, \chi \omega, \pi)(t)$.

8. Application to the Existence of a Solution of Integral Equations

Denote by $\mathcal{C}([0, T], \mathbb{R})$ the class of all continuous functions on $[0, T]$ (where $T > 0$).

Let $Y = C[0, T]$ be considered with the norm $\|\omega\| = \|\omega\|_{\infty} + \|\omega'\|_{\infty}$. Take the usual multiplication, then Y is a BA with the unit 1. Set $\mathfrak{C} = \{\omega \in Y : \omega(t) \geq 0, t \in [a, b]\}$.

Let J be the J -cone metric given as

$$J(\rho, \mathfrak{q}, \sigma)(t) = \sup_{t \in [0, T]} (|\rho(t) - \mathfrak{q}(t)|^2 + |\mathfrak{q}(t) - \sigma(t)|^2 + |\rho(t) - \sigma(t)|^2) e^t, \quad (72)$$

for all $\rho, \mathfrak{q}, \sigma \in \mathcal{C}([0, T], \mathbb{R})$. Note that $(\mathcal{C}([0, T], \mathbb{R}), J)$ is a complete J -CMS over BA $\mathcal{C}([0, T], \mathbb{R})$.

Theorem 36. Let $K, f,$ and a be continuous mappings so that:

- (i) $\sup_{t \in [0, T]} |K(t, s)| = M < 1/\sqrt{T}$, where $K \in \mathcal{C}([0, T] \times [0, T], \mathbb{R})$
- (ii) $a \in \mathcal{C}([0, T], \mathbb{R})$
- (iii) $f \in \mathcal{C}([0, T] \times \mathbb{R}, \mathbb{R})$
- (iv) $|f(s, \rho(s)) - f(s, \mathfrak{q}(s))| \leq \zeta |\rho(s) - \mathfrak{q}(s)|$

for all $\rho, \mathfrak{q} \in \mathcal{C}([0, T], \mathbb{R}), t \in [0, T]$, where $\zeta \in [0, 1/s)$. Then, the integral equation

$$\rho(t) = \int_0^T K(t, s) f(s, \rho(s)) ds + a(t), t \in [0, T], \quad (73)$$

admits a unique solution in $\mathcal{C}([0, T], \mathbb{R})$.

Proof. Let $\mathcal{F} : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathcal{C}([0, T], \mathbb{R})$ be defined by

$$\mathcal{F}(\rho)(t) = \int_0^T K(t, s) f(s, \rho(s)) ds + a(t), \quad (74)$$

for all $t \in [0, T]$. We have

$$\begin{aligned} & J(\mathcal{F}(\rho), \mathcal{F}(\mathfrak{q}), F(\mathfrak{q}))(t) \\ &= \sup_{t \in [0, T]} (|\mathcal{F}(\rho(t)) - \mathcal{F}(\mathfrak{q}(t))|^2 + |\mathcal{F}(\mathfrak{q}(t)) - \mathcal{F}(\mathfrak{q}(t))|^2 + |\mathcal{F}(\rho(t)) \\ &\quad - \mathcal{F}(\rho(t))|^2) e^t = 2 \sup_{t \in [0, T]} (|\mathcal{F}(\rho(t)) - \mathcal{F}(\rho(t))|^2) e^t \\ &= 2 \sup_{t \in [0, T]} \left(\int_0^T |K(t, s)|^2 (|f(s, \rho(s)) - f(s, \rho(s))|^2) ds \right) e^t \\ &\leq M^2 T \left(2 \sup_{s \in [0, T]} \zeta |\rho(s) - \rho(s)|^2 \right) e^t = \zeta J(\rho, \mathfrak{q}, \mathfrak{q}). \end{aligned} \quad (75)$$

Then, all the conditions of Theorem 29 hold, and hence, there is a unique fixed point of \mathcal{F} . So, (73) has a unique solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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